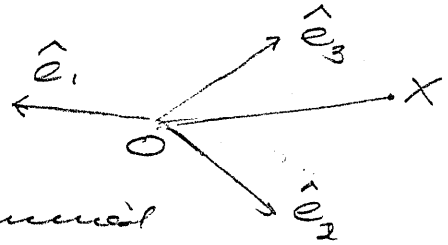
1/12/01 I **Curves in space. Local theory.**

Euclidean space

$\mathbb{E}^3$ - affine space with Euclidean dot product of vectors $\vec{v} \cdot \vec{w}$
- length $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$

Frame: Choice of origin O and an orthonormal basis $\hat{e}_1, \hat{e}_2, \hat{e}_3$.



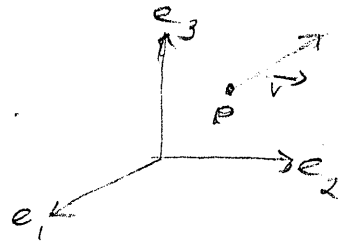
Any point X is determined by three coordinates:

$$\overrightarrow{OX} = \hat{e}_1 x^1 + \hat{e}_2 x^2 + \hat{e}_3 x^3 = [\hat{e}_1\ \hat{e}_2\ \hat{e}_3] \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

Customary to write

$$X = \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix}$$ - position vector of point X

- omit arrows on vectors
- use xyz when convenient

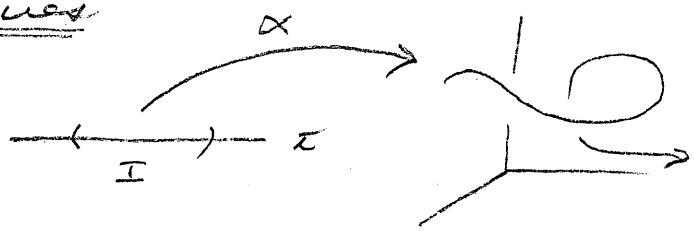


A bound vector or a vector at a point P is a pair $(P, \vec{v})$. A free vector can be "moved" anywhere in $\mathbb{E}^3$. This is special to Euclidean space.

Parametrized curves

A parametrized curve is a smooth function

$$\alpha : I \longrightarrow \mathbb{E}^3$$



I - (open) interval in reals $\mathbb{R}$

t - parameter

Write

$$X = \begin{bmatrix} \alpha^1(t) \\ \alpha^2(t) \\ \alpha^3(t) \end{bmatrix} \quad \text{or} \quad \overrightarrow{OX} = \hat{e}_1 \alpha^1(t) + \hat{e}_2 \alpha^2(t) + \hat{e}_3 \alpha^3(t)$$

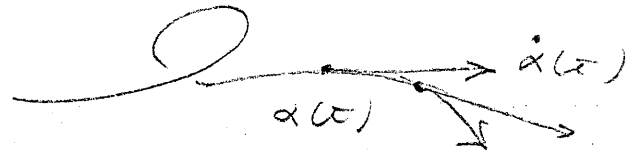
Smooth! $\alpha^1(t), \alpha^2(t), \alpha^3(t)$ are C^∞
- usually C^2 or C^3 suffices

Velocity vector

$X = \alpha(t)$ curve

$V = \frac{d\alpha}{dt}(t)$ or $\dot\alpha(t)$

- "velocity" - a bound vector



Dot denotes $\frac{d}{dt}$.

V is a vector field along α.

Regular curves

- $\alpha(t)$ is class C^2 or C^3
- $\dot\alpha(t)$ is never the zero vector

Note: If $\dot\alpha(t)$ is never zero, then $x = \alpha(t)$ is injective in the small.

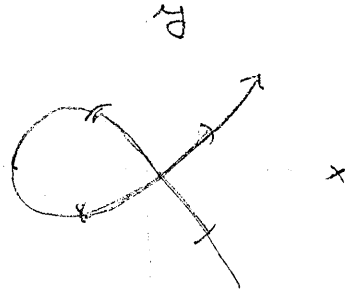
For: $\alpha(t) = \alpha(t_0) + \dot\alpha(t_0)(t-t_0) + O((t-t_0)^2)$

so $\alpha(t) - \alpha(t_0) = (\dot\alpha(t_0) + O(t-t_0))(t-t_0)$

For t near to but $\neq t_0$, we have $\alpha(t) \neq \alpha(t_0)$

Example Folium of Decartes

$$\alpha(t) = \begin{bmatrix} t^2-1 \\ t^3-t \\ 0 \end{bmatrix} \qquad y^2 = x^2 + x^3$$



$$\dot\alpha(t) = \begin{bmatrix} 2t \\ 3t^2-1 \\ 0 \end{bmatrix}$$ This is never zero.

Injective in the small, but not in the large. $\alpha(-1) = \alpha(1)$.

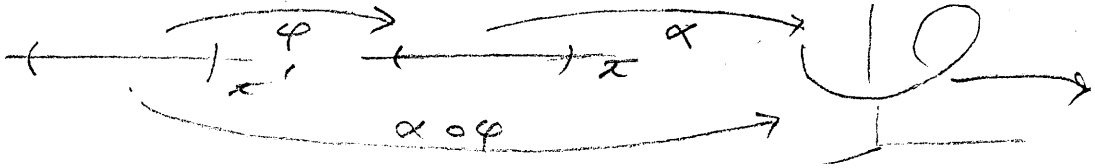
Change of parameter

$X = \alpha(t)$ curve

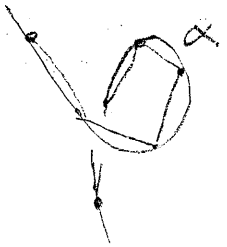
$t = \varphi(t')$ φ monotone and C^2 or smooth $\frac{dt}{dt'}$ never zero

$X = \alpha(\varphi(t')) = (\alpha\circ\varphi)(t')$ is same curve with new parameter t'.

Write $X = \alpha(t')$. This is a different α.



Arc length



curve $X = \alpha(t)$ - regular

$a = x_0 < x_1 < \cdots < x_n = b$

Length of polygonal approximation

$$= \sum_{i=1}^{n} \| \alpha(x_i) - \alpha(x_{i-1}) \|.$$

(length of curve from $\alpha(a)$ to $\alpha(b)$) = sup (length of polygonal approximations)

- sup is over partitions $\{x_i\}$.

From calculus

$$\text{length} = \int_a^b \| \dot{\alpha}(x) \| \, dx.$$

Note Length does not depend on parameterization - see geometric definition. Or, change parameters

$$\int_a^b \left\| \frac{d\alpha}{dx}(x) \right\| dx = \int_{a'}^{b'} \left\| \frac{d\alpha}{dx}(\varphi(x')) \right\| \left\| \frac{d\varphi}{dx}(x') \right\| dx'$$

$$= \pm \int_{a'}^{b'} \left\| \frac{d\alpha}{dx}(\varphi(x')) \frac{d\varphi}{dx}(x') \right\| dx' \quad \left. \begin{array}{l} + \text{ or } - \text{ as} \\ \frac{dx}{dx} > 0 \text{ or } < 0 \end{array} \right\}$$

$$= \pm \int_{a'}^{b'} \left\| \frac{d}{dx} \alpha(\varphi(x')) \right\| dx'$$

Arc length as parameter

$X = \alpha(t)$ $a \le x \le b$ regular curve

$$s = \int_a^x \| \dot{\alpha}(u) \| \, du$$ arc length along α from $\alpha(a)$ to $\alpha(x)$

$\frac{ds}{dx} = \| \dot{\alpha}(x) \|$ is never zero.

$x = \varphi(s)$ - solve for x as function of s

$X = \alpha(\varphi(s))$ $0 \le s \le L$ L = length of α from $\alpha(a)$ to $\alpha(b)$

or $\alpha(s)$

Problem (1.1) Compute the arc lengths of:

a) the line $x = l_x t + x_0$, $y = l_y t + y_0$, $0 \le t \le b$

b) the circle $x = a\cos\omega t$, $y = a\sin\omega t$, $0 \le t \le \frac{2\pi}{\omega}$

c) the logarithmic spiral (in polar coordinates) $r = a e^{-b\theta}$, $0 \le \theta < \infty$

d) the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

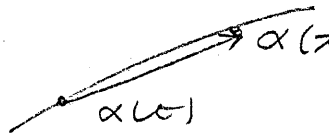
Hint! $x = a\cos^3 t$, $y = a\sin^3 t$.

Parameterize these curves by their arc lengths s, when possible.

1/15/01 OLIDAY 1/17/01

The moving frame

Unit tangent vector



$$\frac{\alpha(t+h)-\alpha(t)}{\|\alpha(t+h)-\alpha(t)\|} = \begin{pmatrix}\text{unit}\\ \text{secant}\\ \text{vector}\end{pmatrix}$$

$$\lim_{h\to 0}\begin{pmatrix}\text{unit}\\ \text{secant}\\ \text{vector}\end{pmatrix} = \frac{\dot\alpha(t)}{\|\dot\alpha(t)\|} = T(t)$$ — bound vector

is the unit tangent vector at $\alpha(t)$

Note: T is in the direction of increasing t or arclength.

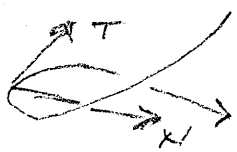
Note: If $f(t)$ is any function, $\frac{df}{dt} = \frac{df}{ds}\frac{ds}{dt}$.

So

$$T = \frac{\frac{d\alpha}{dt}}{\left\|\frac{d\alpha}{dt}\right\|} = \frac{\frac{d\alpha}{dt}}{\frac{ds}{dt}} = \frac{d\alpha}{ds}$$ — unit tangent vector

If $X = \alpha(s)$ is parameterized by arclength s, then $T = \alpha'(s)$, $\|T\| = 1$. $' = \frac{d}{ds}$.

Principal normal

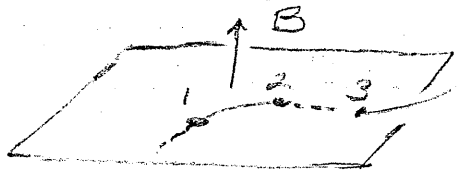


Unit vector $N(s) = \frac{T'(s)}{\|T'(s)\|}$ in the direction of the "acceleration" $\ddot\alpha(t)$.

The principal normal is bound at $\alpha(s)$.

Note: $T(s)\cdot T(s) = 1$ gives $2T(s)\cdot T'(s) = 0$.

Osculating plane



$\alpha(s)$ regular curve
s arclength
s_1, s_2, s_3 near fixed s_0
$\alpha(s_1), \alpha(s_2), \alpha(s_3)$ not collinear } B is a unit vector

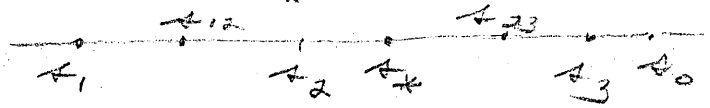
$B(s_1,s_2,s_3)\cdot X - \phi(s_1,s_2,s_3)=0$
– equation of plane through $\alpha(s_1), \alpha(s_2), \alpha(s_3)$

Now! $s \mapsto B(s_1,s_2,s_3)\cdot\alpha(s) - \phi(s_1,s_2,s_3)$
is a function of one variable with zeros at s_1, s_2, s_3 – Apply Rolle's theorem twice!

$B(s_1,s_2,s_3)\cdot\alpha'(s_{12}) = 0$
$B(s_1,s_2,s_3)\cdot\alpha'(s_{23}) = 0$.

Again: $B(s_1,s_2,s_3)\cdot\alpha''(s_*) = 0$

Let $s_1, s_2, s_3 \to s_0$:



$B(s_0) = \lim B(s_1,s_2,s_3)$ exists

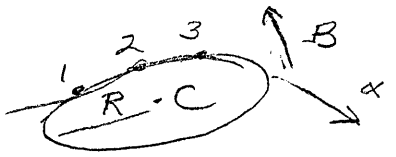
$B(s_0)\cdot\alpha'(s_0) = 0$ and $B(s_0)\cdot\alpha''(s_0)=0$

So: $B(s)$ (any s) is a unit vector parallel to $\alpha'(s)\times\alpha''(s)$. Need $\alpha'\times\alpha'' \neq 0$

Note $\alpha'(s)\cdot\alpha'(s)=1$ gives $\alpha'(s)\cdot\alpha''(s)=0$, so α', α'', $\alpha'\times\alpha''$ are mutually orthogonal.

The unit vector $B(s)$ is the binormal to the curve α at $\alpha(s)$, a bound vector.

Osculating circle



$\alpha(s)$ regular curve
s arclength
s_1, s_2, s_3 near fixed s_0
$\alpha(s_1), \alpha(s_2), \alpha(s_3)$ not collinear

$\alpha(s_1), \alpha(s_2), \alpha(s_3)$ determine a circle in the plane $B(s_1,s_2,s_3)\cdot X - \phi(s_1,s_2,s_3)=0$.

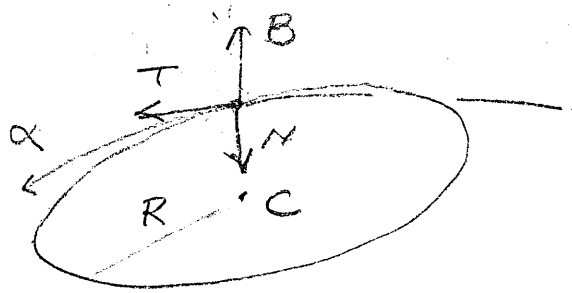
$C(s_1,s_2,s_3)$ center (point)
$R(s_1,s_2,s_3)$ radius (real number) } of circle

Equation of circle in plane above:
$(X - C(t_1,t_2,t_3)) \cdot (X - C(\text{---})) = (R(t_1,t_2,t_3))^2$
The function
$$t \longmapsto (\alpha(t) - C(t_1,t_2,t_3)) \cdot (\alpha(t) - C(t_1,t_2,t_3)) - (R(t_1,t_2,t_3))^2$$
has zeros at t_1, t_2, t_3.

Problem (1.2) Use Rolle's Theorem three times, as before. Take the limit as $t_1, t_2, t_3 \to t_0$ and obtain for
$C(t_0) = \lim C(t_1,t_2,t_3)$
$R(t_0) = \lim R(t_1,t_2,t_3)$
- and hence for any t -:

(0) $B(t) \cdot (\alpha(t) - C(t)) = 0$
(1) $(\alpha(t) - C(t)) \cdot (\alpha(t) - C(t)) = (R(t))^2$
(2) $(\alpha(t) - C(t)) \cdot \alpha'(t) = 0$
(3) $(\alpha(t) - C(t)) \cdot \alpha''(t) + 1 = 0$

Consequences of Prob 1.2:
(0): $\alpha(t) - C(t)$ is perpendicular to $B(t)$
(2): " " to $T(t)$
so $\alpha(t) - C(t)$ is parallel to $N(t)$:
$$\alpha(t) - C(t) = N(t)\lambda$$
By (1): $\lambda^2 = (R(t))^2$, so $\lambda = -R(t)$.



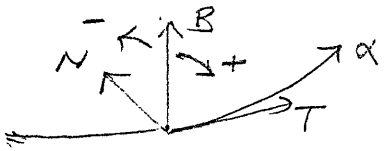
$C(t) = \alpha(t) + N(t)R(t)$
- center of curvature

$R(t)$
- radius of curvature
at $\alpha(t)$.

Now, $\alpha''(t)$ is a multiple of $N(t)$:
$\alpha''(t) = \kappa(t)N(t)$, $\kappa(t) \geq 0$ scalar
Then (3) says, $-\kappa(t)R(t) \cdot N(t)\kappa(t) + 1 = 0$,
so $\boxed{\kappa(t) = \frac{1}{R(t)}}$, curvature of α at $\alpha(t)$.
Note $\alpha''(t) = T'(t)$, so $\boxed{\frac{dT}{ds} = N\kappa}$, $\kappa = \|T'\|$.

Remark $\kappa(s)$ is identically zero iff α is a straight line. For: $\kappa(s)=0$ all s means $T'(s)=0$ or $\alpha''(s)=0$ for all s, so $\alpha(s) = \alpha(0) + Ts$, with T a constant vector.

Torsion



Torsion measures the "twisting" of the osculating plane about T.

We have:

$$\frac{dB}{ds} = (T \times N)' = \underbrace{T' \times N}_{0 \text{ by } T' = N\kappa} + T \times N' = T \times N'$$

so $\frac{dB}{ds}$ is perpendicular to T.

And, $B \cdot B = 1$, $2B \cdot B' = 0$,

so $\frac{dB}{ds}$ is perpendicular to B.

Thus, $\frac{dB}{ds}$ is parallel to N;

$$\boxed{\frac{dB}{ds} = -N\tau}, \quad \tau = \pm \|B'\|.$$

$\tau(s)$ is the torsion of α at $\alpha(s)$.

Remark $\tau(s)$ is identically zero iff α is a plane curve. For: $\tau(s)=0$ all s means $\frac{dB}{ds}=0$, so B = constant vector. Then $\frac{d}{ds}(B \cdot \alpha(s)) = B \cdot \alpha' = B \cdot T = 0$, so $B \cdot \alpha(s)$ = constant. The converse is trivial.

Note The functions $\kappa(s)$ and $\tau(s)$ are invariants under rigid motions - congruent curves have the same curvature and torsion. Later, the converse will be proved.

1/9/01 **Problem (1.3)** Let α be a plane curve. If $\theta(s)$ is the angle between the tangent vector $\alpha'(s)$ at $\alpha(s)$ and a fixed direction in the plane, show that $\kappa(s) = \left|\frac{d\theta}{ds}(s)\right|$. Note: Plane curves can have signed curvature, curves in space do not.

Hint: $T = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$ and $\frac{dT}{ds} = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \frac{d\theta}{ds}$

Example The ellipse $\alpha(t) = \begin{bmatrix} a\cos t \\ b\sin t \\ 0 \end{bmatrix}$.

Use $\frac{d\alpha}{dt} = \frac{d\alpha}{ds}\frac{ds}{dt}$:

$\dot{\alpha} = Ti = \begin{bmatrix} -a\sin t \\ b\cos t \\ 0 \end{bmatrix}$, $T = \frac{d\alpha}{ds}$ is unit vector,

so $i^2 = a^2\sin^2 t + b^2\cos^2 t$

Then $\ddot{\alpha} = (T'i)i + T\ddot{i} = T\ddot{i} + N\kappa i^2 = \begin{bmatrix} -a\cos t \\ -b\sin t \\ 0 \end{bmatrix}$ (CENTRIPETAL ACCELERATION)

($T'i = N\kappa$)

Vector product $\dot{\alpha} \times \ddot{\alpha}$:

$\dot{\alpha} \times \ddot{\alpha} = Ti \times (T\ddot{i} + N\kappa i^2) = B\kappa i^3$

($T \times T = 0$, $T \times N = B$)

$= \begin{bmatrix} -a\sin t \\ b\cos t \\ 0 \end{bmatrix} \times \begin{bmatrix} -a\cos t \\ -b\sin t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ ab \end{bmatrix}$,

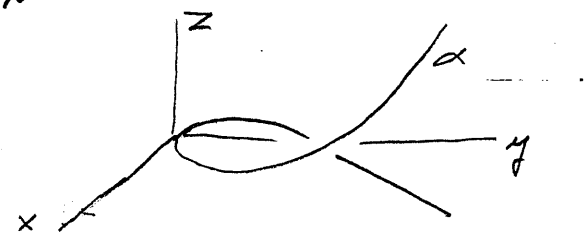
so $\kappa i^3 = ab$. Thus

$$\kappa = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}$$

Remark The same technique using $\dot{\alpha} = Ti$ works for curves in space.

Example The twisted cubic

$\alpha(t) = \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix}$.



(1) $T\dot{s} = \dot{\alpha} = \begin{bmatrix} 1 \\ 2\tau \\ 3\tau^2 \end{bmatrix}$, $\dot{s}^2 = 1 + 4\tau^2 + 9\tau^4$

(2) $(T'\dot{s})\dot{s} + T\ddot{s} = N\kappa\dot{s}^2 + T\ddot{s} = \begin{bmatrix} 0 \\ 2 \\ 6\tau \end{bmatrix}$

Vector product (1) × (2):

$$T\dot{s} \times (N\kappa\dot{s}^2 + T\ddot{s}) = \begin{bmatrix} 1 \\ 2\tau \\ 3\tau^2 \end{bmatrix} \times \begin{bmatrix} 0 \\ 2 \\ 6\tau \end{bmatrix}$$

to

(3) $B\kappa\dot{s}^3 = \begin{bmatrix} 6\tau^2 \\ -6\tau \\ 2 \end{bmatrix}$

Differentiate (3):

$$\left(\frac{dB}{ds}\dot{s}\right)\kappa\dot{s}^3 + B\frac{d}{dt}(\kappa\dot{s}^3) = \frac{d}{dt}\begin{bmatrix} 6\tau^2 \\ -6\tau \\ 2 \end{bmatrix}$$

$\frac{dB}{ds} = -N\tau$

(4) $-N\kappa\tau\dot{s}^4 + B\frac{d}{dt}(\kappa\dot{s}^3) = \begin{bmatrix} 12\tau \\ -6 \\ 0 \end{bmatrix}$

Scalar product (2) · (4):

(5) $-\kappa^2\tau\dot{s}^6 = -12$

Note: T, N, B are right handed orthonormal.

Scalar product (3)·(3):

(6) $\kappa^2\dot{s}^6 = 36\tau^4 + 36\tau^2 + 4$

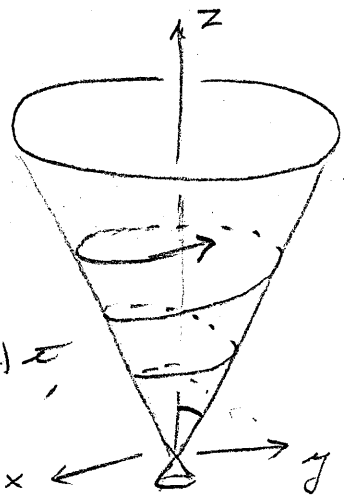
Solve (1), (6), (5) for

$$\kappa^2 = \frac{4(1 + 9\tau^2 + 9\tau^4)}{(1 + 4\tau^2 + 9\tau^4)^3}, \quad \tau = \frac{3}{1 + 9\tau^2 + 9\tau^4}$$

Problem (1.4) Let α be a regular curve parametrized by t. "$\cdot$" denotes $\frac{d}{dt}$. Show

$$\kappa = \frac{\|\dot{\alpha} \times \ddot{\alpha}\|}{\|\dot{\alpha}\|^3}, \quad \tau = \frac{\dot{\alpha} \times \ddot{\alpha} \cdot \dddot{\alpha}}{\|\dot{\alpha} \times \ddot{\alpha}\|^3}.$$

Remark One can compute the curvature and torsion of curves defined implicitly as the intersection of two surfaces $f(x,y,z) = 0$ and $g(x,y,z) = 0$. It begins by noting that the tangent vector T at a point is parallel to $\nabla f \times \nabla g$. (∇ denotes gradient.) See Willmore p. 16 ff.



Problem (1.5) Investigate the helix on a right circular cone given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos Wt \\ \sin Wt \\ 0 \end{bmatrix} (\sin a)\, t + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} (\cos a)\, t,$$

where a is the vertex angle of the cone.

a) Find the curvature and torsion. For a helix on a cylinder, $\kappa/\tau = \text{CONSTANT}$. Is there a relation between κ and τ for the conical helix? Does it characterize such a helix?

b) Reconcile the "limiting cases" of $a = 0$ (helix on a circular cylinder) and $a = 90°$ (planar spiral).

Prob (1.6) For the curve $X(t) = (a + a\cos t,\ a \sin t,\ 2a \sin \frac{t}{2})$, find at each point: the curvature and torsion, the vectors of the Frenet frame, and the equations of the osculating plane and osculating circle. Make a sketch.

Problem (1.7) a) Let κ and τ denote respectively the curvature and torsion of a curve in space. Show that, if the curve lies on a sphere, $\frac{d}{ds}\left(\frac{1}{\tau\kappa^2}\frac{d\kappa}{ds}\right) - \frac{\tau}{\kappa} = 0$, $\frac{d}{ds}$ denoting differentiation with respect to arc length along the curve.

b) Give an example of such a curve which does not lie in a plane.

c) Is there a converse to a)? (Hint: For this, look at the center of the osculating sphere to the curve at a point. This sphere is determined by four points on the curve in the same manner that the osculating circle is determined by three.)

1/22/01 **Equations of Frenet & Serret**

Let T, N, B be the usual vectors associated with a regular curve α - parameterized by arclength s.
Already proved are the first & third equation of

$$\begin{cases} \frac{dT}{ds} = \kappa N \\ \frac{dN}{ds} = -\kappa T + \tau B \\ \frac{dB}{ds} = -\tau N \end{cases}$$ Frenet-Serret equations

To check the second equation:

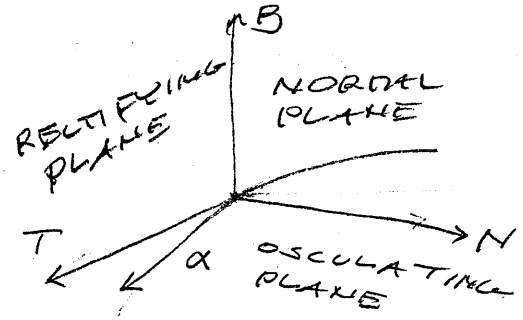
$$\frac{dN}{ds} = \frac{d}{ds}(B \times T) = \frac{dB}{ds} \times T + B \times \frac{dT}{ds}$$

$$= (-\tau N) \times T + B \times (\kappa N) = \tau B - \kappa T.$$

($-\tau N = \frac{dB}{ds}$, $\kappa N = \frac{dT}{ds}$)

These, together with $\frac{d\alpha}{ds} = T$, contain all information for the curve.

Moving frame - or Frenet frame.
or moving trihedron.



Write Frenet equations using matrices:

$$\frac{d}{ds}[T\,N\,B] = [T\,N\,B] \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}$$

3 col. vectors

NOTE T, N, B, κ, τ depend on s.

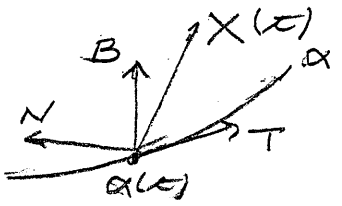
$$\Omega(s) = \begin{bmatrix} 0 & -\kappa(s) & 0 \\ \kappa(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{bmatrix}$$ — CARTAN matrix

Write the vectors of the frame as

$E(s) = [\,T(s)\ N(s)\ B(s)\,]$

3×3 matrix, 3 column vectors, so $\frac{dE}{ds} = E\Omega$

Vector fields in moving frame



$\alpha(t)$ reference curve

$X(t)$ vector field along α — vector is bound at $\alpha(t)$.

$\bar{x}(t) = \begin{bmatrix} x^1(t) \\ x^2(t) \\ x^3(t) \end{bmatrix}$ — components of X relative to T, N, B.

$X = E\bar{x}$

Now

$\frac{dX}{dt}(t) = \lim \frac{X(t+h) - X(t)}{h}$ is relative to fixed basis $\hat{i}, \hat{j}, \hat{k}$.

We have

$$\frac{dX}{dt} = \frac{d}{dt}(E\bar{x}) = \frac{dE}{ds}\frac{ds}{dt}\bar{x} + E\frac{d\bar{x}}{dt}$$

$$\boxed{\frac{dX}{dt} = E\left(\frac{d\bar{x}}{dt} + \Omega\frac{ds}{dt}\bar{x}\right)}$$

$\frac{d\bar{x}}{dt}$: motion relative to frame

$\Omega\frac{ds}{dt}\bar{x}$: instantaneous rotation about axis in frame

Aside One can write

$$\begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}\begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} \tau \\ 0 \\ \kappa \end{bmatrix} \times \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

$\begin{bmatrix} \tau \\ 0 \\ \kappa \end{bmatrix}$ is the Darboux vector.

Canonical representations

α smooth curve

s arclength.

Describe curve near $s=0$:- use Maclaurin expansion:

$$\alpha(s) = \alpha(0) + \alpha'(0)s + \frac{1}{2}\alpha''(0)s^2 + \frac{1}{6}\alpha'''(0)s^3 + \frac{1}{24}\alpha''''(0)s^4 + O(s^5)$$

or $\alpha(s) = \alpha(0) + T(0)s + \frac{1}{2}T'(0)s^2$

$+ \frac{1}{6}T''(0)s^3 + \frac{1}{24}T'''(0)s^4 + O(s^5)$

We have

$T = E\begin{bmatrix}1\\0\\0\end{bmatrix}$, $T' = E\Omega\begin{bmatrix}1\\0\\0\end{bmatrix}$,

$T'' = E(\Omega^2 + \Omega')\begin{bmatrix}1\\0\\0\end{bmatrix}$

$T''' = E(\Omega^3 + \Omega'\Omega + 2\Omega\Omega' + \Omega'')\begin{bmatrix}1\\0\\0\end{bmatrix}$

Write

$\alpha(s) = \alpha(0) + T(0)u(s) + N(0)v(s) + B(0)w(s)$.

One obtains (evaluated at $s=0$):

$u(s) = s - \frac{\kappa^2}{6}s^3 - \frac{\kappa\kappa'}{8}s^4 + O(s^5)$

$v(s) = \frac{\kappa}{2}s^2 + \frac{\kappa'}{6}s^3 + \frac{\kappa'' - \kappa^3 - \kappa\tau^2}{24}s^4 + O(s^5)$

$w(s) = \frac{\kappa\tau}{6}s^3 + \frac{2\kappa'\tau + \kappa\tau'}{24}s^4 + O(s^5)$

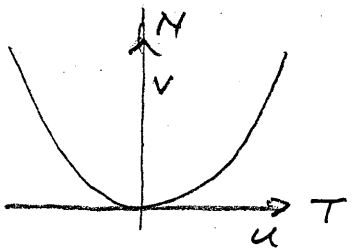
Eg.

$\Omega^2 + \Omega' = \begin{bmatrix}-\kappa^2 & -\kappa' & \kappa\tau\\ \kappa' & -\kappa^2 - \tau^2 & -\tau'\\ \kappa\tau & \tau' & -\tau^2\end{bmatrix}$.

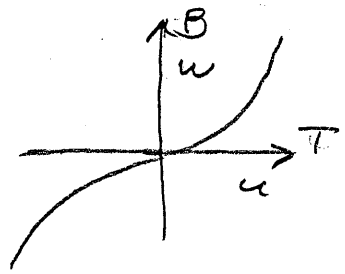
Note T'' should be checked.

For small s:

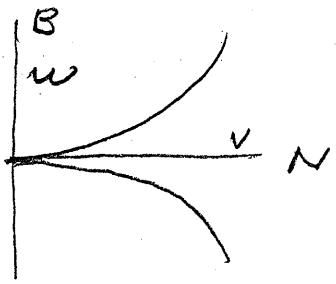
$v^2 \approx \frac{\kappa}{2}u^2$ | $w \approx \frac{\kappa\tau}{6}u^3$ | $w^2 \approx \frac{2\tau^2}{9\kappa}v^3$



OSCULATING PLANE

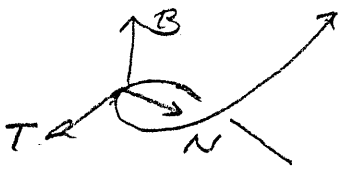


RECTIFYING PLANE



NORMAL PLANE

Note



In the small, a curve looks like a twisted cubic.

Intrinsic equations of a space curve

The functions $\kappa(s)$ and $\tau(s)$ are *invariants* of a space curve - unchanged by rigid motions in $\mathbb{E}^3$.

Conversely: Prescribing $\kappa(s)$, $\tau(s)$ and a frame $\alpha(0)$, $T(0)$, $N(0)$, $B(0)$ yields a unique curve - locally.

For:

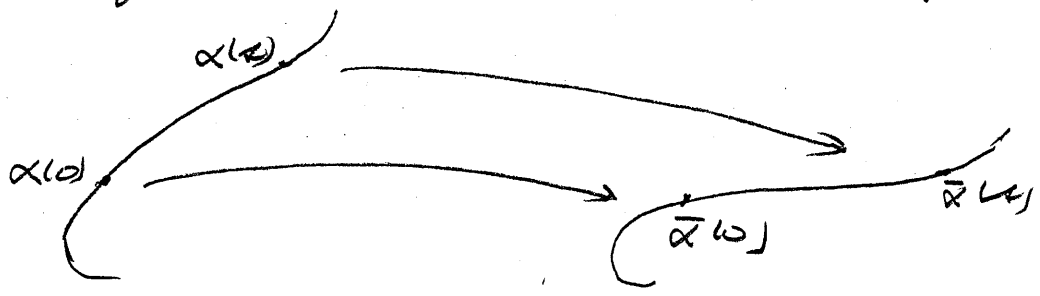
$$\begin{cases} \frac{d\alpha}{ds} = T \\ \frac{dT}{ds} = \kappa N \\ \frac{dN}{ds} = -T\kappa + B\tau \\ \frac{dB}{ds} = -N\tau \end{cases}$$

12 first order scalar ODEs - with 12 initial conditions $\alpha(0)$, $T(0)$, $N(0)$, $B(0)$.

Solutions to such a system exist and are unique - locally. SEE PROB 1.8 - NEXT PAGE

Aside Solve the Frenet equations, then $\alpha(s) = \alpha(0) + \int_0^s T(u)\,du$.

Uniqueness - another proof.



Assume: $\kappa(s)$ and $\tau(s)$ the same for two curves $\alpha(s)$ and $\bar{\alpha}(s)$ as measured from $\alpha(0)$ and $\bar{\alpha}(0)$, respectively. Move curves rigidly so that at $s=0$ the frames $\alpha(0)$, $T(0)$, $N(0)$, $B(0)$ and $\bar{\alpha}(0)$, $\bar{T}(0)$, $\bar{N}(0)$, $\bar{B}(0)$ coincide.

Consider the function

$$f(s) = T(s)\cdot\bar{T}(s) + N(s)\cdot\bar{N}(s) + B(s)\cdot\bar{B}(s).$$

Then

$$(T\cdot\bar{T})' = N\kappa\cdot\bar{T} + T\cdot\bar{N}\kappa$$

$$(N\cdot\bar{N})' = (-T\kappa + B\tau)\cdot\bar{T} + N\cdot(-\bar{T}\kappa + \bar{B}\tau)$$

$$(B\cdot\bar{B})' = (-N\tau)\cdot\bar{B} + B\cdot(-\bar{N}\tau),$$

Add to obtain $f'(s) = 0$ all s, so $f(s) =$ constant. $f(0) = 3$, so $f(s) = 3$ all s. Since $f(s)$ is the sum of three cosines, each summand must be constantly 1. Hence $T(s) = \bar{T}(s)$ all s, so $\alpha(0) = \bar{\alpha}(0)$ implies $\alpha(s) = \bar{\alpha}(s)$ for all s. QED

Problem (1.9) Find the "coördinate function" $\alpha(s)$ for the cylindrical helix whose intrinsic equations are $\kappa(s) = 1/2$ and $\tau(s) = 1/2$. Make a sketch.

Problem (1.8) Let $T(s)$, $N(s)$, $B(s)$ be a solution to Frenet's equations. If T, N, B are orthonormal at $s=0$, they are orthonormal for all s.

Hint: T, N, B are orthonormal exactly when $E(s) = [T(s)\ N(s)\ B(s)]$ is an orthogonal matrix: ${}^tE\cdot E =$ identity. Compute $\frac{d}{ds}{}^tE\cdot E$ using $E' = E\Omega$.