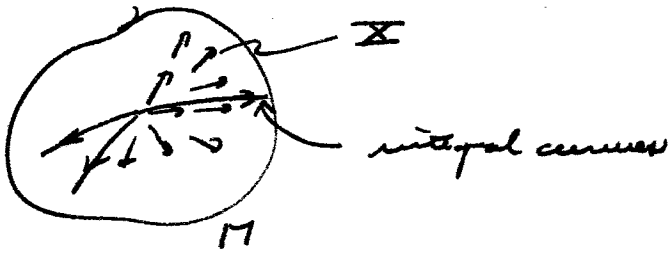
Lecture 13. Abstract surfaces. Vector fields on surfaces

Vector fields

M abstract surface (with metric).



X vector field on M (differentiable)

$X = X^1(u,v)\frac{\partial}{\partial u} + X^2(u,v)\frac{\partial}{\partial v}$ in local coordinates.

Recall: An integral curve of X through $P=(u_0,v_0)$ is a solution of the ODE:

$$\begin{cases} \frac{du}{dt} = X^1(u,v) \\ \frac{dv}{dt} = X^2(u,v) \\ u(t_0)=u_0\ ,\ v(t_0)=v_0 \end{cases}$$

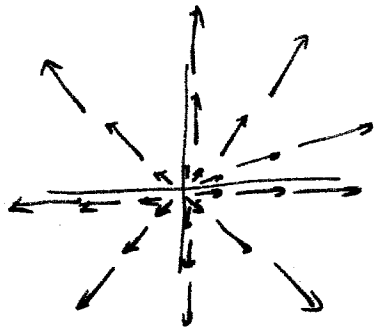
Singularity

P is singular point of X if $X=0$ at P.

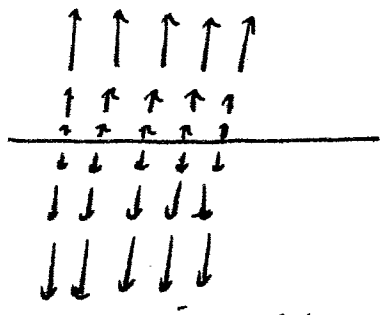
i.e. $X^1(u_0,v_0)=0,\ X^2(u_0,v_0)=0$ for $P=(u_0,v_0)$.

In this case: $C(\tau)=P_0$ (constant curve) is an integral curve of X.

P is isolated singularity if points near P and $\neq P$ are not singularities of X.



isolated singularity

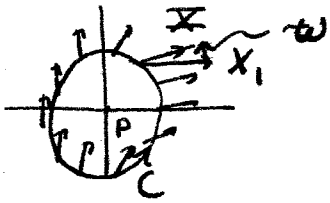


non-isolated singularity.

Index

X vector field on M

P isolated singularity of X



(u,v) coordinates near $P=(0,0)$

$X_1 = \frac{\partial}{\partial u}$, $X_2 = \frac{\partial}{\partial v}$

C small curve in (u,v) coordinates going once around P in $a \le t \le b$.

ω angle X_1 to X on C.

i.e. $\tau(t)$ at $C(t)$.

$\omega(b) - \omega(a)$ = multiple of 2π since X and X_1 are in same position for $t=a$ and $t=b$.

$$I_P(X) = \frac{1}{2\pi}\left(\omega(b) - \omega(a)\right).$$

Note: 1) $I_P(X)$ does not depend on choice of coordinates. If (u',v') used, angle between $X_1 = \frac{\partial}{\partial u}$ and $X_1' = \frac{\partial}{\partial u'}$, measured by adding to $\omega(b)$ and $\omega(a)$ would then cancel. (QUERY)

Note! Still true if $\frac{\partial(u',v')}{\partial(u,v)} < 0$. Viewed as map to unit circle, our orientation on the latter has been fixed by choice of sign of $\omega(b) - \omega(a)$ rather than $\omega(a) - \omega(b)$.

2) $I_p(X)$ does not depend on choice of C.

Suppose C_0 and C_1 are used. In $u'=u$, $w^2=v$ curves C_α is $u' = u'_\alpha(\tau)$ $(\alpha = 0, 1)$.

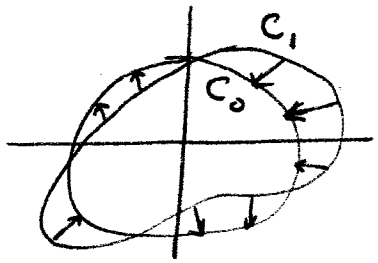
Consider C_α : $u' = (1-\alpha)u'_0(\tau) + \alpha u'_1(\tau)$

for $0 \le \alpha \le 1$. C_α is C_0 for $\alpha = 0$, C_1 for $\alpha = 1$.

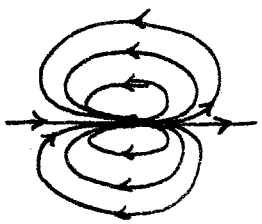
Obtain $\varpi(\tau, \alpha)$ as before. Then

$$\frac{1}{2\pi}\left(\varpi(b,\alpha) - \varpi(a,\alpha)\right)$$

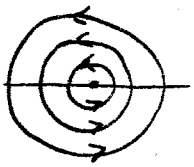
is continuous and integer-valued; hence the same integer for $\alpha = 0$ and $\alpha = 1$.



Examples. Images of X are sketched.



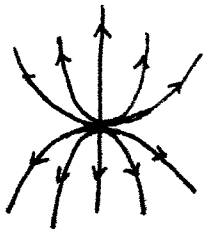
$I_p = 2$



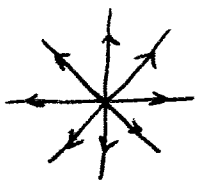
$I_p = 1$



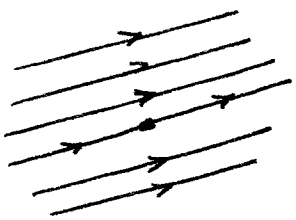
$I_p = 1$



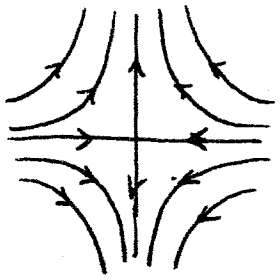
$I_p = 1$



$I_p = 1$

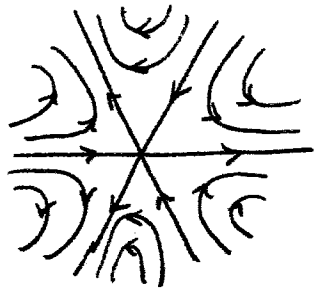


$I_p = 0$



$I_p = -1$

essentially only picture with $I_p = -1$



$I_p = -2$.

Eg. In the Euclidean plane, the field X with $X^1(x,y) = \mathrm{Re}((x+\sqrt{-1}y)^4)$

$X^2(x,y) = \mathrm{Im}((x+\sqrt{-1}y)^4)$

has index 4 at origin for $u>0$. The field X with $X^1(x,y) = \mathrm{Re}((x-\sqrt{-1}y)^4)$

$X^2(x,y) = \mathrm{Im}((x-\sqrt{-1}y)^4)$

has index -4 at the origin for $u>0$.

Total index

M closed surface

X vector field with only isolated singularities.

X has only finitely many singularities p (by compactness of M).

$I(X) = \sum_p I_p(X)$ = total index of X.

summed over the finitely many singularities p of X.