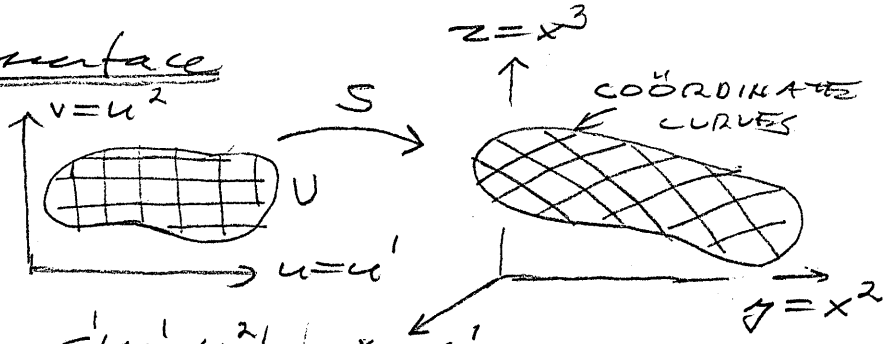
1/29/01 [3]. **Surfaces in space. First fundamental form.**

This part of the theory of surfaces is:
*local* - deals with a "piece" of a surface
*extrinsic* - surfaces lie in Euclidean space.

**Parameterized surface**

U open set in $\mathbb{R}^2$

$S : U \to \mathbb{E}^3$




Write

$X = S(u,v)$ or

$$\begin{cases} x^1 = S^1(u^1, u^2) \\ x^2 = S^2(u^1, u^2) \\ x^3 = S^3(u^1, u^2) \end{cases}$$

**Regular surface** S satisfies:
i) S is class $C^3$ on open set U
ii) the matrix derivative

$$S_*(u,v) = \begin{bmatrix} \frac{\partial S^1}{\partial u}(u,v) & \frac{\partial S^1}{\partial v}(u,v) \\ \frac{\partial S^2}{\partial u}(u,v) & \frac{\partial S^2}{\partial v}(u,v) \\ \frac{\partial S^3}{\partial u}(u,v) & \frac{\partial S^3}{\partial v}(u,v) \end{bmatrix}$$

3 × 2 matrix

which is a linear transformation -

$(S_*)^i_j = \frac{\partial S^i}{\partial u^j}$ - has rank 2.

That is, the vectors

$S_1 = \frac{\partial S}{\partial u} = \frac{\partial S}{\partial u^1}$ and $S_2 = \frac{\partial S}{\partial v} = \frac{\partial S}{\partial u^2}$

are linearly independent: $S_1 \times S_2 \neq 0$.

**Example** A surface is in *range form* if

$$X = \begin{bmatrix} x \\ y \\ f(x,y) \end{bmatrix}, \; f \text{ class } C^3.$$

**Fact** Every regular surface can be expressed locally in Monge form.

**Pf:**

$$S_1 \times S_2 = \pm\left[\frac{\partial(S^2,S^3)}{\partial(u,v)}, \frac{\partial(S^3,S^1)}{\partial(u,v)}, \frac{\partial(S^1,S^2)}{\partial(u,v)}\right]$$

is not zero, so one of the components is not zero; say, $\frac{\partial(S^1,S^2)}{\partial(u,v)} \neq 0$ at $(u_0,v_0)$.

By the Inverse Function theorem, we can find

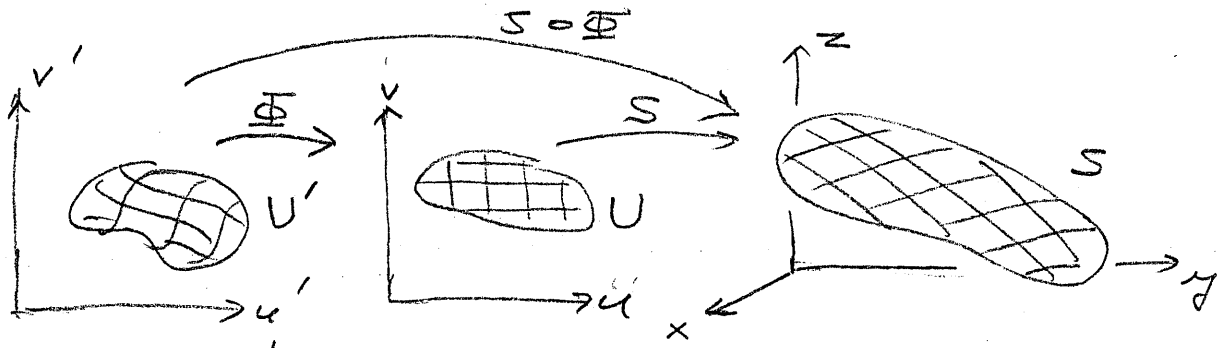
$\Phi$: $\begin{cases} u = \Phi^1(x,y) \\ v = \Phi^2(x,y) \end{cases}$ near $(x_0,y_0)$ corresponding to $(u_0,v_0)$.

Set $f(x,y) = S^3(\Phi^1(x,y), \Phi^2(x,y))$.

Then $(S\circ\Phi)(x,y) = \begin{bmatrix} x \\ y \\ f(x,y) \end{bmatrix}$ and $S\circ\Phi$ is injective near $(x_0,y_0)$.

**Consequence** For a regular surface $S$, $(u,v) \mapsto S(u,v)$ is locally injective.

**Change of parameters** (or coördinates)




$\Phi$: $\begin{cases} u = \Phi^1(u',v') \\ v = \Phi^2(u',v') \end{cases}$. Write $S(u',v')$ for $(S\circ\Phi)(u',v')$.

Assume change of parameters is a smooth local bijection, so $\Phi_*$ (derivative) is $\neq 0$:

i.e. $\frac{\partial(\Phi^1,\Phi^2)}{\partial(u',v')} \neq 0$ or $\begin{vmatrix} \frac{\partial\Phi^1}{\partial u} & \frac{\partial\Phi^1}{\partial v} \\ \frac{\partial\Phi^2}{\partial u} & \frac{\partial\Phi^2}{\partial v} \end{vmatrix} \neq 0$ (determinant)

Then: $S \circ \Phi$ is a regular surface:

$$(S\circ\Phi)_*(u',v') = S_*(\Phi(u',v'))\ \Phi_*(u',v')$$

$3\times 2$ rank 2 $\quad 2\times 2$ rank 2

Or:

$$\frac{\partial S}{\partial u'} = \frac{\partial S}{\partial u}\frac{\partial u}{\partial u'} + \frac{\partial S}{\partial v}\frac{\partial v}{\partial u'}$$

$$\frac{\partial S}{\partial v'} = \frac{\partial S}{\partial u}\frac{\partial u}{\partial v'} + \frac{\partial S}{\partial v}\frac{\partial v}{\partial v'}$$

$\frac{\partial u}{\partial u'}$ means $\frac{\partial \Phi^1}{\partial u'}$, etc.

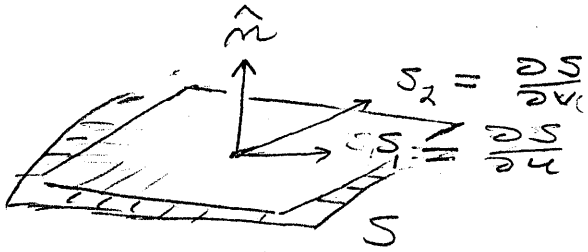
to

$$\frac{\partial S}{\partial u'} \times \frac{\partial S}{\partial v'} = \left(\frac{\partial S}{\partial u} \times \frac{\partial S}{\partial v}\right)\frac{\partial(u,v)}{\partial(u',v')}.$$

1/31/01 Note The parameterized surface $S$ and $S\circ\Phi$ are the same geometric surface. Quantities are well-defined on "geometric" when they do not depend on parameterization. This needs to be checked, but is usually immediate.

Tangent plane

$S(u,v)$ and the vectors $S_1(u,v)$ & $S_2(u,v)$ are a frame for the tangent plane to $S$ at $S(u,v)$.




$$\hat{n}(u,v) = \frac{S_1(u,v)\times S_2(u,v)}{\|S_1(u,v)\times S_2(u,v)\|} = \left(\begin{array}{c}\text{unit normal to } S \text{ at} \\ \text{point } S(u,v)\end{array}\right)$$

Check independence on parameterization:

$$\frac{\partial S}{\partial u'} \times \frac{\partial S}{\partial v'} = \left(\frac{\partial S}{\partial u} \times \frac{\partial S}{\partial v}\right)\frac{\partial(u,v)}{\partial(u',v')}$$

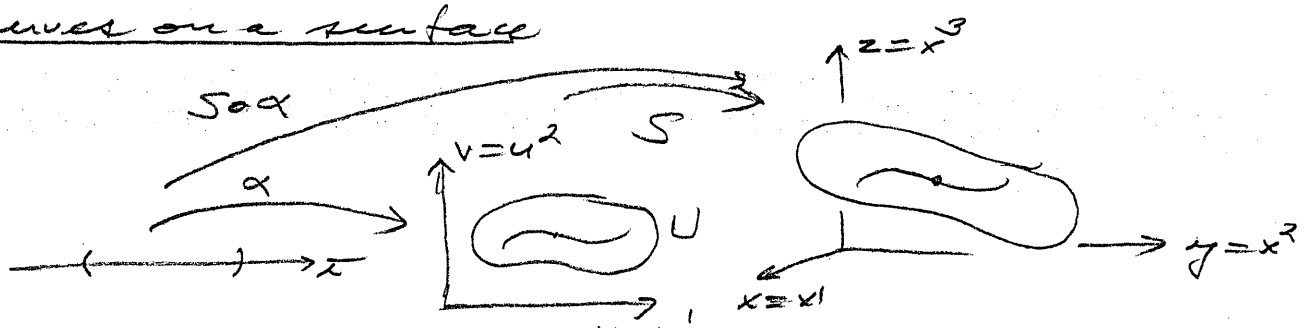
$$\left\|\frac{\partial S}{\partial u'} \times \frac{\partial S}{\partial v'}\right\| = \left\|\frac{\partial S}{\partial u} \times \frac{\partial S}{\partial v}\right\| \left|\frac{\partial(u,v)}{\partial(u',v')}\right|$$

so

$$\hat{n}(u',v') = \hat{n}(u,v)\cdot \operatorname{sign}\left(\begin{array}{c}\text{Jacobian} \\ \text{determinant}\end{array}\right).$$

Thus: The normal is reversed when the Jacobian is negative.

## Curves on a surface




Parameterized curve in $uv$-plane:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \alpha(t) \quad \text{or} \quad \begin{cases} u^1 = \alpha^1(t) \\ u^2 = \alpha^2(t) \end{cases}$$

Parameterized curve in surface $S$:

$$X = (S \circ \alpha)(t) = S(\alpha(t)) \quad \text{or} \quad x^i = S^i(\alpha^1(t), \alpha^2(t)) \qquad i = 1,2,3$$

Velocity vector:

$$\frac{d}{dt} S(\alpha(t)) = S_*(\alpha(t))\, \dot{\alpha}(t)$$

($3\times 2$ $\quad$ $2\times 1$)

$$\text{or} \quad \frac{\partial S}{\partial u}\frac{du}{dt} + \frac{\partial S}{\partial v}\frac{dv}{dt}$$

$$\text{or} \quad \frac{\partial S}{\partial u^1}\frac{du^1}{dt} + \frac{\partial S}{\partial u^2}\frac{du^2}{dt}$$

$$\text{or} \quad S_1 \dot{u}^1 + S_2 \dot{u}^2 \leftarrow$$

Components of velocity vector with respect to $S_1, S_2$.

**Thus:** The tangent vector to the curve $S(\alpha(t))$ at $\alpha(t_0)$ lie in the tangent plane to $S$ at $S(\alpha(t_0))$.

**And:** The curve is regular unless $\dot{\alpha}(t) \neq 0$ since $S_*(\alpha(t))$ (linear) is injective.

## First fundamental form

$S$ regular surface

$S\circ\alpha$ regular curve on $S$

$s$ = arc length along $S\circ\alpha$

$T_\alpha$ = unit tangent vector to $S\circ\alpha$ at $S(\alpha(t))$

We have

$$T_\alpha \frac{ds}{dt} = \frac{d}{dt} S\circ\alpha(t) = \frac{\partial S}{\partial u}\frac{du}{dt} + \frac{\partial S}{\partial v}\frac{dv}{dt}$$

so

$$\left(\frac{ds}{dt}\right)^2 = E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2$$

$E(\alpha^1(t), \alpha^2(t))$, etc.

where

$$E(u,v) = \frac{\partial S}{\partial u}(u,v) \cdot \frac{\partial S}{\partial u}(u,v)$$

$$F(u,v) = \frac{\partial S}{\partial u}(\text{—}) \cdot \frac{\partial S}{\partial v}(\text{—}) = \frac{\partial S}{\partial v}(\text{—}) \cdot \frac{\partial S}{\partial u}$$

$$G(u,v) = \frac{\partial S}{\partial v}(\text{—}) \cdot \frac{\partial S}{\partial v}(\text{—}) .$$

Also written

$$\dot{s}^2 = g_{11}(\dot{u}^1)^2 + 2g_{12}\dot{u}^1\dot{u}^2 + g_{22}(\dot{u}^2)^2$$

$$= \sum_{i,j=1}^{2} g_{ij}(u^1,u^2)\,\dot{u}^i\dot{u}^j$$

where

$$g_{ij}(u^1,u^2) = \frac{\partial S}{\partial u^i}(u^1,u^2) \cdot \frac{\partial S}{\partial u^j}(u^1,u^2)$$

and $\dot{\alpha}(t) = \begin{bmatrix} \dot{u}^1 \\ \dot{u}^2 \end{bmatrix}$. $\quad \begin{matrix} g_{11} = E \\ g_{12} = g_{21} = F \\ g_{22} = G \end{matrix}$

Or, in terms of matrices;

$$\left(\frac{ds}{dt}\right)^2 = (S_*\dot{\alpha}) \cdot (S_*\dot{\alpha}) = {}^t(S_*\dot{\alpha})(S_*\dot{\alpha})$$

$a \cdot b = {}^t a b$ — property of $\mathbb{E}^3$

$$= {}^t\dot{\alpha}\,{}^tS_*S_*\dot{\alpha}$$

and

$${}^tS_*S_* = {}^t[S_1\ S_2][S_1\ S_2] = \begin{bmatrix} {}^tS_1S_1 & {}^tS_1S_2 \\ {}^tS_2S_1 & {}^tS_2S_2 \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$$

Length of a curve $S\circ\alpha$ is

$$\int_0^L ds = \int_a^b \frac{ds}{dt}\,dt = \int_a^b \sqrt{E(\alpha^1(t),\alpha^2(t))\,\dot{\alpha}^1(t)\,\dot{\alpha}^1(t) + \cdots}\,dt$$

$$= \int_a^b \sqrt{\sum_{i,j=1}^{2} g_{ij}(u^1(t),u^2(t))\,\dot{u}^i(t)\,\dot{u}^j(t)}\,dt$$

First fundamental form, metric, or element of arc length on $S$:

$$ds^2 = E(u,v)\,du^2 + 2F(u,v)\,du\,dv + G(u,v)\,dv^2$$

or $ds^2 = \sum_{i,j=1}^{2} g_{ij}(u^1, u^2)\, du^i du^j$

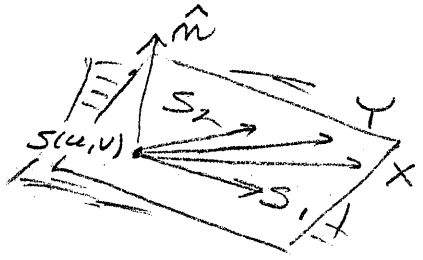
This is a quadratic form on the tangent space to $S$ at $S(u,v)$. Its matrix is

$$\begin{bmatrix} E(u,v) & F(u,v) \\ F(u,v) & G(u,v) \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} g_{11}(u^1,u^2) & g_{12}(-) \\ g_{21}(-) & g_{22}(-) \end{bmatrix}.$$

Note This form is used at $S(u,v)$ only. As $(u,v)$ changes, the space and form change - a field of quadratic forms.

2/2/01 Inner products

Fix a point $S(u,v)$ on the surface $S$.




$S_1 = \frac{\partial S}{\partial u}$ & $S_2 = \frac{\partial S}{\partial v}$ are a basis for the tangent space to $S$ at $S(u,v)$.

Any two vectors $X$ and $Y$ of this space may be written

$$X = S_1(u,v)X^1 + S_2(u,v)X^2$$
$$Y = S_1(u,v)Y^1 + S_2(u,v)Y^2$$

$(u,v)$ fixed; $X^1, \ldots, Y^2$ scalars

Then, in $E^3$:

$$X \cdot Y = (S_1X^1 + S_2X^2)\cdot(S_1Y^1 + S_2Y^2)$$
$$= EX^1Y^1 + F(X^1Y^2 + X^2Y^1) + GY^1Y^2$$

or

$$X \cdot Y = \sum_{i,j=1}^{2} g_{ij}(u,v)X^iY^j.$$

Positive definiteness of the metric

At a point $S(u,v)$ of $S$:

$$X \cdot X = E(X^1)^2 + 2FX^1X^2 + G(X^2)^2.$$

If $X \neq 0$, then $X \cdot X > 0$ in $E^3$.

This is reflected in

$E = S_1 \cdot S_1 > 0$ , $G = S_2 \cdot S_2 > 0$

and

$$EG - F^2 = (S_1 \cdot S_1)(S_2 \cdot S_2) - (S_1 \cdot S_2)^2$$
$$= (S_1 \times S_2) \cdot (S_1 \times S_2) > 0$$

This characterizes the quadratic form

$$\begin{bmatrix} x^1 & x^2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = \sum_{i,j=1}^{2} g_{ij} x^i x^j$$

($g_{11} = E$, etc.) being positive definite.

---

Note lengths of curves can be obtained from using the metric only. For the velocity vector along a curve $S \circ \alpha$ is a vector field with components $\dot{\alpha}^1(t)$ and $\dot{\alpha}^2(t)$ with respect to

$S_1(\alpha^1(t), \alpha^2(t))$ and $S^2(-,-)$ at $S(\alpha(t))$.

The length of this vector is

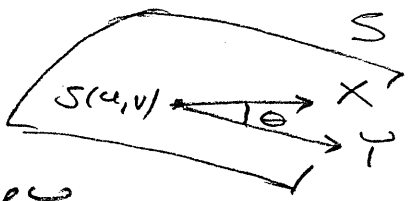
$$\sqrt{E(\alpha^1(t), \alpha^2(t))(\dot{\alpha}^1(t))^2 + 2F(-)\dot{\alpha}^1\dot{\alpha}^2 + G(-)(\dot{\alpha}^2)^2}$$

This is $\frac{ds}{dt}$ along the curve. The integral $\int (\ ) dt$ gives length. One does not need to know the extrinsic Euclidean space $E^3$.

Angles

$X$ and $Y$ vectors tangent to $S$ at point $S(u,v)$.

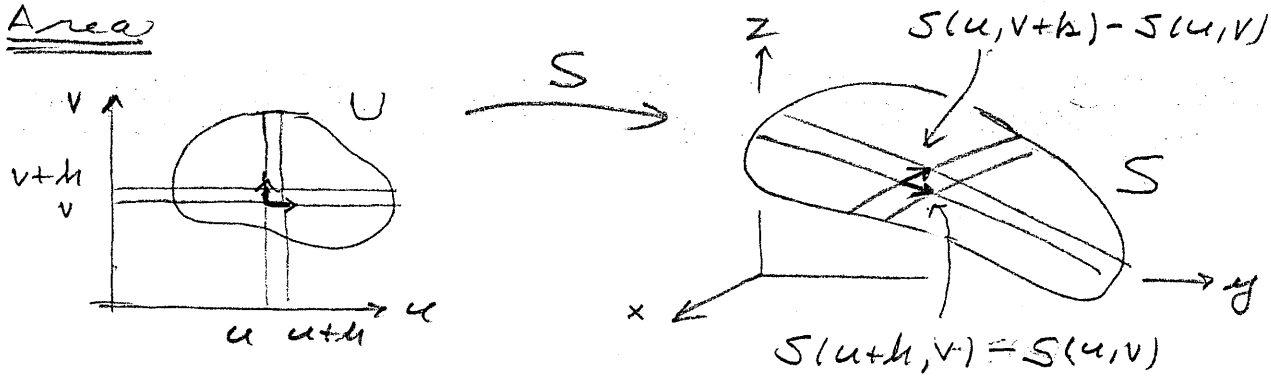
$\theta$ = angle between $X$ & $Y$




$$\cos\theta = \frac{X \cdot Y}{\|X\| \, \|Y\|}$$

This can be computed knowing only $E, F, G$ or $g_{ij}$ at $(u,v)$.

## Area




$$\left(\begin{array}{c}\text{area of}\\ \text{parallelo-}\\ \text{gram}\end{array}\right) \underset{\text{approx}}{=} \left\| \left(S(u+h,v) - S(u,v)\right) \times \left(S(u,v+h) - S(u,v)\right) \right|$$

$$S(u+h,v) - S(u,v) = S_1(u,v)\,h + O(h^2)$$
$$S(u,v+h) - S(u,v) = S_2(u,v)\,h + O(h^2)$$

so

$$\text{area} = \| S_1(u,v) \times S_2(u,v) \|\, hk + O(3)$$

Now $\| S_1 \times S_2 \| = \sqrt{EG - F^2}$,

so define

$$\left(\begin{array}{c}\text{element}\\ \text{of area}\\ \text{on } S\end{array}\right) = dS = \sqrt{E(u,v)\,G(u,v) - \left(F(u,v)\right)^2}\, du\, dv$$

One also writes

$$dS = \sqrt{\det \begin{bmatrix} g_{11}(u^1,u^2) & g_{12}(-) \\ g_{21}(-) & g_{22}(u) \end{bmatrix}}\, du^1\, du^2$$

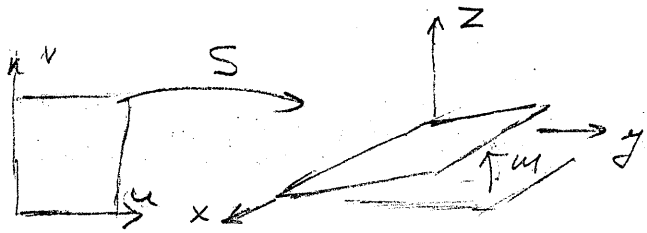
or $dS = \left\| \frac{\partial S}{\partial u}(u,v) \times \frac{\partial S}{\partial v}(u,v) \right\| du\, dv$.

The area of the parametrized surface is

$$\int_S dS = \int_U \sqrt{E(u,v)\,G(-) - F(-)^2}\, du\, dv$$

Note Length, angle, and area can be computed knowing only the matrix of the first fundamental form.

Example Plane in space making slope $m$ from the horizontal ($z = my$).




$$S(u,v) = \begin{bmatrix} u \\ v \\ mv \end{bmatrix} \qquad S_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad S_2 = \begin{bmatrix} 0 \\ 1 \\ m \end{bmatrix}$$

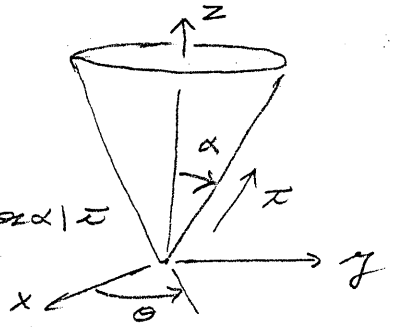
$$ds^2 = du^2 + (1+m^2)\,dv^2$$

$$dS = \sqrt{1+m^2}\,du\,dv$$

Reparameterize by $u' = u$, $v' = \sqrt{1+m^2}\,v$.

Then $ds^2 = du'^2 + dv'^2$, $dS = du'\,dv'$. Plane is isometric with usual Euclidean plane.

Example Cone with vertex angle $\alpha$.




$$S(\theta, r) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} (\sin\alpha)\, r + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} (\cos\alpha)\, r$$

Set $a = \sin\alpha$.

$$S_1 = \frac{\partial S}{\partial \theta} = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} ar, \qquad S_2 = \frac{\partial S}{\partial r} = \begin{bmatrix} (\cos\theta)\, a \\ (\sin\theta)\, a \\ \sqrt{1-a^2} \end{bmatrix}$$

$$ds^2 = a^2 r^2\, d\theta^2 + dr^2$$

$$dS = a r\, d\theta\, dr$$

Introduce new parameters:

$$\begin{cases} u = (\cos a\theta)\, r \\ v = (\sin a\theta)\, r \end{cases} \qquad \begin{cases} r = \sqrt{u^2+v^2} \\ \theta = \frac{1}{a} \operatorname{Arctan} \frac{v}{u} \end{cases}$$

Then

$$du = -a(\sin a\theta)\, r\, d\theta + (\cos a\theta)\, dr$$

$$dv = a(\cos a\theta)\, r\, d\theta + (\sin a\theta)\, dr$$

so

$$du^2 + dv^2 = a^2 r^2\, d\theta^2 + dr^2$$

gives $ds^2$.

To obtain $dS$ use $du\,dv = \left|\frac{\partial(u,v)}{\partial(\theta,z)}\right| d\theta\,dz$:

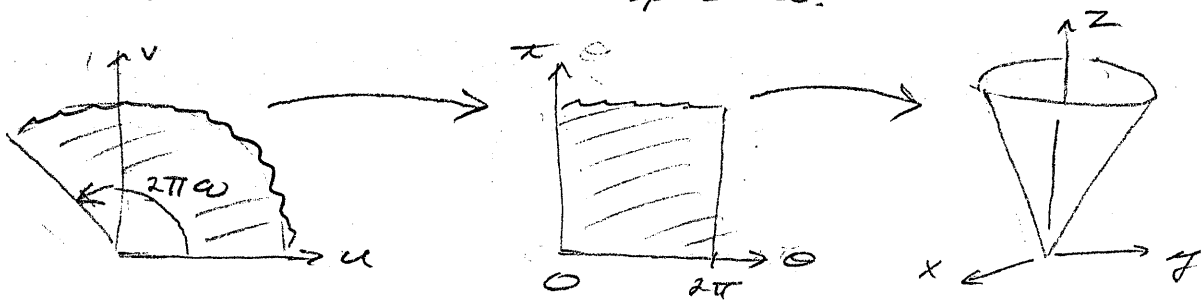
$$\frac{\partial(u,v)}{\partial(\theta,z)} = \begin{vmatrix} -a(\sin a\theta)z & \cos a\theta \\ a(\cos a\theta)z & \sin a\theta \end{vmatrix} = -az,$$

so $du\,dv = az\,d\theta\,dz$.

In the new parameters

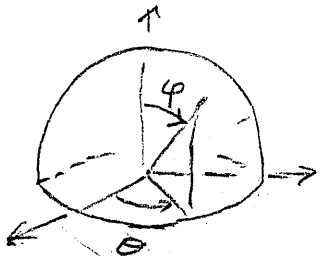
$$S(u,v) = \begin{bmatrix} \cos\left(\frac{1}{a}\operatorname{Arctan}\frac{v}{u}\right) \\ \sin\left(\text{---}''\text{---}\right) \\ 0 \end{bmatrix} a\sqrt{u^2+v^2} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \sqrt{1-a^2}\sqrt{u^2+v^2}$$

Note this cone is locally isometric with a Euclidean plane.




Problem (3.1) A hemisphere of radius $a$ is given in spherical coordinates by

$$S(\theta,\varphi) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} a\sin\varphi + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} a\cos\varphi$$




a) Show $ds^2 = a^2\sin^2\theta\,d\theta^2 + a^2 d\varphi^2$ and $dS = a^2\sin\theta\,d\theta\,d\varphi$.

b) Find the length of the (small) circle of latitude $\lambda$ and the area on $S$ which it encloses.




Note: This hemisphere is not isometric with the Euclidean plane.

**Problem (3.2)** A hemisphere of radius $a$ is given in Monge form by $z = \sqrt{a^2 - x^2 - y^2}$.

a) Find $ds^2 = E(x,y)\,dx^2 + 2F(x,y)\,dx\,dy + G(x,y)\,dy^2$ and $dS = \sqrt{EG - F^2}\,dx\,dy$.

b) Find the length and area of 3.1b) using this metric.

**Problem (3.3)** Let the metric on a surface is given by $ds^2 = E(u,v)\,du^2 + 2F(-)\,du\,dv + G(-)\,dv^2$. If new parameters are introduced by $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \Phi^1(u',v') \\ \Phi^2(u',v') \end{bmatrix}$, show that the coefficients of the metric $ds^2 = E'(u',v')\,du'^2 + \cdots$ are given by

$$\begin{bmatrix} E' & F' \\ F' & G' \end{bmatrix} = \begin{bmatrix} \frac{\partial \Phi^1}{\partial u'} & \frac{\partial \Phi^1}{\partial v'} \\ \frac{\partial \Phi^2}{\partial u'} & \frac{\partial \Phi^2}{\partial v'} \end{bmatrix}^T \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} \frac{\partial \Phi^1}{\partial u'} & \frac{\partial \Phi^1}{\partial v'} \\ \frac{\partial \Phi^2}{\partial u'} & \frac{\partial \Phi^2}{\partial v'} \end{bmatrix}$$

at $(u',v')$ at $\Phi(u',v')$

This is succinctly written

$$g'_{ij} = \sum_{a,b=1}^{2} g_{ab} \frac{\partial u^a}{\partial u'^i} \frac{\partial u^b}{\partial u'^j}, \quad i,j = 1,2.$$

Hint:

$$\begin{bmatrix} S_1' & S_2' \end{bmatrix} = \begin{bmatrix} S_1 & S_2 \end{bmatrix} \begin{bmatrix} \frac{\partial \Phi^1}{\partial u'} & \frac{\partial \Phi^1}{\partial v'} \\ \frac{\partial \Phi^2}{\partial u'} & \frac{\partial \Phi^2}{\partial v'} \end{bmatrix}.$$

**Problem (3.4)** Use problem 3.3 to reconcile Problems 3.1 and 3.2.