The Maximum Upper Density of a Set of Positive Real Numbers with no solutions to $x + y = kz$

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January 11, 1996

Abstract

If k is a positive real number, we say that a set S of real numbers is k-sum-free if there do not exist x, y, z in S such that $x + y = kz$. For $k \geq 4$ we find the maximum upper density of a k-sum-free subset of the set of positive real numbers. We also show that if k is an integer greater than 3 then the set of positive real numbers and the set of positive integers are each the union of three (but not two) k-sum-free sets.

1. Introduction

We say that a set S of real numbers is sum-free if there do not exist x, y, z in S such that $x + y = z$. If k is a positive real number, we say that a set S is k-sum-free if there do not exist x, y, z in S such that $x + y = kz$ (we assume not all x, y, and z are equal to each other to avoid a triviality when $k = 2$). Many problem in group theory and number theory focus on sum-free sets. In work related to Fermat's Last Theorem, Schur [Sc] proved that the positive integers cannot be partitioned into finitely many sum-free sets. Van der Warden [W] proved that the positive integers cannot be partitioned into finitely many 2-sum-free sets.

If S is a subset of the positive integers we define the upper density $\delta_U(S)$ and lower density $\delta_L(S)$ of S to be the limit superior and limit inferior respectively of $\left\{\frac{|S \cap \{1,2,\ldots,n\}|}{n} \mid n \in Z^+\right\}$. If k is a positive integer let $U(k)$ and $L(k)$ denote the supremum of $\delta_U(S)$ and $\delta_L(S)$ respectively over all k-sum-free subsets S of the positive integers. Let $f(n,k)$ be the maximum size of a k-sum-free subset of $\{1, 2, \ldots, n\}$ and let $G(k)$ denote the limit superior over the positive integers of $\frac{f(n,k)}{n}$. For any k we clearly have the relationship $L(k) \leq U(k) \leq G(k)$. Since the odd positive integers are sum-free, $L(1) \geq \frac{1}{2}$. It is easy to show that $G(1) \leq \frac{1}{2}$, so

$L(1) = U(1) = G(1) = \frac{1}{2}$. Roth [Ro] showed that $G(2) = 0$. His results were strengthened by Szemerédi [Sz], Salem and Spencer [SS], and Heath-Brown [H].

If k is a positive integer and S is a k-sum-free subset of the positive integers with $x \in S$ and $y \in S \cap \{1, 2, \ldots, kx\}$, then $kx - y \notin S$, so $L(k) \le \frac{1}{2}$. If k is odd then, since the odd integers are then k-sum-free, $L(k) = \frac{1}{2}$. If $k \neq 2$ is even, then the set of all positive integers whose mod k congruence class is in $\left\{1, 2, \ldots, \frac{k-2}{2}\right\}$ is k-sum-free. Hence $\frac{1}{2} - \frac{1}{k} \le L(k) \le \frac{1}{2}$ for even k.

Chung and Goldwasser [CG1] showed that if $n \ge 23$ then the set of all odd positive integers less than or equal to n is the unique maximum 3-sum-free subset of $\{1, 2, \ldots, n\}$. Hence $L(3) = U(3) = G(3) = \frac{1}{2}$.

The above density functions have analogs over k-sum-free subsets of the positive real numbers where k is any positive real number. If S is a (Lebesgue) measurable k-sum-free subset of the positive real numbers, we define the upper density $\delta_u(S)$ and lower density $\delta_l(S)$ of S to be the limit superior and limit inferior respectively of $\left\{\frac{\mu(S \cap (0,x])}{x} \mid x \in \mathbb{R}^+\right\}$ where μ denotes measure. Let $u(k)$ and $l(k)$ denote the least upper bound of $\delta_u(S)$ and $\delta_l(S)$ respectively over all measurable k-sum-free subsets S of the positive real numbers. Let $g(k)$ denote the maximum size of a measurable k-sum-free subset of $(0, 1]$. Clearly we have $l(k) \le u(k) \le g(k)$ for any positive real number k. It is obvious that $g(1) = \frac{1}{2}$ and can be shown that $g(2) = 0$. Chung and Goldwasser [CG2] found $g(k)$ for all $k \ge 4$ and showed that there is an essentially unique maximum set, the union of three intervals:

$$(e_1, f_1] \cup (e_2, f_2] \cup (e_3, f_3]$$

where

$$f_1 = \frac{4}{k^4 - 2k^2 - 4} \qquad f_2 = \frac{2(k^2 - 2)}{k^4 - 2k^2 - 4} \qquad f_3 = 1 \tag{1.1}$$

and $e_i = \frac{2}{k} f_i \quad (i = 1, 2, 3)$.

In this paper we will find $u(k)$ for $k \ge 4$. We will generalize a result of Rado [R, R1] by showing that for any positive integer k greater than 3 the positive real numbers and the positive integers are each the union of three (but not two) k-sum-free sets and that the positive real numbers and the positive integers are each the union of four (but not three) 3-sum-free sets.

2. Maximum upper density of a k-sum-free set

Lemma 1. *Suppose $k \geq 4$ is a real number, c and w are positive real numbers with $c \leq \frac{2}{k^2}w$, and S is a measurable k-sum-free subset of the positive real numbers which contains c. Then*

$$\mu\left(S \cap \left(\frac{2}{k^2}w + \frac{c}{k}, w\right)\right) \leq \left(1 - \frac{2}{k}\right) w \ ,$$

with equality if and only if

$$\mu\left(S \cap \left(\frac{2}{k}w, w\right)\right) = \left(1 - \frac{2}{k}\right) w \ .$$

Proof. Let S be a set satisfying the hypotheses and suppose $S \cap \left(\frac{1}{k}w, \frac{2}{k}w\right) \neq \phi$. If $x \in S \cap \left(\frac{1}{k}w, \frac{2}{k}w\right)$ then $0 < kx - w < w$ and there is a "forbidden pairing" with respect to x of $[kx - w, w]$: if $z \in S \cap [kx - w, w]$ then $kx - z \in [kx - w, w]$ but $kx - z \notin S$. Hence

$$\mu(S \cap [kx - w, w]) \leq \frac{1}{2}[w - (kx - w)] \tag{2.1}$$

and if y is the infemum of $\left\{\frac{x}{w} \mid x \in S \cap \left(\frac{1}{k}w, \frac{2}{k}w\right)\right\}$ then

$$\mu(S \cap [kyw - w, w]) \leq \frac{1}{2}[w - (kyw - w)] \ . \tag{2.2}$$

If there exists $x \in S \cap \left(\frac{1}{k}w, \frac{2}{k}w\right)$ such that $kx - w \leq \frac{2}{k^2}w + \frac{c}{k}$ then, letting $v = \mu\left(S \cap \left(\frac{2}{k^2}w + \frac{c}{k}, w\right)\right)$, by (2.1)

$$v \leq \frac{1}{2}[w - (kx - w)] < \frac{1}{2}w \leq \left(1 - \frac{2}{k}\right) w \ ,$$

so the conclusion of the Lemma holds if $kyw - w < \frac{2}{k^2}w + \frac{c}{k}$. If $kyw - w \geq \frac{2}{k^2}w + \frac{c}{k}$, there are three cases to consider.

Case (i). Suppose $\frac{1}{k} \leq y \leq \frac{1}{k-1}$. Then $kyw - w \leq yw$ and, since $S \cap \left(\frac{1}{k}w, yw\right) = \phi$,

$$\begin{aligned} v &= \mu\left(\cap\left(\frac{2}{k^2}w + \frac{c}{k}, \min\left\{\frac{1}{k}w, kyw - w\right\}\right)\right) + \mu(S \cap (kyw - w, w)) \\ &\leq \min\left\{\frac{1}{k}w, kyw - w\right\} - \left(\frac{2}{k^2}w + \frac{c}{k}\right) + \frac{1}{2}[w - (kyw - w)] \ . \end{aligned} \tag{2.3}$$

The right-hand side of (2.3) is clearly a maximum when $kyw - w = \frac{1}{k}w$, so

$$\begin{aligned} v &\leq \frac{1}{k}w - \left(\frac{2}{k^2}w + \frac{c}{k}\right) + \frac{1}{2}\left[w - \frac{1}{k}w\right] = \frac{k^2 + k - 4}{2k^2}w - \frac{c}{k} \\ &\leq \frac{k^2 + k - 4 + (k-4)(k-1)}{2k^2}w - \frac{c}{k} = \left(1 - \frac{2}{k}\right) w - \frac{c}{k} \\ &< \left(1 - \frac{2}{k}\right) w \ . \end{aligned}$$

Case (ii). Suppose $\frac{1}{k-1} < y \leq \frac{1}{k} + \frac{2}{k^2}$. Then $y < ky - 1$ and

$$\begin{aligned} v &= \mu\left(S \cap \left(\frac{2}{k^2}w + \frac{c}{k}, \frac{1}{k}w\right)\right) + \mu(S \cap (yw, kyw - w)) + \mu(S \cap (kyw - w, w)) \\ &\leq \frac{1}{k}w - \left(\frac{2}{k^2}w + \frac{c}{k}\right) + (kyw - w) - yw + \frac{1}{2}[w - (kyw - w)] \\ &= \left(\frac{k}{2} - 1\right)\left(y + \frac{2}{k^2}\right)w - \frac{c}{k} \leq \left(\frac{k-2}{2}\right)\left(\frac{1}{k} + \frac{2}{k^2} + \frac{2}{k^2}\right)w - \frac{c}{k} \\ &= \frac{k-2}{k}\ \frac{k+4}{2k}w - \frac{c}{k} \leq \frac{k-2}{k}w - \frac{c}{k} < \left(1 - \frac{2}{k}\right)w\ . \end{aligned}$$

Case (iii). Suppose $y > \frac{1}{k} + \frac{2}{k^2}$. Then $ky - 1 > \frac{2}{k}$ and $v = a + b + d + e$ where $a = \mu\left(S \cap \left(\frac{2}{k^2}w + \frac{c}{k}, \frac{1}{k}w\right)\right)$, $b = \mu\left(S \cap \left(yw, \frac{2}{k}w\right)\right)$, $d = \mu\left(S \cap \left(\frac{2}{k}w, w - c\right)\right)$, and $e = \mu(S \cap (w - c, w))$. Then $b \leq \left(\frac{2}{k} - y\right)w$ and by (2.2).

$$\begin{aligned} d + e &\leq \left(1 - \frac{2}{k}\right)w - \frac{1}{2}[w - (kyw - w)] \\ &= \left(1 - \frac{2}{k}\right)w - \frac{k}{2}\left(\frac{2}{k} - y\right)w \leq \left(1 - \frac{2}{k}\right)w - \frac{k}{2}b\ . \end{aligned}$$

Hence

$$\frac{k}{2}b +_d +e \leq \left(1 - \frac{2}{k}\right)w\ . \tag{2.4}$$

Since $c \in S$, if $x \in S \cap \left(\frac{2}{k^2}w + \frac{c}{k}, \frac{1}{k}w\right)$ then $\frac{2}{k}w < kx - c < w - c$ but $kx - c \notin S$. By this forbidden pairing,

$$ka + d \leq w - c - \frac{2}{k}w\ . \tag{2.5}$$

Thus we have

$$\begin{aligned} v &\leq \left(\frac{k}{2}a + \frac{d}{2}\right) + \left(\frac{k}{4}b + \frac{d}{2} + \frac{e}{2}\right) + \frac{e}{2} \\ &\leq \frac{1}{2}\left[\left(1 - \frac{2}{k}\right)w - c\right] + \frac{1}{2}\left(1 - \frac{2}{k}\right)w + \frac{c}{2} = \left(1 - \frac{2}{k}\right)w \end{aligned} \tag{2.6}$$

by (2.4), (2.5), and the fact that $e \leq c$. For equality to hold in the Lemma, we must be in Case (iii) and (2.4), (2.5), and (2.6) all must be equalities, which completes the proof.

Theorem 1. *If $k \geq 4$ is a positive real number and S is a measurable k-sum-free subset of the positive real numbers, then the upper density of S is at most $\frac{k^2-2k}{k^2-2}$.*

Proof. Let S be a measurable k-sum-free subset of the positive real numbers containing c and z where $c \leq \left(\frac{2}{k^2}\right)z$. Let m be the largest positive integer such that $\left(\frac{2}{k^2}\right)^m z \geq c$, that is let

$$m = \left\lfloor \frac{\log \frac{c}{z}}{\log \frac{2}{k^2}} \right\rfloor\ ,$$

and let $w_i = \left(\frac{2}{k^2}\right)^i z$ for $i = 0, 1, \ldots, m$. By Lemma 1,

$$\mu\left(S \cap \left(\frac{2}{k^2} w_i, w_i\right)\right) \le \left(1 - \frac{2}{k}\right) w_i + \frac{c}{k} \quad \text{for} \quad i = 0, 1, \ldots, m-1 \; .$$

Hence

$$\begin{aligned} \mu(S \cap (0, z]) &= \mu(S \cap (0, w_m)) + \mu\left(S \cap \bigcup_{i=0}^{m-1} \left(\frac{2}{k^2} w_i, w_i\right)\right) \\ &\le w_m + z\left(1 - \frac{2}{k}\right) \sum_{i=0}^{m-1} \left(\frac{2}{k^2}\right)^i + m\frac{c}{k} \end{aligned}$$

and

$$\frac{\mu(S \cap (0, z])}{z} \le \frac{k^2}{2}\frac{c}{z} + \left[1 - \left(\frac{2}{k^2}\right)^m\right] \frac{k^2 - 2k}{k^2 - 2} + \frac{m}{k}\frac{c}{z} \; .$$

Taking the limit as z goes to infinity (so m goes to infinity since c is fixed) then gives the result.

The set

$$T_k(\infty) = \bigcup_{i \in Z} \left(\frac{2}{k}\left(\frac{2}{k^2}\right)^i, \left(\frac{2}{k^2}\right)^i\right] \tag{2.7}$$

has upper density $\frac{k^2-2k}{k^2-2}$ for $k > 2$ so the bound in Theorem 1 is best possible. Hence for any real number k greater than 2, $g(k) \ge \frac{k^2-2k}{k^2-2} + \frac{8(k-2)}{k(k^2-2)(k^4-2k^2-4)}$ (the three intervals defined in equations (1.1)), $u(k) \ge \frac{k^2-2k}{k^2-2}$, and these are both equalities if $k \ge 4$.

These constructions can be used to produce k-sum-free subsets of the positive integers as well. Let k be a positive integer greater than 2 and let J_k be the union of the three intervals defined in equations (1.1). Define a subset $H_k(n)$ of $\{1, 2, \ldots, n\}$ by $H_k(n) = \{1, 2, \ldots, n\} \cap \{nx \mid x \in J_k\}$ and a subset $H_k(\infty)$ of the positive integers Z by $H_k(\infty) = Z \cap T_k(\infty)$. Then $\lim_{n\to\infty} \frac{|H_k(n)|}{n} = \mu(J_k)$ and $\delta_U(H_k(\infty)) = \frac{k^2-2k}{k^2-2}$, so $G(k) \ge \mu(J_k)$ and $U(k) \ge \frac{k^2-2k}{k^2-2}$. We conjecture that these are both equalities if k is an integer greater than 3 (not for $k = 3$ because these values are then both less than $\frac{1}{2}$ and the odd integers give values equal to $\frac{1}{2}$). Theorem 1 does not apply for $2 < k < 4$, but we suspect $U(k) = \frac{k^2-2k}{k^2-2}$ for these values of k as well. Chung and Goldwasser discuss some conjectures about $g(k)$ for $2 < k < 4$ in [CG2].

3. The positive integers and real numbers as the union of k-sum-free sets

In contrast to the situation for $k = 1$ and $k = 2$, if k is an integer greater than 2 then the positive real numbers and the positive integers are each the union of finitely many k-sum-free sets.

Theorem 2. *If k is an integer greater than or equal to 4 then the positive integers and the positive real numbers are each the union of three k-sum-free sets, but not of two. The positive integers and the positive real numbers are each the union of four 3-sum-free sets, but not of three.*

Proof. First we show that for any positive integer k, the set of all positive integers, and hence the set of all positive real numbers, is not the union of two k-sum-free sets. Suppose A and B are k-sum-free sets whose union is the positive integers. Let x be an integer such that $kx \in A$ and $k(x+1) \in B$ (such an integer exists because the multiples of k greater than any fixed number is not a k-sum-free set) and let y be an integer greater than x such that $y \in A$ and $y+1 \in B$. Then $k(y-x) = ky - kx = k(y+1) - k(x+1)$ cannot be in either A or B.

If x is any positive real number and $k \geq 3$ we define the k-sum-free set $T_k(x, \infty)$ by

$$T_k(x, \infty) = \{xy \mid y \in T_k(\infty)\} .$$

where $T_k(\infty)$ is defined in (2.7). We note that $T_k\left(\frac{k^2}{2}, \infty\right) = T_k(\infty)$. If $k \geq 4$ then $T_k(\infty) \cup T_k\left(\frac{k}{2}, \infty\right) \cup T_k\left(\frac{k^2}{4}, \infty\right)$ is the set of all positive real numbers because $\frac{k^2}{4} \geq k$ (if $k = 4$ the three sets are actually disjoint). Thus for $k \geq 4$ the positive reals, and hence the positive integers, are the union of three k-sum-free sets. If $k = 3$ the above union of three sets does not cover the positive real numbers (because $\frac{3^2}{4} < 3$). However, if we add $T_k\left(\frac{k^3}{8}, \infty\right)$ to the union, then the four sets do cover the positive reals (because $\frac{3^3}{8} > 3$). To complete the proof we need only show that the set of positive integers is not the union of three 3-sum-free sets. This could be done by a direct argument with many cases. Instead we chose to do it by computer. It turns out that 53 is the largest integer n such that hte set of positive integers less than or equal to n is the union of three 3-sum-free sets. One such partition is

$$\begin{aligned} A &= \{1, 3, 4, 7, 10, 12, 13, 16, 19, 21, 22, 25, 28, 30, 31, 34, 37, 39, 40, 43, 46, 48, 49, 52\}, \\ B &= \{2, 5, 6, 15, 18, 24, 33, 35, 38, 41, 42, 44, 45, 47, 50, 51, 53\}, \\ C &= \{8, 9, 11, 14, 17, 20, 23, 26, 27, 29, 32, 36\} . \end{aligned}$$

4. Open problems

The set $\bigcup_{i=1}^{\infty} [3i-2, 3i-1)$ is a sum-free subset of the positive reals with upper and lower density equal to $\frac{1}{3}$. Hence $\frac{1}{3} \leq l(1) \leq u(1)$.

Conjecture 1. $l(1) = u(1) = \frac{1}{3}$.

We restate here the conjectures of Section 2:

Conjecture 2. *$G(k) = g(k)$ and $U(k) = u(k)$ for $k \geq 4$.*

Conjecture 3. *$u(k) = \frac{k^2-2k}{k^2-2}$ for $k > 2$.*

The bound $\frac{1}{2} - \frac{1}{k} \leq L(k)$ for the maximum lower density of a k-sum-free subset of the positive itnegers for even $k \neq 2$ is not best possible. One can improve it slightly by considering integers $(\mathrm{mod}\ k^t)$ for various positive integral values of t, but we do not have a conjecture as to the actual value of $L(k)$ for even integers greater than 4. The positive integers congruent to 2 or 3 (mod 5) (or congruent to 1 or 4 (mod 5)) are 4-sum-free, so $L(4) \geq \frac{2}{5}$.

Conjecture 4. *$L(4) = \frac{2}{5}$.*

The set $T_k(\infty)$ has lower density $\frac{k-2}{k^2-2}$ for $k \geq 3$, which is $\frac{1}{k}$ times its upper density. One can obtain a k-sum-free set S of the real numbers for $k \geq 3$ by uniformly "fattening up" the odd integral points as much as possible and translating the resulting set. The set $\bigcup_{i=0}^{\infty} \left(\frac{2}{k-2} + (k+2)i,\ \frac{2}{k-2} + (k+2)i + 1\right)$ has lower (and upper) density $\frac{1}{k+2}$ (if $k \geq 3$), which is a little more than $\frac{k-2}{k^2-2}$.

Conjecture 5. *$l(k) = \frac{1}{k+2}$ for $k \neq 2$.*

Conjecture 6. *If $k < 4$ the positive real numbers are not the union of three k-sum-free sets.*

Of course Theorem 2 shows Conjecture 6 is true for $k \leq 3$.

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