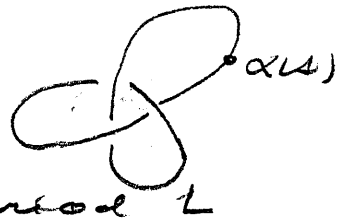
2. Curves in space. Some global theorems

Fenchel's Theorem

Closed curves in space

α regular curve
s = arc length
L = length of α
$\alpha(s)$ periodic with period L



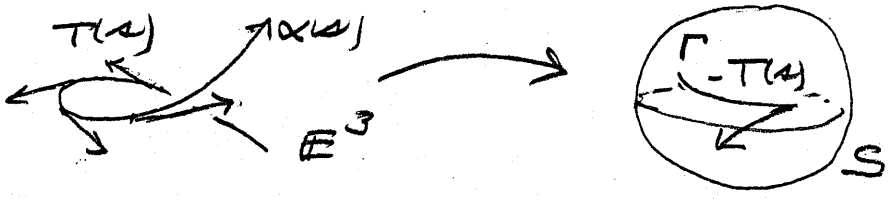
Total curvature of α:

$$k = \int_0^L \chi(s)\, ds \quad . \quad \text{Note: } \chi(s) \geq 0.$$

Theorem (Fenchel, 1929). The total curvature of a closed curve in space is $k \geq 2\pi$.

Proof - to follow

Tangent indicatrix



Translate $T(s)$ in E^3 from $\alpha(s)$ to the center of the unit sphere, S. The ensuing curve $s \mapsto T(s)$ is the **tangent indicatrix** Γ of α. - a closed curve.

Note s is arc length on α, not on Γ.
Γ need not be regular - since $\frac{dT}{ds} = \kappa\chi$ can be zero.
But: Γ is **rectifiable** - it has length!

$$\left(\begin{matrix}\text{length}\\ \text{of } \Gamma\end{matrix}\right) = \int_0^L \left\| \frac{dT}{ds}(s) \right\| ds = \int_0^L \kappa(s)\, ds$$

$$= h = (\text{total curvature of } \Gamma)$$

Preliminaries:

S - unit sphere in E^3.

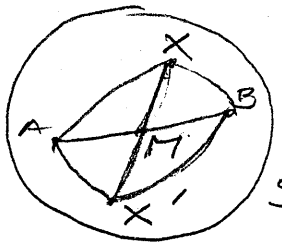
$\overline{AB}$ distance between points A & B of S as measured along a great circle arc. (so $\overline{AB} \leq \pi$).

Observation: Assume

A & B on S

$\overline{AB} \underset{\neq}{<} \pi$

M = midpoint of arc AB,



so $\overline{AM} = \overline{MB} = \frac{1}{2}\overline{AB} < \frac{\pi}{2}$.

Claim: If X is a point of S such that $\overline{MX} \leq \frac{\pi}{2}$, then $2\overline{MX} \leq \overline{AX} + \overline{BX}$.

For: Obtain X' by a half-turn (180° rotation) of S about M, so $\overline{AX'} = \overline{BX}$. Then

$$2\overline{MX} = \overline{XM} + \overline{MX'} = \overline{XX'}$$

$$\underset{\uparrow}{\leq} \overline{XA} + \overline{AX'} = \overline{AX} + \overline{BX}$$

} DISTANCES ARE UNSIGNED

triangle inequality

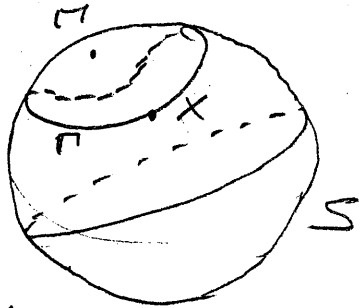
- needs all points in one hemisphere

Lemma: Assume:

Γ closed rectifiable curve on S

h = length of Γ

$h \underset{\neq}{<} 2\pi$

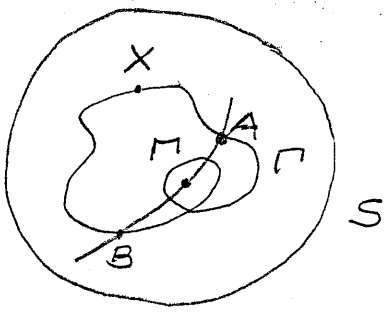


Then: There is a point M in S so that

$$\overline{MX} \leq \frac{h}{4} < \frac{\pi}{2} \quad \text{for all } X \text{ on } \Gamma.$$

1/26/01 **Proof** of) For A and B points on Γ, let $\widehat{AB}$ denote the length of arc on Γ between A and B.

1) Choose A and B on Γ so that A & B divide Γ into two arcs of equal length: $\widehat{AB} = \frac{h}{2} < \pi$.



Then $\overline{AB} \leq \widehat{AB} < \pi$.

2) Let M be the midpoint of arc AB. If X is any point on Γ such that $\overline{MX} < \frac{\pi}{2}$ (note, A & B are such points), then

$$2\,\overline{MX} \underset{\text{by claim}}{\leq} \overline{AX} + \overline{BX} \leq \widehat{AX} + \widehat{BX} = \widehat{AB} = \frac{h}{2} < \pi,$$

(arc length along Γ)

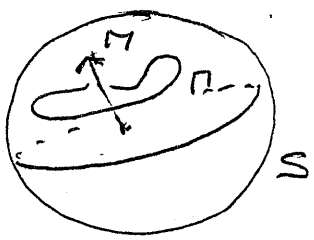
Hence, for any X on Γ,

either $\overline{MX} \geq \frac{\pi}{2}$ or $\overline{MX} \leq \frac{h}{4} \leq \frac{\pi}{2}$.

3) The function $f(X) = \overline{MX}$ is a continuous function from Γ to $[0, \frac{h}{4}] \cup [\frac{\pi}{2}, \pi]$ (disjoint union since $\frac{h}{4} < \frac{\pi}{2}$).

Now, $f(\Gamma)$ = image of Γ is connected - by the Intermediate Value Theorem, so $f(\Gamma) \subseteq [0, \frac{h}{4}]$. Hence $\overline{MX} \leq \frac{h}{4} < \frac{\pi}{2}$ for all X on Γ. QED

Consequence M on S is a "unit vector". The lemma says: Γ does not meet the great circle lying in the plane (through the center of S) orthogonal to M.



For: Points on this great circle are at distance $\frac{\pi}{2}$ from M.

Proof of Fenchel's Theorem

1) Let Π be an arbitrary point on S and view Π as a unit vector. The function $f(s) = \Pi \cdot \alpha(s)$ is differentiable, so has a maximum at some $s = s_0$. Then

$f'(s) = \Pi \cdot \alpha'(s) = \Pi \cdot T(s)$ is 0 at s_0

Π is tangent to α - point of Γ

Hence, Γ meets the great circle in the plane orthogonal to Π at the point $T(s_0)$.

2) Since Π in 1) is arbitrary, Γ meets every great circle of S. By the Lemma, necessarily Γ has length $L \geq 2\pi$. QED

Aside Formally:

Lemma $(L < 2\pi) \Rightarrow (\exists \Pi \in S)(\forall X \in \Gamma)\left(\overline{\Pi X} < \frac{\pi}{2}\right)$

or: $(L < 2\pi) \Rightarrow (\exists \Pi \in S)(\Pi^{\perp} \cap \Gamma = \phi)$

Contrapositive: $(\forall \Pi \in S)(\Pi^{\perp} \cap \Gamma \neq \phi) \Rightarrow (L \geq 2\pi)$.

Corollary A closed regular space curve whose curvature satisfies $\kappa(s) \leq \frac{1}{a}$ (for all s) has length $L \geq 2\pi a$

Proof
$$L = \int_0^L ds \geq \int_0^L a\kappa(s)\,ds = a\kappa \geq 2\pi a. \quad \text{QED}$$



Bends less rapidly than a circle of radius a, so cannot close up before length $2\pi a$.

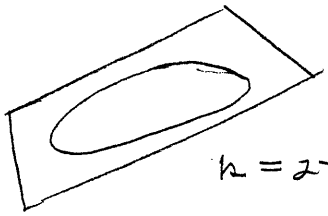
$\kappa(s) \leq \frac{1}{a}$

$R(s) \geq a$.

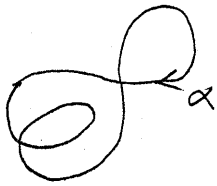
Refinements - of Fenchel's Theorem

Fenchel The total curvature of a space curve is $k \geq 2\pi$ with equality iff the curve is a plane convex curve.

Compare this with the theorem of turning Tangents!



$k = 2\pi$



$$\int_\alpha \chi(s)\, ds = 2\pi j$$

$\chi(s)$ signed planar curvature
j = integer = rotation index

$j = \pm 1$ iff curve is convex

Fary-Milnor (1950). The total curvature of a knot is $k \geq 4\pi$.



$k \geq 4\pi$.

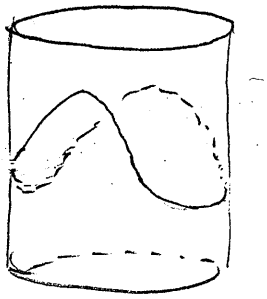
One shows that, for $2\pi \leq k < 4\pi$, the curve spans a "disk", so it unknotted.



Ref S. S. Chern, Curves and surfaces in Euclidean space. MAA Studies in Mathematics, Vol 27. Global Differential Geometry. 1989. pp. 99-139.

Problem (2.1) Consider the curve

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \\ 0 \end{bmatrix} a + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} h \cos 2t$$

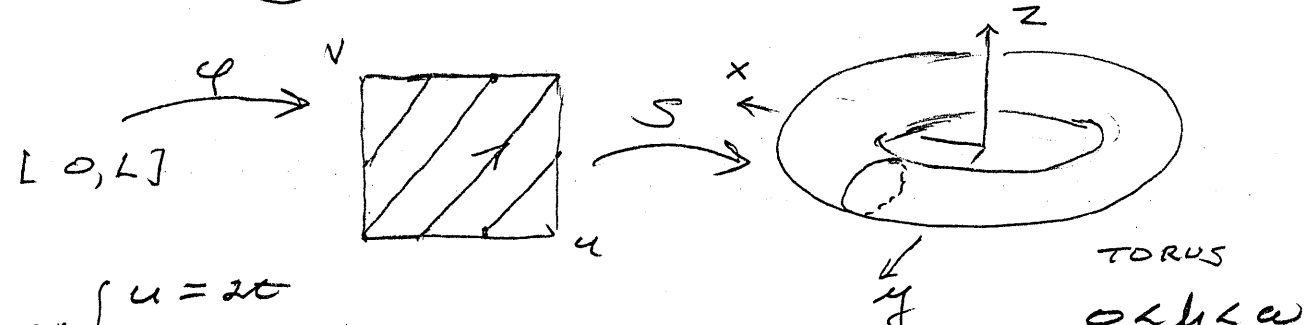


lying on a right circular cylinder of radius a, $0 \le t < 2\pi$

a) Find the length and total curvature.

b) How does this example relate to Fary-Milnor for large h?

Problem (2.2) Consider the *trefoil knot*:



$$\varphi: \begin{cases} u = 2t \\ v = 3t \end{cases}$$

$$S: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos u \\ \sin u \\ 0 \end{bmatrix} (a + h \cos v) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} (h \sin v)$$

$0 < h < a$

a) Find the total curvature k of $S \circ \varphi$.
By Fenchel, it will be $k \ge 4\pi$.

b) What happens as $h \to 0$?

c) Make an *accurate* sketch.