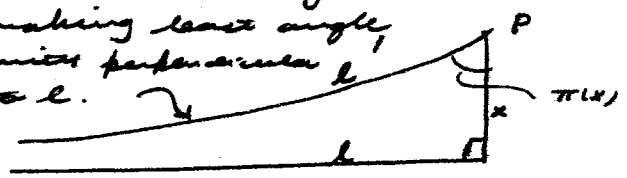
Angle of parallelism

x = distance P to l

$\pi(x)$ = angle between l' and the perpendicular.
= angle of parallelism

line not meeting l making least angle with perpendicular to l.



l' = limiting parallel

Angle of parallelism formula (Lobachefsky)

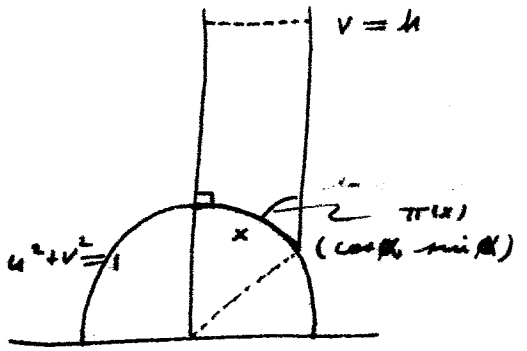
$$\tan \frac{\pi(x)}{2} = e^{-\frac{x}{k}}$$

Proof

1) Compute the length x.

$u = \cos\theta$, $v = \sin\theta$

$\alpha \le \theta \le \frac{\pi}{2}$



$$ds^2 = k^2 \frac{du^2 + dv^2}{v^2}$$

$$= k^2 \frac{(-\sin\theta\, d\theta)^2 + (\cos\theta\, d\theta)^2}{(\sin\theta)^2} = k^2 (\operatorname{csc}\theta)^2 d\theta$$

$$x = k \int_\alpha^{\frac{\pi}{2}} \operatorname{csc}\theta\, d\theta = k \Big| \log \tan\frac{\theta}{2} \Big]_\alpha^{\frac{\pi}{2}}$$

$$= -k \log \tan\frac{\alpha}{2}$$

$$\tan\frac{\alpha}{2} = e^{-\frac{x}{k}}$$

(Note: $\alpha = \pi(x)$ as evident from figure.)

2) Compute area of region enclosed as $h \to \infty$.

$$dA = \frac{h^2}{v^2}\, du\, dv$$

$$A = h^2 \int_0^{\cos\alpha} \left(\int_{\sqrt{1-u^2}}^{\infty} \frac{1}{v^2}\, dv \right) du$$

$$= h^2 \int_0^{\cos\alpha} \frac{1}{\sqrt{1-u^2}}\, du = h^2 \left[\arcsin u \right]_0^{\cos\alpha}$$

$$= h^2 \left(\frac{\pi}{2} - \alpha \right)$$

3) Apply Gauss-Bonnet to :



$$\left(-\frac{1}{h^2}\right) A + \int_{\text{segment } v=h} \kappa_g\, ds + 3\left(\pi - \frac{\pi}{2}\right) + \left(\pi - \pi(x)\right) = 2\pi.$$

κ_g is bounded on this segment $v=h$ (actually $\kappa_g \equiv 0$) and length of segment $\leq \frac{h}{h} \to 0$ as $h \to \infty$. Hence $\int \kappa_g\, ds \to 0$ as $h \to \infty$.

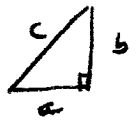
$$\left(-\frac{1}{h^2}\right) A + 3\left(\pi - \frac{\pi}{2}\right) + \left(\pi - \pi(x)\right) = 2\pi$$

or $$A = h^2 \left(\frac{\pi}{2} - \pi(x) \right)$$

4) Thus $\pi(x) = \alpha$ and $\tan \frac{\pi(x)}{2} = e^{-\frac{x}{h}}$. QED

The Pythagorean theorem

Euclidean geometry



$$c^2 = a^2 + b^2$$

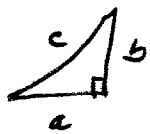
Elliptic geometry



$K = \frac{1}{R^2}$

$$\cos\frac{c}{R} = \cos\frac{a}{R}\cos\frac{b}{R}$$

Hyperbolic geometry



$K = -\frac{1}{h^2}$

$$\cosh\frac{c}{h} = \cosh\frac{a}{h}\cosh\frac{b}{h}.$$

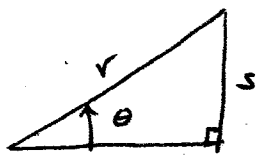
Note: Using $\cos\frac{c}{R} = 1 - \frac{1}{2}\frac{c^2}{R^2} + \cdots$

gives Euclidean case as $R \to \infty$. Similarly

$\cosh\frac{c}{h} = 1 + \frac{1}{2}\frac{c^2}{h^2} + \cdots$ and letting $h \to \infty$.

Proof - hyperbolic case.

1) Geodesic polar coordinates:



$$ds^2 = dr^2 + (g(r))^2 d\theta^2$$

$$g(r) = h \sinh\frac{r}{h}.$$

Christoffel symbols:

$$\Gamma^1_{11} = \frac{1}{2}\frac{E_1}{E} = 0 \qquad \Gamma^2_{11} = -\frac{1}{2}\frac{E_2}{G} = 0$$

$$\Gamma^1_{12} = \frac{1}{2}\frac{E_2}{E} = 0 \qquad \Gamma^2_{12} = \frac{1}{2}\frac{G_1}{G} = \frac{g'}{g}$$

$$\Gamma^1_{22} = -\frac{1}{2}\frac{G_1}{E} = -gg' \qquad \Gamma^2_{22} = \frac{1}{2}\frac{G_2}{G} = 0$$

where $g' = \frac{dg}{dr}$.

Geodesic eqns:

$$\frac{d^2r}{ds^2} - gg'\left(\frac{d\theta}{ds}\right)^2 = 0$$

$$\frac{d^2\theta}{ds^2} + 2\frac{g'}{g}\frac{dr}{ds}\frac{d\theta}{ds} = 0.$$

Two first integrals:

$$\left(\frac{dr}{ds}\right)^2 + g^2\left(\frac{d\theta}{ds}\right)^2 = 1 \qquad (s \text{ is arc length})$$

- obtain by substituting geodesic eqns in $\frac{d}{ds}\left(\left(\frac{dr}{ds}\right)^2 + g^2\left(\frac{d\theta}{ds}\right)^2\right)$ and get 0.

$$\frac{d}{ds}\left(g^2\frac{d\theta}{ds}\right) = g^2\frac{d^2\theta}{ds^2} + 2gg'\frac{dr}{ds}\frac{d\theta}{ds}$$

$$= g^2\left(\frac{d^2\theta}{ds^2} + 2\frac{g'}{g}\frac{dr}{ds}\frac{d\theta}{ds}\right) = 0.$$

Thus: $g^2\frac{d\theta}{ds} = \text{constant} = c$

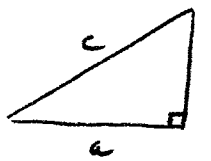
along a geodesic.

Eliminate $\frac{d\theta}{ds} = \frac{c}{g^2}$,

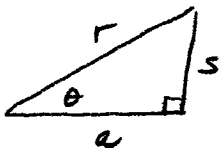
obtain

$$\left(\frac{dr}{ds}\right)^2 + \frac{c^2}{g^2} = 1.$$

2)



from



For perpendicular geodesic chords:

when $\theta = 0$, $r = a$, $\frac{dr}{ds} = 0$.

$$0^2 + \frac{c^2}{\left(h \sinh \frac{a}{h}\right)^2} = 1 \quad , \quad c = h \sinh \frac{a}{h}.$$

$$\frac{dr}{ds} = \sqrt{1 - \left(\frac{h \sinh \frac{a}{h}}{h \sinh \frac{r}{h}}\right)^2}$$

$r = c$ when $s = b$; $r = a$ when $s = 0$.

$$\int_a^c \frac{dr}{\sqrt{1 - \left(\frac{h \sinh \frac{a}{h}}{h \sinh \frac{r}{h}}\right)^2}} = b.$$

3) Evaluate integral;

$$\int_a^c \frac{dr}{\sqrt{1 - \frac{\sinh^2 \frac{a}{h}}{\sinh^2 \frac{r}{h}}}} = \int_a^c \frac{\sinh \frac{r}{h}\, dr}{\sqrt{\sinh^2 \frac{r}{h} - \sinh^2 \frac{a}{h}}}$$

$$= \begin{matrix} u = \cosh \frac{r}{h} \end{matrix} \quad h \int_{\cosh \frac{a}{h}}^{\cosh \frac{c}{h}} \frac{du}{\sqrt{u^2 - 1 - \sinh^2 \frac{a}{h}}}$$

$$= h \int_{\cosh\frac{a}{h}}^{\cosh\frac{c}{h}} \frac{du}{\sqrt{u^2 - \cosh^2\frac{a}{h}}}$$

$$= h \left[\operatorname{arcosh}\left(\frac{u}{\cosh\frac{a}{h}}\right) \right]_{\cosh\frac{a}{h}}^{\cosh\frac{c}{h}}$$

$\int \frac{du}{\sqrt{u^2 - A^2}} = \operatorname{arcosh}\frac{u}{A} + C$

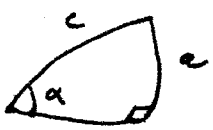
$$= h \operatorname{arcosh} \frac{\cosh\frac{c}{h}}{\cosh\frac{a}{h}} \qquad (\operatorname{arcosh} 1 = 0)$$

So $\cosh\frac{c}{h} = \cosh\frac{a}{h}\cosh\frac{b}{h}$. QED.

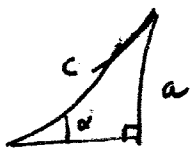
Rmk Other trigonometric identities may be derived similarly.

N.B. One can pass from elliptic to hyperbolic trigonometric formulae by setting $R = \sqrt{-1}\,h$.

Eg.



$\sin\frac{c}{R} = \sin\frac{a}{R}\sin\alpha$



$\sinh\frac{c}{h} = \sinh\frac{a}{h}\sin\alpha$.

(These formulae follow from $g^2\frac{d\theta}{ds} = \text{const.}$
A version of *Clairaut's theorem.)