Theorem of Poincaré-Hopf

(Poincaré - 1885
Hopf - 1926)

M closed abstract surface

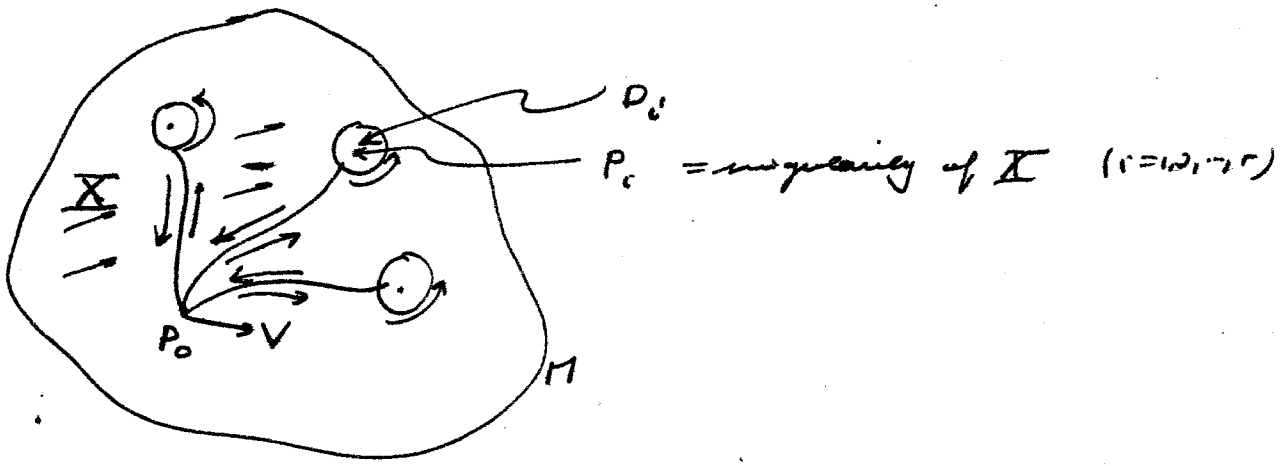
X differentiable vector field on M with only isolated singularities

Thm: $I(X) = \chi(M)$.

N.B. $I(X)$ is the same for all such vector fields X.

Proof

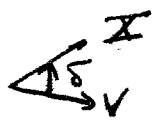
1) Case when M orientable.



Begin at P_0 and follow path shown, making small loops around each singularity of X.

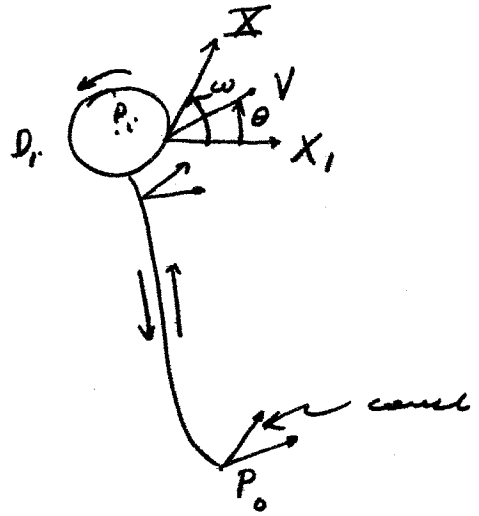
Parallel transport V along this path from P_0.

Let δ = angle from V to X



Compute the change in δ along this path in two ways.

i)



$\delta = \omega - \theta$

$$\begin{pmatrix}\text{increase in } \delta \\ \text{around loop only}\end{pmatrix} = 2\pi I_{P_i}(\bar{X}) - \int_{D_i} K\,dA$$

↑ change in θ

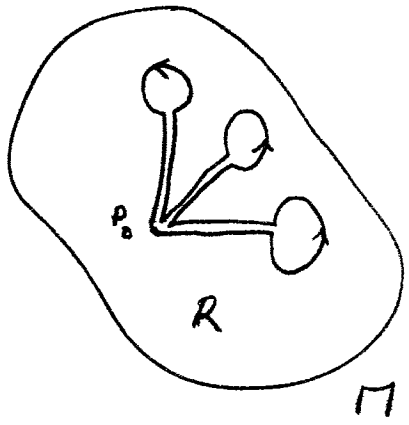
Change in ω out and back cancel.
Change in θ along entire path same as for around loop - angle between two parallel fields along a curve is constant.

Thus:

$$\begin{pmatrix}\text{increase in } \delta \\ \text{along entire} \\ \text{path}\end{pmatrix} = \sum_{i=1}^{r} 2\pi I_{P_i}(\bar{X}) - \sum_{i=1}^{r} \int_{D_i} K\,dA$$

$$= 2\pi I(\bar{X}) - \sum_{i=1}^{r} \int_{D_i} K\,dA.$$

ii) Vein parts as bounding the following region:



$$R = M - \overset{r}{\underset{i=1}{\cup}} D_i$$

Change in $\mathbb{X}$ around path is zero since R can be triangulated and change in angle of $\mathbb{X}$ around each triangle is zero - no singularities of $\mathbb{X}$ in R. (Compare remark $\chi_1 = \frac{\partial}{\partial u}$ in each coord. patch as in proof of angle change of V formula p. 9-14).

Change in V at P_0 after parallel transport once around path:

$$\text{change} = -\int_{M - \overset{r}{\underset{i=1}{\cup}} D_i} K \, dA$$

↗ traverse of boundary is opposite to direction used in usual formula.

Thus:

$$\begin{pmatrix} \text{increase in } \delta \\ \text{once around} \\ \text{path} \end{pmatrix} = \int_{M - \overset{r}{\underset{i=1}{\cup}} D_i} K \, dA$$

$$= \int_M K \, dA - \sum_{i=1}^{r} \int_{D_i} K \, dA$$

$$= 2\pi \chi(M) - \sum_{i=1}^{r} \int_{D_i} K \, dA$$

Equating: $I(\tilde{X}) = \chi(M)$.

2) Case when M non-orientable.

Let M' = two-fold orientable cover of M. X' a field on M' giving X on M.

Then $I(X') = 2\,I(X)$ (since there are two preimages of the same index for X' for each one of X)

and $\chi(M') = 2\chi(M)$. QED.

(Note. Compare: Stokes Ch VIII, Sect 11)

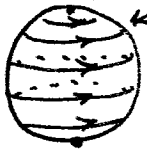
Remarks 1) Index can be defined and the Poincaré-Hopf theorem proved without the aid of the metric. i.e. theorem remains true for closed two-dimensional manifolds.

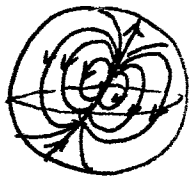
2) A two-manifold always admits metrics; this may be used to prove the theorem of the first remark.

Examples.

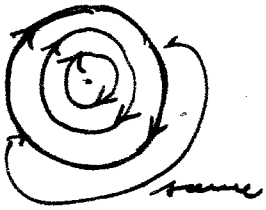
1) Sphere $I = \chi = 2$



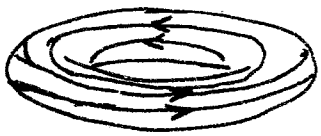


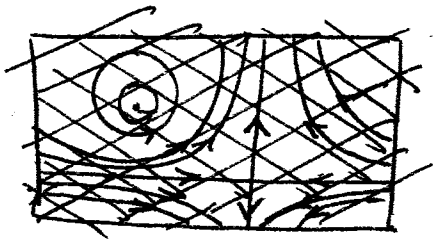


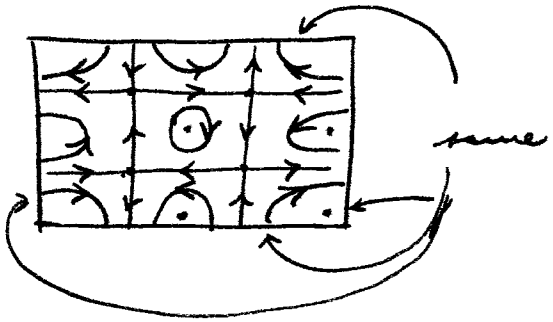
2) Projective plane. $I = \chi = 1$



3) Torus. $I = \chi = 0$



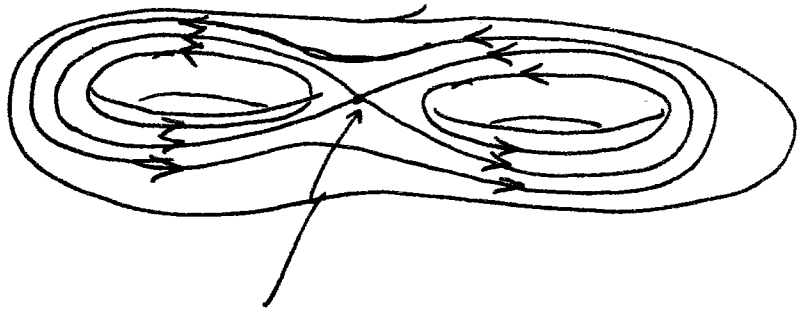




4 singularities of index $+1$

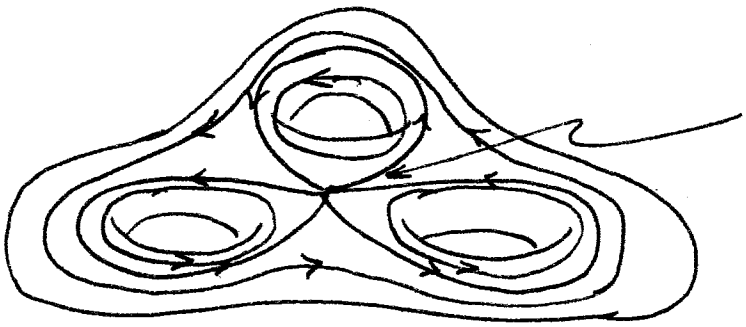
4 " " -1.

4) Surface of genus 2. $I = \chi = -2$



Singularity of index $= -1$
Another such on bottom.

5) Surface of genus 3. $I = \chi = -4$.



Singularity of index -2. Another such on bottom.