10. Abstract surfaces. Theorem of Gauss & Bonnet

Preliminaries
- a proof in the case of a simply-connected region.

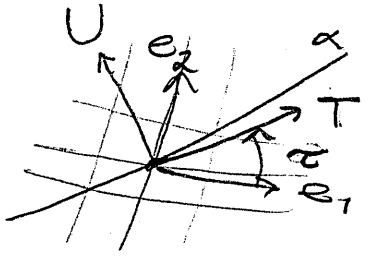
A formula of Liouville

u^1, u^2 orthogonal coördinates

$F = 0$ as before

$\alpha : \begin{cases} u^1 = \alpha^1(s) \\ u^2 = \alpha^2(s) \end{cases}$ curve

s = arc length along α



Field of orthonormal frames:

$$e_1(u^1, u^2) = \frac{1}{\sqrt{E(u^1, u^2)}} \frac{\partial}{\partial u^1}\Big|_{(u^1, u^2)}$$

$$e_2(u^1, u^2) = \frac{1}{\sqrt{G(u^1, u^2)}} \frac{\partial}{\partial u^2}\Big|_{(u^1, u^2)}$$

Unit tangent to α at s:

$$T(s) = e_1 \cos\tau + e_2 \sin\tau$$

($e_1(\alpha^1(s), \alpha^2(s))$, $\tau(s)$)

Unit vector normal to curve - and tangent to surface:

$$U(s) = -e_1 \sin\tau + e_2 \cos\tau$$

Now:

$$\frac{DT}{ds} = \frac{De_1}{ds}\cos\tau + e_1 \frac{d}{ds}\cos\tau + \frac{De_2}{ds}\sin\tau + e_2 \frac{d}{ds}\sin\tau$$

(EQUATION CONTINUED)

$$\frac{DT}{ds} = (e_2\,\omega)\cos\tau + e_1(-\sin\tau)\frac{d\tau}{ds}$$
$$+ (-e_1\,\omega)\sin\tau + e_2(\cos\tau)\frac{d\tau}{ds}$$
$$= (-e_1\sin\tau + e_2\cos\tau)\left(\omega + \frac{d\tau}{ds}\right).$$

But $\frac{DT}{ds} = U\kappa_g$, κ_g = geodesic curvature.

so

$$\boxed{\kappa_g = \omega + \frac{d\tau}{ds}}$$

$$\begin{pmatrix}\text{rotation}\\ \text{of curve}\end{pmatrix} = \begin{pmatrix}\text{rotation}\\ \text{of frame}\end{pmatrix} + \begin{pmatrix}\text{rotation}\\ \text{relative}\\ \text{to frame}\end{pmatrix}.$$

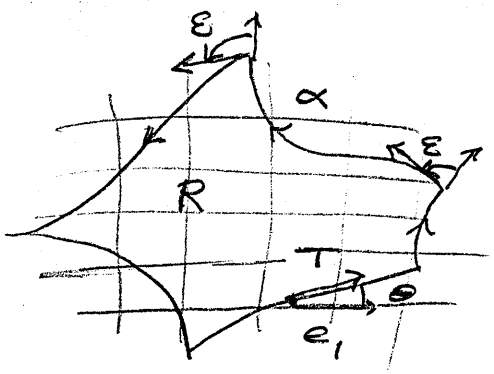
$\kappa_g = \frac{d}{ds}(\tau - \theta) = \frac{d}{ds}\measuredangle$ (parallel field to T)

<u>And</u>: Replace T by a (unit) vector X which is parallel along α. Let θ = angle e_1 to X. Since $\omega + \frac{d\theta}{ds} = 0$ (p. 9-8), we have

$$h = \begin{pmatrix}\text{holonomy}\\ \text{angle}\end{pmatrix} = \begin{pmatrix}\text{change}\\ \text{in } \theta\\ \text{around } \alpha\end{pmatrix}$$
$$= \int_\alpha d\theta = \int_0^l \frac{d\theta}{ds}\,ds = \int_0^l (-\omega)\,ds.$$

l = length α.

<u>Turning tangents</u>



R simply connected region within one chart

α boundary of R

R & α coherently oriented

ε exterior angles through which tangent to α turns at "vertices".

$\theta(s)$ angle $e_1(\alpha^1(s), \alpha^2(s))$ to tangent $T(s)$

Now, $\int_\alpha d\theta + \sum \varepsilon = 2\pi$. Valid using any fixed parameter curves.

$\int_0^L \frac{d\theta}{ds}\, ds$ (θ = total exterior angles)

Tangent turns through 2π once around α.

And by Liouville, $\frac{d\theta}{ds} = -\omega + \kappa_g$,

so

$$\int_\alpha (-\omega + \kappa_g)\, ds + \sum \varepsilon = 2\pi.$$

But

$$\int_\alpha (-\omega)\, ds = h = \int_R K\, dA \quad \text{by Bonnet},$$

so

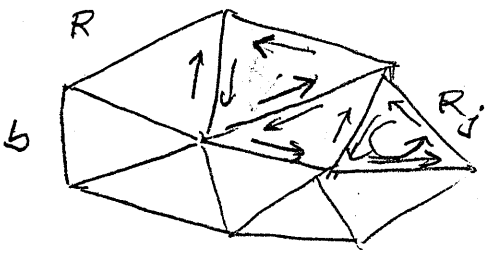
$$\int_R K\, dA + \int_\alpha \kappa_g\, ds + \sum \varepsilon = 2\pi.$$

This is Gauss-Bonnet for a simply connected region.

Note. This holds for regions made of several pieces - by additivity and cancellation. to be done in general next.

The Theorem of Gauss & Bonnet

Let R be made up of "triangles", each lying within a single chart - so the formula above holds. The triangular regions R_j are to be "coherently oriented."



b_j = boundary of R_j - $j = 1, 2, 3, \ldots, F$

$\varepsilon_j^1, \varepsilon_j^2, \varepsilon_j^3$ exterior angles of R_j.

F = number of faces

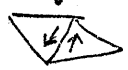
E = number of edges

$V = V_{int} + V_{bdry}$ = number of vertices interior boundary

For R_j :

$$\int_{R_j} K\,dA + \int_{b_j} \kappa_g\,ds + (\varepsilon_j^1 + \varepsilon_j^2 + \varepsilon_j^3) = 2\pi$$

Add over $j = 1, 2, \dots F$ noting $\sum \int_{R_j} = \int_R$ and $\int_{b_j} \kappa_g\,ds$ cancel on interior edges :



$$\int_R K\,dA + \int_b \kappa_g\,ds + \sum_{j=1}^{F} (\varepsilon_j^1 + \varepsilon_j^2 + \varepsilon_j^3) = 2\pi F.$$

To evaluate the sum, use the interior angles $\pi - \varepsilon_j^1$, $\pi - \varepsilon_j^2$, $\pi - \varepsilon_j^3$ of the triangles. Now :

$$\sum_{j=1}^{F} \left((\pi - \varepsilon_j^1) + (\pi - \varepsilon_j^2) + (\pi - \varepsilon_j^3) \right)$$

$$= \left(\begin{array}{c} \text{sum of all interior} \\ \text{angles of all triangles} \end{array} \right)$$

$$= \left(\sum_{\substack{\text{interior} \\ \text{vertices}}} + \sum_{\substack{\text{boundary} \\ \text{vertices}}} \right) \left((\pi - \varepsilon_j^1) + (\pi - \varepsilon_j^2) + (\pi - \varepsilon_j^3) \right)$$

$$= 2\pi V_{int} + (\pi V_{bdry} - \Sigma \varepsilon)$$

each contributes 2π to sum

each contributes $\pi - \left(\begin{array}{c} \text{exterior angle} \\ \text{at vertex} \end{array} \right)$

Thus $\sum_{j=1}^{F} (\varepsilon_j^1 + \varepsilon_j^2 + \varepsilon_j^3)$

$= 3F \cdot \pi - (2\pi V_{int} + \pi V_{bdry} - \Sigma\varepsilon)$

$= 3\pi F - 2\pi V_{int} - \pi V_{bdry} + \Sigma\varepsilon,$

so

$$\int_R K\,dA + \int_b \kappa_g\,ds + \Sigma\varepsilon$$

$= 2\pi F - (3\pi F - 2\pi V_{int} - \pi V_{bdry})$

$= -\pi F + 2\pi V_{int} + \pi V_{bdry}$

Now,

(number of boundary edges) = (number of boundary vertices),

so $2E = 3F + V_{bdry}$ ← counts each edge twice

Add

$$2\pi(3F - 2E + V_{bdry}) = 0$$

to equation above:

$$\begin{pmatrix}\text{RIGHT}\\ \text{HAND}\\ \text{SIDE}\end{pmatrix} = -\pi F + 2\pi V_{int} + \pi V_{bdry}$$

$$+ 3\pi F - 2\pi E + \pi V_{bdry}$$

$$= 2\pi(F - E + V_{int} + V_{bdry}).$$

$\underbrace{\qquad\qquad}_{V}$

But $\chi(R) = V - E + F$ is the Euler characteristic of R. So

$$\boxed{\int_R K\,dA + \int_b \kappa_g\,ds + \Sigma\varepsilon = 2\pi\chi(R)}$$

- the general Gauss-Bonnet Theorem for abstract surfaces.

Remarks

1) This shows $\chi(R)$ does not depend on the triangulation of R.

2) Energy is intrinsic - nothing depends on charts or coördinates employed.

3) The region R can be quite general.

4) If R is orientable, then $\chi = 2 - 2g$, g = genus.

5) If R is orientable and closed (no boundary), then

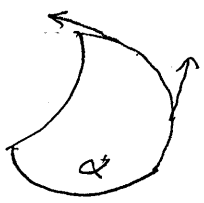
$$\int_R K \, dA = 2\pi(2 - 2g).$$

GEOMETRY TOPOLOGY

6) Total turning of tangent around a curve γ

$$\int d\tau + \sum \varepsilon = 2\pi n$$

n = integer



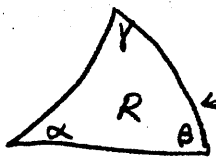
To show $n = 1$, "flatten" the surface by "homotopy"

$$E(u,v;t) = (1-t) + t\,E(u,v),$$
$$G(u,v;t) = (1-t) + t\,G(u,v).$$

$t = 1$ is original surface, $t = 0$ is Euclidean plane $ds^2 = du^2 + dv^2$. n is an integer-valued continuous function of t; $n = 1$ at $t = 0$, so $n = 1$ for $t = 1$ also. The total angle $= 2\pi$.

Applications

1) Geodesic triangles



← geodesics ($\chi_g \equiv 0$).

R topologically equivalent to a disc so $\chi(R) = 1$.

$$\int_R K\,dA + (\pi-\alpha) + (\pi-\beta) + (\pi-\gamma) = 2\pi$$

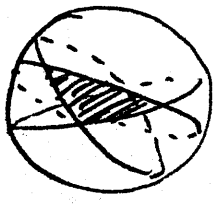
or

$$\int_R K\,dA = \alpha + \beta + \gamma - \pi.$$

a) If $K = \text{const.}$:

$$\alpha + \beta + \gamma - \pi = K \cdot (\text{area of } R)$$

Eg. Sphere of radius 1 :



$$\begin{pmatrix}\text{area of a}\\ \text{spherical}\\ \text{triangle}\end{pmatrix} = \alpha + \beta + \gamma - \pi.$$

b) In general : $\alpha + \beta + \gamma \geq \pi$, $= \pi$, or $\leq \pi$

if $K \geq 0$, $K \equiv 0$, or $K \leq 0$.

b) Let $A = \int_R dA$ be the area of R.

As R shrinks to a point P,

$$\lim \frac{1}{A} \int_R K\,dA = K_0 = \text{Gauss curvature at the point P.}$$

Thus:

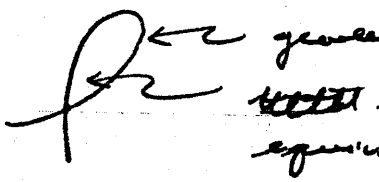
$$K_0 = \lim \frac{\alpha+\beta+\gamma-\pi}{A} \quad \text{(Gauss)};$$

Gauss curvature at P as limit as vertices of the triangle tend to P. This is Gauss's original definition of curvature.

N.B. shows K intrinsic.

2) Intersection of geodesics - negative curvature.

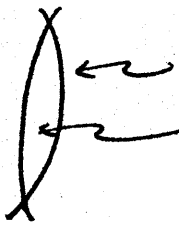
Assume $K < 0$.

a) 

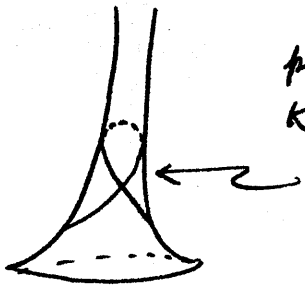
This situation cannot occur.

For

would be a geodesic triangle with $\alpha + \pi + \pi < \pi$. $(0 < \alpha < 2\pi)$.

b) 

This situation cannot occur.

Eg 

Note: Geodesics on the pseudo sphere will be determined later.

3) Admissible metrics on a surface

a) Suppose M is a closed surface with $K \equiv 0$. Then M is a torus or Klein bottle.

Pf $\int_M K \, dA = 2\pi \chi(M)$ gives $\chi(M) = 0$.

b) Suppose M is a closed surface with $K > 0$ everywhere. Then M is a sphere or projective plane.

Pf $\chi(M) > 0$.

Note: K needn't be constant, so M is only topologically a sphere or projective plane.

Note: If M is a surface in space, then M is topologically a sphere and is convex in the sense it bounds a convex set in space. (Hadamard).

c) Suppose M is a closed surface with $K < 0$. Then: M has genus $g \geq 2$ if it is orientable;

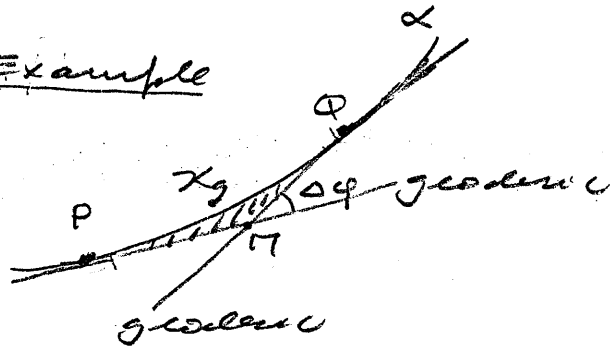
or: M is a projective plane or Klein bottle with at least one handle if it is non-orientable.

Pf $\chi(M) < 0$.

Rmk Orientable closed surfaces with $K < 0$ constant and $g \geq 2$ can be constructed using non-Euclidean (Lobachevski) geometry.

N.B. The above a global [illegible]

Example



α curve
P & Q points on α
κ_g geodesic curvature
Geodesics tangent to α at P & Q meet at M with angle $\Delta\varphi$.

For triangle MQP we have: s = arc length

$$\int_{\Delta MPQ} K\,dA + \left(\int_P^M + \int_M^Q + \int_Q^P\right)\kappa_g\,ds$$

on geodesics $\kappa_g = 0$ $\qquad -\int_P^Q \kappa_g\,ds$

(holonomy angle h) $\quad + (\Delta\varphi + \pi + \pi) = 2\pi$

exterior angles at tangent points

so

$$\frac{1}{\Delta s}\int_{\Delta MPQ} K\,dA + \frac{\Delta\varphi}{\Delta s} = \frac{1}{\Delta s}\int_P^Q \kappa_g\,ds$$

where $\Delta s = \int_P^Q ds$ along α. From figure above,

$$\int_{\Delta MPQ} K\,dA \approx K\,\text{Area}(\Delta MPQ) = O((\Delta s)^2).$$

Let $Q \to P$ to obtain

$$\boxed{\frac{d\varphi}{ds} = \kappa_g}.$$

This extends to abstract surfaces the elementary definition of the curvature of a curve in the Euclidean plane. And, gives a simple description of geodesic curvature in terms of quantities in the surface.

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