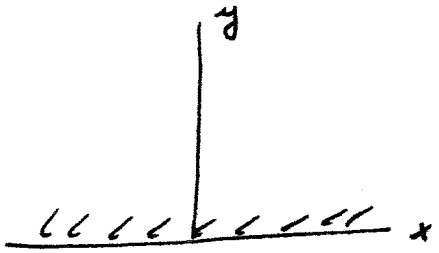
Models of the Lobachevsky plane

In each case model below, the space is topologically equivalent to the open Euclidean plane. All have const curvature $K = -\frac{1}{k^2}$, $k>0$, and "lines" are geodesics: geodesic curvatures $\kappa_g \equiv 0$. All are complete and isometric in the large.

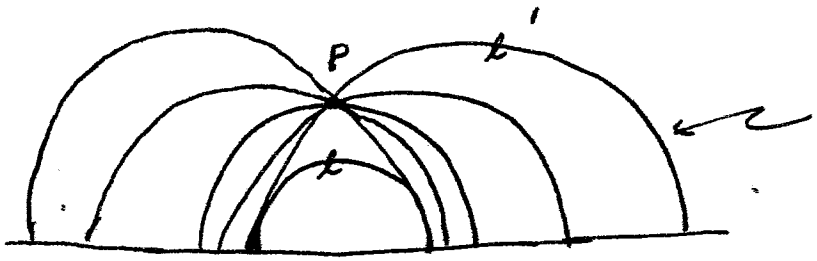
Poincaré's model

The space is the upper half plane $y>0$.



Metric: $ds^2 = k^2 \frac{dx^2+dy^2}{y^2}$

Geodesics: Ordinary circles which intersect the x-axis orthogonally — including vertical lines.



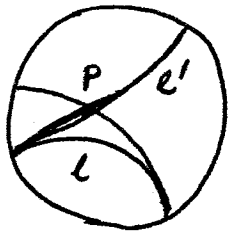
Lines not meeting l and passing through P.

Note: This model will be studied in detail later.

Riemann's model

Space: Interior of the disc of radius $2k$:

$$u^2 + v^2 < 4k.$$



Metric:

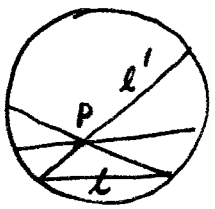
$$ds^2 = \frac{du^2 + dv^2}{\left(1 - \frac{1}{4k^2}(u^2+v^2)\right)^2}$$

Geodesics: Ordinary circles orthogonal to the boundary of the disc - including straight lines.

Klein's model

Space: Interior of the disc of radius a:

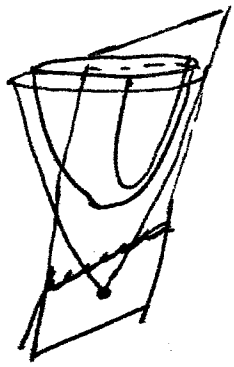
$$u^2 + v^2 < a^2$$



Metric: $$ds^2 = k^2 \frac{(a^2-v^2)\,du^2 + 2uv\,du\,dv + (a^2-u^2)\,dv^2}{(a^2-u^2-v^2)^2}.$$

Geodesics: Ordinary straight line segments lying within the disc.

Weierstrass's model



Space: The sheet of the hyperboloid of two sheets

$$x^2 + y^2 - z^2 = -h^2$$

where $z \geq h$.

Metric: $ds^2 = dx^2 + dy^2 - dz^2$

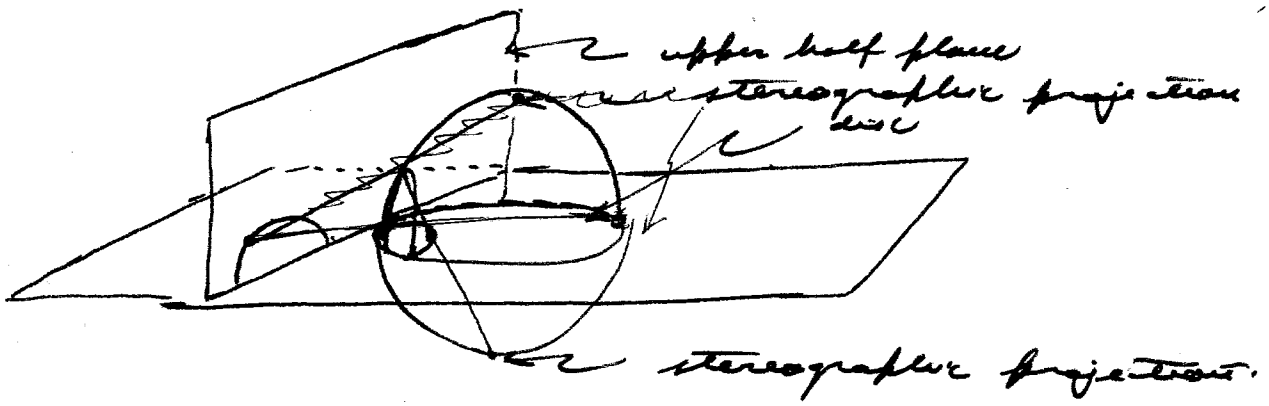
(with x, y, z subject to $x^2+y^2-z^2=-h^2$).

Geodesics: Intersections of the hyperboloid with planes ~~through the vertex~~ of its asymptotic cone.

Isometries

The isometries between the above models can be written down explicitly. Two have nice geometric descriptions.

Poincaré and Riemann models.



Poincaré and Klein models.

