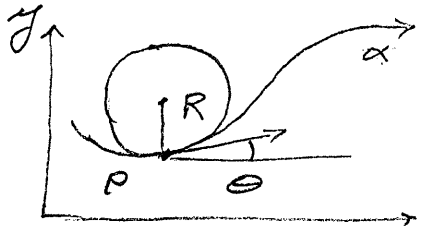
2/10/

0. Introduction - The theorem of Gauss - Bonnet and Poincaré - Hopf

The goal is to establish the connection between geometry, topology, and analysis.

Geometry

curvature of plane curves



α plane curve
$x = f(t)$
$y = g(t)$
circle: "best fitting" or osculating circle to α at P.

R = radius of curvature of α at P

$\kappa = \frac{1}{R}$ the curvature of α at P

From calculus:

$\kappa = \frac{d\theta}{ds}$ θ = angle (x-axis to tangent $\frac{d\alpha}{dt}$)
s = arc length along α

Signs:) $\kappa > 0$ / $\kappa = 0$ ($\kappa < 0$

Problem (0.1) Taking $\kappa = \frac{d\theta}{ds}$ as the definition of curvature, show:

a) The curvature of a curve given by $x = f(t)$, $y = g(t)$ is $\kappa(t) = \frac{\dot{f}\ddot{g} - \ddot{f}\dot{g}}{(\dot{f}^2 + \dot{g}^2)^{3/2}}$ (at t)

$\cdot$ or $'$ denote the derivative d/dt.

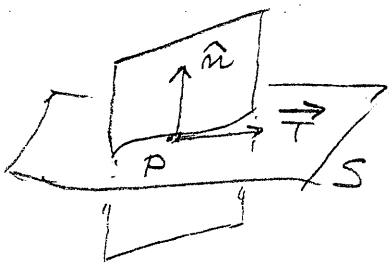
b) The curvature of the level curves given by $F(x,y) =$ constant is

$$\kappa(x,y) = -\frac{F_1^2 F_{22} - 2F_1 F_2 F_{12} + F_2^2 F_{11}}{(F_1^2 + F_2^2)^{3/2}} \text{ at } (x,y)$$

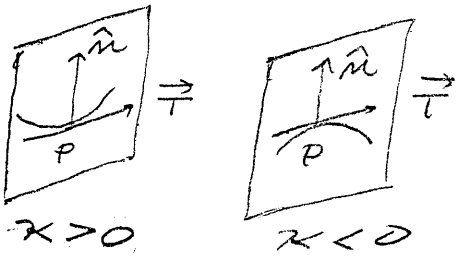
where subscripts 1 and 2 denote the partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$

Curvature of surfaces in space.

Extrinsic



S surface in Euclidean E^3
P point on S
$\hat{n}$ unit normal to S at P
$\vec{T}$ a vector tangent to S at P
κ = curvature at P of curve in plane of $\hat{n}$ and $\vec{T}$ where this plane meets S.



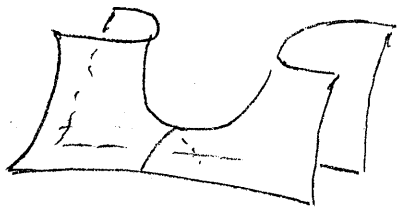
$\kappa(P, \vec{T})$ depends only on P and $\vec{T}$.

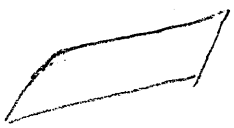
$\kappa_1(P) = \min \kappa(P, \vec{T})$
$\kappa_2(P) = \max \kappa(P, \vec{T})$
} as $\vec{T}$ varies over vectors tangent to S at P.

κ_1 and κ_2 are the principal curvatures - depend on P only.

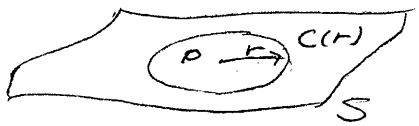
$K(P) = \kappa_1 \kappa_2$ Gauss curvature of S at P - depends on P only.


$K > 0$
sphere


$K < 0$
saddle surface


$K = 0$
plane

Intrinsic



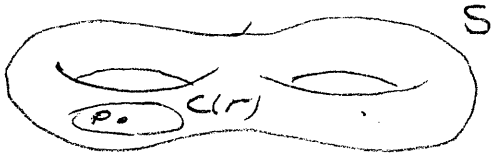
$C(r)$ circle of radius r viewed from within the surface S
$L[r]$ length of $C(r)$

$$K(P) = \lim_{r \to 0} \frac{2\pi r - L(r)}{\frac{1}{3}\pi r^3}$$

$K > 0$ - circumference too short
$K < 0$ - " " too long
$K = 0$ - " " just right (planar)

Note K is intrinsic - can be computed from metric on S - i.e. K is a "bending invariant". "Theorema Egregium" - Gauss.

Curvature of abstract surfaces



Intrinsic K for any surface - in E^3 or not. Need only to measure lengths of vectors and curves in S.

$$K(P) = \lim_{r \to 0} \frac{2\pi r - L(r)}{\frac{1}{3}\pi r^3}$$ defines K at P

Problem (0.2) Using this definition of curvature, show that a sphere of radius a has constant curvature $K = 1/a^2$.

Total curvature

$\int_S K\,dA$ - a number depending on the geometry of S.

S a connected compact or "closed" surface
dA element of surface area of S.

Example S a sphere of radius a.

$$\int_S K\,dA = \int_S \frac{1}{a^2}\,dA = \frac{1}{a^2}\int_S dA$$

$$= \frac{1}{a^2}\left(\text{area of } S\right) = \frac{1}{a^2}\cdot 4\pi a^2 = 4\pi$$

1/10/01 Topology

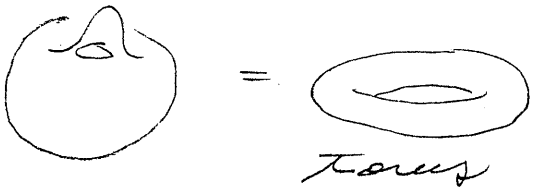
Genus of a surface

S orientable surface, closed

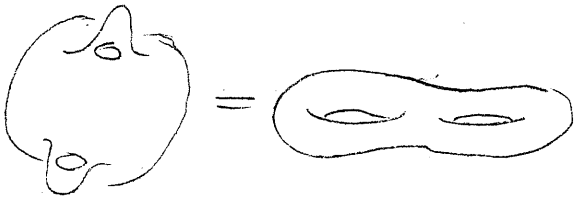
g = genus = number of "handles" needed to obtain the surface from a sphere by attaching handles



= sphere $\quad g = 0$



= torus $\quad g = 1$



$g = 2$

g = number of "holes"

Theorem of Gauss - Bonnet

S closed oriented surface

g genus of S

K curvature calculated from any prescription of distance measurement on S - the "metric"

Then

$$\int_S K \, dA = 2\pi(2-2g)$$

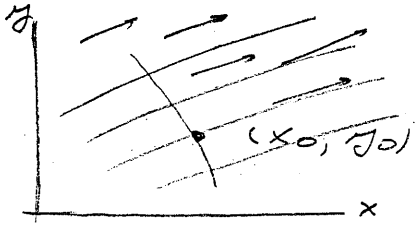
Eg For a sphere, $g=0$ gives 4π.

Analysis

Vector fields in the plane

$$\begin{cases} \frac{dx}{dt} = f(x,y) & x = x_0 \\ \frac{dy}{dt} = g(x,y) & y = y_0 \\ & \text{when} \\ & t = t_0 \end{cases}$$

System of two first autonomous ODEs.



One-parameter family of curves fill a region.

Integral curves
x = function of t
y = " " of t
are solutions of initial value problem.

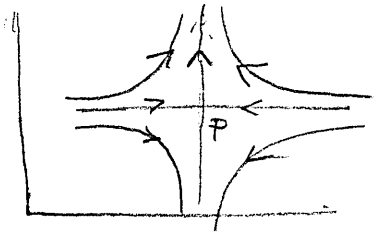
$\vec{X} = \hat{i} f(x,y) + \hat{j} g(x,y)$ is vector field on plane $\mathbb{R}^2$. (Euclidean metric not needed.)

Vector fields on surfaces

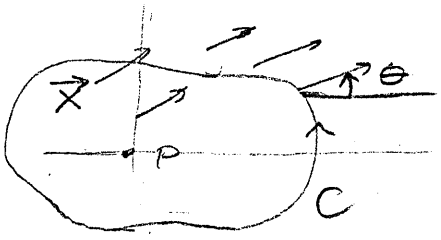


integral curves

Index of a vector field



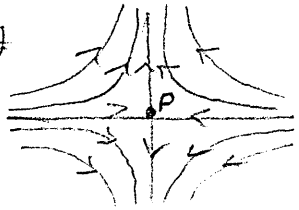
Singular point:
$\vec{X} = 0$ at P
Isolated:
$\vec{X} \neq 0$ near P.



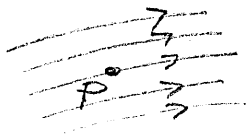
- θ = angle field makes with fixed direction - eg. the x-axis.
- δ = change in θ upon traversing once closed loop about P once.
- δ = integer multiple of 2π

$I_P(\vec{X}) = \frac{1}{2\pi}\delta$, the index of the vector field $\vec{X}$ at P.

Ex.

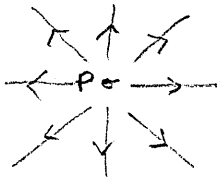


$I_P = -1$



$I_P = 0$

NOT NECESSARILY SINGULARITY.



$I_P = +1$

Problem (0.3) a) Show that the index at P of the smooth vector field

$$\vec{X} = \hat{i}\, f(x,y) + \hat{j}\, g(x,y)$$

is $$\frac{1}{2\pi} \int_C \frac{f\,dg - g\,df}{f^2+g^2},$$

where the smooth curve C encloses P once in the counter-clockwise direction and $\vec{X}$ is defined and non-vanishing on C and its interior less P.

Hint: Compute $\frac{d}{ds} \arctan \frac{g}{f}$ along C.

b) Find (polynomial) functions $f(x,y)$ and $g(x,y)$ such that the field $\hat{i} f(x,y) + \hat{j} g(x,y)$ has index -3 at $(0,0)$.

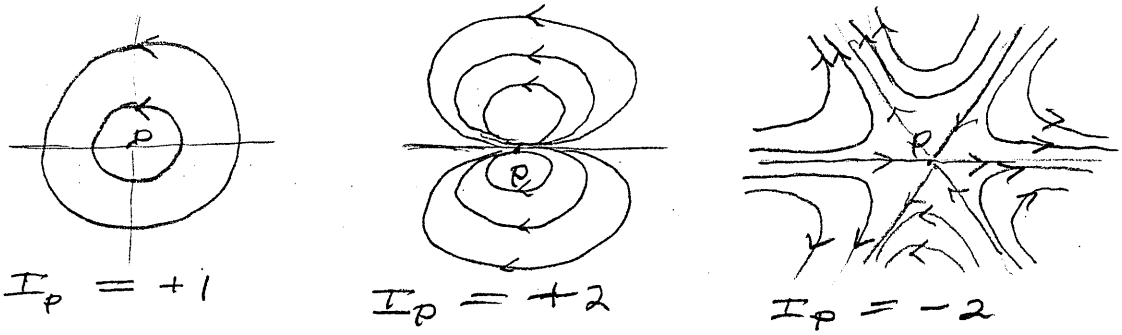
Hint: Examine the fields

$$\hat{i}\, Re((x+\sqrt{-1}y)^n) + \hat{j}\, Im((x+\sqrt{-1}y)^n)$$

and

$$\hat{i}\, Re((x-\sqrt{-1}y)^n) + \hat{j}\, Im((x-\sqrt{-1}y)^n),$$

where n is a positive integer.



$I_P = +1$ $I_P = +2$ $I_P = -2$

Total index

S closed orientable surface

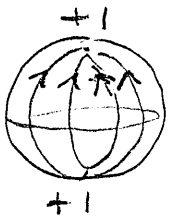
$\vec{X}$ smooth vector field on S with finitely many isolated singularities

- any such field on S

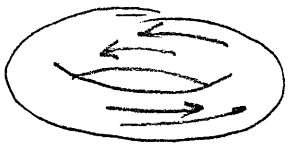
Total index:

$$I(\vec{X}) = \sum_{p \text{ singular}} I_p(\vec{X})$$

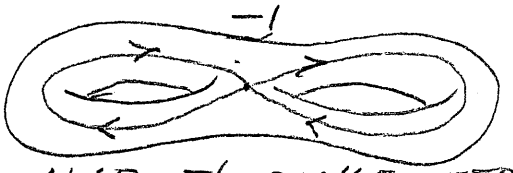
Eg



$I(\vec{X}) = +2$



$I(\vec{X}) = 0$



ALSO -1 ON "BOTTOM"

$I(\vec{X}) = -2$

Theorem of Poincaré-Hopf

Assumptions as above.

$$I(\vec{X}) = 2 - 2g$$

Summary

$$\frac{1}{2\pi} \int_S K \, dA = I(\vec{X}) = 2 - 2g \quad (\text{INTEGER})$$

$\frac{1}{2\pi} \int_S K \, dA$: calculated from any geometry on S

$I(\vec{X})$: calculated from any vector field on S (analysis)

$2 - 2g$: calculated from the topology of S