114

Universal Graphs and Induced-Universal Graphs

Fan R. K. Chung
BELLCORE
MORRISTOWN, NEW JERSEY 07960

ABSTRACT

We construct graphs that contain all bounded-degree trees on *n* vertices as induced subgraphs and have only *cn* edges for some constant *c* depending only on the maximum degree. In general, we consider the problem of determining the graphs, so-called universal graphs (or induced-universal graphs), with as few vertices and edges as possible having the property that all graphs in a specified family are contained as subgraphs (or induced subgraphs). We obtain bounds for the size of universal and induced-universal graphs for many classes of graphs such as trees and planar graphs. These bounds are obtained by establishing relationships between the universal graphs and the induced-universal graphs.

1. Introduction

A fundamental problem in extremal graph theory is to find a "minimum" graph that "contains" all graphs in a specified family of graphs. By "minimum" we mean, for example, having the minimum possible number of vertices and/or edges, or in general, minimizing certain weight functions associated with the graphs. There are several different notions of "containment." We say a graph H is contained in G if the vertex set $V(H)$ of H is a subset of $V(G)$ and the edge set $E(H)$ of H is a subset of $E(G)$. We say a graph H is contained in G as an induced subgraph if $V(H) \subseteq V(G)$ and $E(H)$ consists exactly of all edges $\{u, v\}$ in $E(G)$ with $u, v \in V(H)$. Of course, there are other types of containment (such as homomorphic subgraphs) that we will not discuss here. For undefined terminology, the reader is referred to [1].

Let $\boldsymbol{F}$ denote a family of graphs. A graph G is said to be *$\boldsymbol{F}$-universal* if G contains all graphs in $\boldsymbol{F}$ as subgraphs. A graph G is said to be *$\boldsymbol{F}$-induced-universal* if it contains all graphs in $\boldsymbol{F}$ as induced subgraphs. The problem of

Journal of Graph Theory, Vol. 14, No. 4, 443–454 (1990)
 CCC 0364-9024/90/040443-12$04.00

interest is to find the minimum universal graphs for various classes of graphs. Many problems in extremal graph theory can be described in the context of universal graphs. In addition, many optimization problems arising in data representations [13], data structures [22], and circuit design [23] can be viewed as problems of determining certain universal graphs.

Let $f_v(\boldsymbol{F})$ denote the minimum number of vertices in an $\boldsymbol{F}$-universal graph. Analogously, let $f_e(\boldsymbol{F})$ denote the minimum number of edges in an $\boldsymbol{F}$-universal graph. Clearly, $f_v(\boldsymbol{F})$ is at least as large as the maximum number of vertices of the graphs in $\boldsymbol{F}$. Let $f(\boldsymbol{F})$ denote the minimum number of edges in an $\boldsymbol{F}$-universal graph on $f_v(\boldsymbol{F})$ vertices. Also, let $g_v(\boldsymbol{F})$ denote the minimum number of vertices in an $\boldsymbol{F}$-induced-universal graph and $g_e(\boldsymbol{F})$ denote the minimum number of edges in an $\boldsymbol{F}$-induced-universal graph. It can be easily seen that $f_e \leq f$, $f_v \leq g_v$ and $f_e \leq g_e$.

In the early 1960s, Rado investigated induced universal graphs for infinite graphs [21]. Since then many results have been obtained for universal graphs for various families of graphs such as cycles [5], trees [7–11,14] bounded-degree trees [3,4,15], caterpillars [12,17], graphs on n edges [2], planar graphs [2], and bounded-degree planar graphs [3,4]. Moon [19] considered induced universal graphs for the family of all graphs on n vertices. Recently Kannan, Noas, and Rudich [16] considered induced-universal graphs for trees on n vertices, planar graphs on n vertices, and n-vertex graphs with arboricity k.

In this paper we establish relations between the universal graphs and the induced-universal graphs. We then use these relations together with the following known results to improve the bounds for the sizes of the induced-universal graphs for trees and planar graphs:

(i) There is a graph with n vertices and $cn \log n$ edges for some constant c that contains the family $\boldsymbol{T}_n$ of all trees on n vertices. (This is within a constant factor of the minimum number of edges in a universal graph for T_n) (see [9]).

(ii) For the family $\boldsymbol{T}_{n,d}$ of all trees with n vertices with maximum degree d, there is a universal graph on n vertices with bounded degree (depending only on d) (see [3,4]).

(iii) There is a graph with n vertices and $cn^{3/2}$ edges for some constant c that contains the family $\boldsymbol{Pl}_n$ of all planar graphs on n vertices (see [2]).

(iv) There is a graph with cn vertices and $c'n \log n$ edges that contains the family $\boldsymbol{Pl}_{n,d}$ of all planar graphs on n vertices with maximum degree d where the constants c and c' depend only on d (see [3,4]).

(v) There is a graph with n^2 vertices and n^3 edges that contains all trees on n vertices as induced subgraphs and there is a graph with n^4 vertices and n^7 edges that contains all planar graphs on n vertices as induced subgraphs [16]. Both of these results follow from a more general result for the family $\boldsymbol{A}_{k,n}$ of graphs of arboricity k on n vertices (the edge set of which can be partitioned into k parts, but no fewer, each of which forms an acyclic graph). In [16] it was shown that there is a graph on n^{1+k} vertices and n^{1+2k} edges that contains all graphs in $\boldsymbol{A}_{k,n}$.

In this paper we prove the following:

(a) There is a graph with $cn \log n$ vertices and n^2 edges that contains all trees on n vertices as induced subgraphs for some constant c.
(b) For the family $T_{n,d}$ of all n-vertex trees with maximum degree d, there is a graph on cn vertices with bounded degree (depending only on d) that is an induced-universal graph for $T_{n,d}$.
(c) There is a graph with $c(n \log n)^3$ vertices and $c'n^6$ edges that contains all planar graphs on n vertices as induced subgraphs for some constants c and c'.
(d) There is a graph with cn^2 vertices and $c'n^{3.5}$ edges that contains all bounded-degree planar graphs on n vertices as induced subgraphs, where the constants c and c' depend only on the upper bound for the degrees.
(e) There is a graph with $c(n \log n)^k$ vertices and $c'n^{2k}$ edges for some constants c and c' that contains all n-vertex graphs of arboricity k as induced subgraphs.

In a paper by G. Chartrand, D. Geller, and S. Hedetniemi [6] it was conjectured that any planar graph can be edge-partitioned into two outerplanar graphs. If this conjecture is true, then our results (c) and (d) can be further improved by using the following results on universal graphs for outplanar graphs:

(vi) There is a graph on n vertices and $c'n \log n$ edges for some constant c that contains the family of all outerplanar graphs on n vertices [4].
(vii) There is a graph on n vertices with bounded degrees that contains the family $OP_{n,d}$ of all outerplanar graphs on n vertices with maximum degree d.

Both (vi) and (vii) are derived from the fact that an outplanar graph has small separators. That is, one can remove two vertices from an outplanar graph so that the vertices of the remaining graph can be partitioned into two parts A and B such that $|A| \leq |B| < 2|A|$ and there is no edge between A and B.

Using Theorem 1 together with (vi) and (vii), the following results can be shown:

(f) There is a graph on cn^2 vertices and $c'n^3$ edges for some constants c and c' that contains all outerplanar graphs on n vertices as induced subgraphs.
(g) There is a graph on cn vertices with maximum degree c' that contains as induced subgraphs all n-vertex outplanar graphs with maximum degree d, where c and c' depend only on d.

If the conjecture on outplanar graphs by Chartrand, Geller, and Hedetniemi holds, (d) can be improved further as follows:

(d′) There is a graph on cn^2 vertices and $c'n^3$ edges that contains all bounded-degree planar graphs on n vertices as induced subgraphs where c and c' depend only on the maximum degree.

2. RELATIONS BETWEEN UNIVERSAL GRAPHS AND INDUCED UNIVERSAL GRAPHS

Theorem 1. Let A_k denote a family of graphs with arboricity at most k. Let G be a universal graph for A_k. Then

$$g_v(A_k) \leq \sum_i (d_i + 1)^k$$

and

$$g_e(A_k) \leq \sum_{v_i \sim v_j} (d_i + 1)^k d_j^{k-1}$$

where d_i denotes the degree of the ith vertex in G and "$v_i \sim v_j$" denotes that v_i and v_j are adjacent.

Proof. An induced-universal graph H for A_k can be constructed from the universal graph G as follows:

The vertex set of H consists of all $(k + 1)$-tuples $(u_0, u_1, \ldots, u_k)$, where u_0 is a vertex of G and u_i, $i \neq 0$, either is $*$ (a special symbol) or is a neighbor of u_0. Furthermore, all $u_i \neq *$ are distinct. A vertex $(u_0, u_1, \ldots, u_k)$ is adjacent to $(u_0', u_1', \ldots, u_k')$ if $u_0 = u_i'$ or $u_0' = u_i$ for some $i \neq 0$. It is easy to check that H has at most $\Sigma_i (d_i + 1)^k$ vertices. The number of edges in H is no more than $\Sigma_{v_i \sim v_j} (d_i + 1)^k \cdot d_j^{k-1}$. To see that H is A_k-induced-universal, let F be a graph in A_k. Since G is A_k-universal, there is a one-to-one mapping α from $V(F)$ to $V(G)$ so that $\alpha(u)$ is adjacent to $\alpha(v)$ in G if u is adjacent to v in F. Since F is of arboricity k, we can partition the edges of F into k parts, each of which forms an acyclic graph. Let $F_1, \ldots, F_k$ denote these acyclic graphs. In each F_i, we orient the edges so that each connected component is an out-tree with a root. Now each vertex u in F is mapped into the vertex $(\alpha(u), \alpha(v_1), \ldots, \alpha(v_k))$ where v_i is the parent of u in F_i if u is not a root and $\alpha(v_i) = *$ otherwise. It is straightforward to verify that F is contained in H as an induced subgraph. This completes the proof of Theorem 1.

Corollary 1.1. Let A denote a family of acyclic graphs and let G be a universal graph for A. Then there is an induced-universal graph with $2|E(G)| + |V(G)|$ vertices and $\sum d_i^2$ edges where d_i denotes the degree of the ith vertex of G. In other words,

$$g_v(A) \leq 2|E(G)| + |V(G)|,$$

$$g_e(A) \leq \sum_i d_i^2.$$

Proof. We remark that the stronger bound $g_e(A) \leq \Sigma_i d_i^2$ can be established by a modification of the argument used to prove Theorem 1 when $k = 1$.

Corollary 1.2. Let A denote a family of acyclic graphs. Then

$$g_v(A) \leq 2f(A) + f_v(A).$$

In [16] complete subgraphs are used to construct induced-universal graphs for $A_{k,n}$ instead of the more efficient universal graphs that we used here. Using the techniques in [16] together with the relationship between universal graphs and induced-universal graphs, we can then prove the following result that sometimes gives better upper bounds than Theorem 1 (and sometimes worse).

Theorem 2. Let G be an induced-universal graph for a family $\boldsymbol{F}$ of graphs. Suppose every graph in the family $\boldsymbol{H}$ can be edge-partitioned into k parts, each of which forms a graph in $\boldsymbol{F}$. Then

$$g_v(\boldsymbol{H}) \leq |V(G)|^k$$

and

$$g_e(\boldsymbol{H}) \leq k|V(G)|^{2k-2} \cdot |E(G)|.$$

Proof. We construct an $\boldsymbol{H}$-induced-universal graph U as follows:

The vertex set of U consists of all k-tuples $(u_1, u_2, \ldots, u_k)$ where u_i's are (not necessarily different) vertices in G and the vertices $(u_1, u_2, \ldots, u_k)$ and $(u_1', u_2', \ldots, u_k')$ are adjacent in U if u_i and u_i' are adjacent in G for some i, $1 \leq i \leq k$.

To see that U is $\boldsymbol{H}$-induced-universal, we consider a graph H in $\boldsymbol{H}$. The edges of H can be edge-partitioned into k parts, which form graphs $F_1, F_2, \ldots, F_k$ in $\boldsymbol{F}$ and the vertex sets of F_i's are the same as $V(H)$. Let λ_i denote the mapping from $V(F_i)$ to $V(G)$ such that $\lambda_i(u)$ and $\lambda_i(v)$ are adjacent in G if and only if u and v are adjacent in F_i. We define the embedding λ that maps a vertex v in H to $(\lambda_1(v), \lambda_2(v), \ldots, \lambda_k(v))$ in U. Clearly $\lambda(u)$ and $\lambda(v)$ are adjacent in U if and only if u and v are adjacent in H. Therefore, U is $\boldsymbol{H}$-induced universal. The number of vertices of U is just $|V(G)|^k$. The vertices $(u_1, u_2, \ldots, u_i, \ldots, u_k)$ and $(u_1', u_2', \ldots, u_i', \ldots, u_k')$ are said to be i-adjacent in U if u_i and u_i' are adjacent in G. For each i, the number of i-adjacent edges is no more than $|V(G)|^{2k-2}|E(G)|$ since for each edge $\{u_i, u_i'\}$ of G, at most $|V(G)|^{2k-2}$ possible pairs of vertices in U contain u_i, u_i' as the ith coordinates, respectively. Hence, the total number of edges in U is no more than $k|V(G)|^{2k-2}|E(G)|$. This completes the proof of Theorem 2.

III. INDUCED-UNIVERSAL GRAPHS FOR VARIOUS FAMILIES OF GRAPHS

In this section, we will use Theorems 1 and 2 together with known constructions for universal graphs to derive bounds for induced-universal graphs for various families of graphs.Throughout this section, c, c', and the c_i's denote some generic constants that might not have the same value from one appearance to another.

Theorem 3. Let T_n denote the family of all trees on n vertices. Then

$$c_1 n \leq g_v(T_n) \leq c_2 n \log n$$

and

$$c_3 n \log n \leq g_e(T_n) \leq c_4 n^2.$$

Proof. The lower bounds follow from (i) since $g_v \geq f_v$ and $g_e \geq f_e$.

We will use the following constructions of universal graphs which contain all trees on n vertices (see [7,9]).

First we consider a graph $G^{(k)}$ on $2^{k+1} - 1$ vertices, which are named by (0,1)-strings of length no more than k. (There is one vertex, denoted by $*$, which can be viewed as having length 0). For $i \geq 0$, we say $a_1 a_2 \cdots a_t \cdots a_{t+i}$ is a descendent of $a_1 a_2 \cdots a_t$. The edges of $G^{(k)}$ can be described recursively. The edges among all vertices of the form $0a_2 \cdots a_k$ are as in $G^{(k-1)}$; so are the edges among all vertices of the form $1a_2 \cdots a_k$. In addition for every t, $t \geq 1$, there are edges from any

$$1\,\overbrace{00\ldots00}^{t} \quad \text{and} \quad 11\,\overbrace{00\ldots0}^{t-1} \quad \text{to} \quad 0\,\overbrace{11\ldots1}^{t-1}1b_1 \cdots b_r$$

for all choices of b_i's and there are edges from $*$ and 1 to all vertices.

An induced subgraph G of $G^{(k)}$ is said to be admissible if for any vertex $v = a_1 \cdots a_t 1$ in H all descendents of $a_1 \cdots a_t 0$ are in G. In [7] it was proved that all admissible subgraphs G of $G^{(k)}$ are universal for the family of trees with $|V(G)|$ vertices.

Since $|V(G)| = n$ and G has $cn \log n$ edges, using Corollary 1.1 we get $g_v(T_n) \leq cn \log n$.

Let $h(n)$ denote $\Sigma_i d_i^2$ where d_i, $i = 1, \ldots, n$, denotes the degree of the ith vertex in G. We want to show that $h(n) \leq cn^2$. From the preceding construction and the fact that a vertex labeled by a string of length i has degree at most 2^{k-i+2}, we have

$$h(2^{k+1}) \leq \sum_{i=0}^{k} 2^{2k-i+4} < 2^{2k+5}$$

For any n with $2^{k+1} < n \leq 2^k$, we have

$$h(n) \leq h(2^{k+1}) \leq cn^2.$$

Since $h(n)$ is an upper bound for $g_e(T_n)$, we have $g_e(T_n) \leq cn^2$.

Theorem 4. Let $T_{n,d}$ denote the family of all trees on n vertices with maximum degree d. Then

$$c_1 n \leq g_v(T_{n,d}) \leq c_2 n$$

and

$$c_3 n \leq g_e(T_{n,d}) \leq c_4 n .$$

Proof. Since there is a graph on cn edges with bounded-degrees which contain all trees in $T_{n,d}$, by Corollary 1.1 we have

$$g_v(T_{n,d}) \leq cn$$

and

$$g_e(F_{n,d}) \leq c'n .$$

The lower bounds are immediate.

Theorem 5. Let F_n denote the family of all forests on n vertices. Then

$$c_1 n \leq g_v(F_n) \leq c_2 n \log n$$

and

$$c_3 \log n \leq g_e(F_n) \leq c_4 n^2.$$

Proof. It is easy to see that a universal graph for T_n is also a universal graph for F_n and vice versa. Therefore using Theorem 1, we can establish the desired inequalities as in the proof of Theorem 5. Similarly, we have the following:

Theorem 6. Let $F_{n,d}$ denote the family of all forests on n vertices with maximum degree d. Then

$$c_1 n \leq g_v(F_{n,d}) \leq c_2 n$$

and

$$c_3 n \leq g_e(F_{n,d}) \leq c_4 n .$$

Theorem 7. Let Pl_n denote the family of all planar graphs on n vertices. Then for some constants c and c'

$$cn \leq g_v(Pl_n) \leq c'(n \log n)^3$$

and

$$cn \log n \leq g_e(Pl_n) \leq c'n^6.$$

Proof. The lower bounds follow readily from Theorem 3. We recall that a planar graph has arboricity at most 3 [20]. Thus it follows from Theorem 2 and 5 that $g_v(Pl_n) \leq c(n \log n)^3$ and that $g_e(Pl_n) \leq cn^6 \log^4 n$. To obtain a better upper bound for $g_e(Pl_n)$ we proceed as follows:

The universal graph G for Pl_n consists of two copies of the universal graph for $Pl_{\lceil n/2 \rceil}$ together with a clique of size $c\sqrt{n}$; every vertex in the clique is adjacent to all other vertices in G. It is easily checked (see [2]) that G is Pl_n-universal since a planar graph has a separator of size $c\sqrt{n}$ [18]. Theorem 1 states $g_e(Pl_n) \leq \sum_{v_i \sim v_j} (d_i + 1)^3 d_j^2$, where d_i denotes the degree of the ith vertex in G.

Now, we go back to the Pl_n-universal graph G, which has n vertices and $cn^{3/2}$ edges. The vertices of G can be viewed as being partitioned into clusters, which can then be grouped into levels. For $i = 0, 1, \ldots, \log_2 n$, the ith level consists of 2^i clusters each of size at most $2c\sqrt{n/2^i}$ in which each vertex has degree at most $cn/2^i$ in G.

Therefore

$$\begin{aligned} g_e(Pl_n) &\leq \sum_{v_i \sim v_j} (d_i + 1)^3 d_j^2 \\ &\leq c' \sum_i 2^i (2n^{1/2} 2^{-i/2}) (n \cdot 2^{-i})^3 \sum_{j \geq i} 2^{j-i} (n^{1/2} 2^{-j/2}) (n \cdot 2^{-j})^2 \\ &\leq c'n^6. \end{aligned}$$

Theorem 8. Let $Pl_{n,d}$ denote the family of all planar graphs on n vertices with maximum degree d. Then

$$cn \leq g_v(Pl_{n,d}) \leq c'n^2$$

and

$$cn \leq g_e(Pl_{n,d}) \leq c'n^{3.5}.$$

Proof. The lower bounds are trivial and we will only consider the upper bounds. We need the following construction of universal graphs H for $Pl_{n,d}$ (see [4]). The vertices of H can be partittioned into clusters, which can then be paritioned into levels. For $i = 0, 1, \ldots, \log_2 n$, the ith level consists of 2^i

clusters each of size at most $2c\sqrt{n/2^i}$ (where c is the constant with the property that every planar graph on m vertices has a separator of size at most $c\sqrt{m}$). The clusters on the ith level can be labelled by a binary string of length i. Every vertex in a cluster, labeled by w, is adjacent to every vertex in a cluster w' if w' is of the form xw where x is a string of length $2\log_2(dc)$. Each vertex in a cluster of the ith level has degree at most $n^{1/2}2^{-i/2}dc$. Therefore it follows from Theorem 1 that

$$\begin{aligned} g_v(Pl_{n,d}) &\le \sum_j (d_j + 1)^3 \\ &\le \sum_i 2^i(2cn^{1/2}2^{-i/2})(n^{1/2}2^{-i/2}dc)^3 \\ &\le c'n^2. \end{aligned}$$

And

$$\begin{aligned} g_e(Pl_{n,d}) &\le \sum_{v_i \sim v_j} (d_i + 1)^3(d_j + 1)^2 \\ &\le \sum_{v_i \sim v_j} (d_i + 1)^3 \max((d_i + 1)^2, (d_j + 1)^2) \\ &\le 2\sum_i (d_i + 1)^6 \\ &\le 2\sum 2^i(2cn^{1/2}2^{-i/2})(n^{1/2}2^{-i/2}dc)^6 \\ &\le c''n^{3.5}. \end{aligned}$$

Theorem 9. Let $A_{k,n}$ denote the family of n-vertex graphs with arboricity k. Then

$$g_v(A_{k,n}) \le c(n \log n)^k$$

and

$$g_e(A_{k,n}) \le cn^{2k}.$$

The proof is very similar to the preceding proofs and will be omitted here.

Theorem 10. Let OP_n denote the family of all outerplanar graphs on n vertices. Then

$$cn \le g_v(OP_n) \le c'n^2$$

and

$$n \log n \le g_e(OP_n) \le c'n^3.$$

Proof. This follows from the known fact (vi) and Theorem 1 together with the fact that any outplanar graph has arboricity 2.

Theorem 11. Let $OP_{n,d}$ denote the family of all outerplanar graphs on n vertices with maximum degree d. Then we have

$$cn \le g_v(OP_{n,d}) \le c'n$$

and

$$cn \le g_e(OP_{n,d}) \le c'n\,.$$

Proof. This follows from the known fact (vii) and Theorem 1.

If the conjecture of Chartrand, Geller, and Hedetniemi holds, that is, every planar graph can be edge-partitioned into two outerplanar graphs, we can then use Theorem 11 together with Theorem 2 to deduce that there exists an induced-universal graph for bounded degree planar graphs with cn^2 vertices and $c'n^3$ edges.

4. CONCLUDING REMARKS

As we can see most of the results in this paper still have considerable gaps between the upper and lower bounds for the induced-universal graphs (with the exception of the induced-universal graphs for bounded-degree trees). For example we showed that there are T_n-induced-universal graphs with $cn \log n$ vertices and such universal graphs must have at least $c'n$ vertices. That is,

$$c'n \le g_v(T_n) \le cn \log n\,.$$

What is the correct order of magnitude for $g_v(T_n)$? For universal graphs of T_n, we used the degree requirements to establish lower bounds. However, it seems that the degree considerations are not enough to generate good lower bounds for $g_v(T_n)$ as evidenced by the following example:

Let F denote a family of star forests F_i, i.e., F_i is a vertex-disjoint union of $\lfloor n/i \rfloor$ copies of stars on i vertices. How large must an F-induced universal graph be? We will construct an F-induced universal graph H having at most $4n$ vertices.

The vertices of H can be partitioned into two parts A and B, each with at most $2n$ vertices. A can be further partitioned into $A_1, A_2, \ldots, A_{\lfloor \log_2 n \rfloor}$ where $|A_i| = \lfloor n/2^i \rfloor$. The induced subgraph on $A_i \cup B$ is a star forest consisting of

$\lfloor n/2^i \rfloor$ copies of stars each with 2^{i+1} edges. The centers of the stars are in A_i and the leaves are in B.

It is easy to see that the induced subgraph of H on $A_i \cup B$ contains F_k with $2^i \leq k < 2^{i+1}$ as an induced subgraph. Therefore H contains all F_k as induced subgraphs. H is $\boldsymbol{F}$-induced universal and has at most $4n$ vertices.

It would be of particular interest to sharpen the bounds for $g_v(\boldsymbol{T}_n)$ and $g_v(\boldsymbol{Pl}_n)$.

References

[1] C. Berge, *Graphs and Hypergraphs*. North-Holland, Amsterdam (1973).

[2] L. Babai, F. R. K. Chung, P. Erdös, R. L. Graham, and J. Spencer, On graphs which contain all sparse graphs. *Ann. Discrete Math.* **12** (1982) 21–26.

[3] S. N. Bhatt, F. R. K. Chung, T. Leighton, and A. L. Rosenberg, Optimal simulations of tree machines. *27th Annual Symposium on Foundations of Computer Science* (1986) 274–282.

[4] S. N. Bhatt, F. R. K. Chung, T. Leighton, and A. L. Rosenberg, Universal graphs for bounded-degree trees and planar graphs. *SIAM J. Discrete Math* **2** (1989) 145–155.

[5] J. A. Bondy, Pancyclic graphs. *J. Combinat. Theory B* **11** (1971) 80–84.

[6] G. Chartrand, D. P. Geller, and S. Hedetniemi, Graphs with forbidden subgraphs. *J. Combinat. Theory* **10** (1971) 12–41.

[7] F. R. K. Chung and R. L. Graham, On graphs which contain all small trees, *J. Combinat. Theory B* **24** (1978) 14–23.

[8] F. R. K. Chung, R. L. Graham, and N. Pippenger, On graphs which contain all small trees II. *Colloquia Mathematica*. Societatis János Bolyai, Keszthely, Hungary (1976) 213–223.

[9] F. R. K. Chung and R. L. Graham, On universal graphs for spanning trees. *J. London Math. Soc.* **27** (1983) 203–211.

[10] F. R. K. Chung and R. L. Graham, On universal graphs. *Ann. Acad. Sci.* **319** (1979) 136–140.

[11] F. R. K. Chung, R. L. Graham, and D. Coppersmith, On trees which contain all small trees. *The Theory and Applications of Graphs*. John Wiley & Sons, New York (1981) 265–272.

[12] F. R. K. Chung, R. L. Graham, and J. Shearer, Universal caterpillars. *J. Combinat. Theory B* **31** (1981) 348–355.

[13] F. R. K. Chung, A. L. Rosenberg, and L. Snyder, Perfect storage representations for families of data structures. *SIAD 4* (1983) 548–565.

[14] P. C. Fishburn, Minimum graphs that contain all small trees. *Ars Combinat.* **25** (1985) 133–165.

[15] J. Friedman and N. Pippenger, Expanding graphs contain all small trees. *Combinatorica* **7** (1987) 71–76.

[16] S. Kannan, M. Naos, and S. Rudich, Implicit representation of graphs. *Proceedings of the Twentieth Annual ACM Symposium on Theory of Computing* (1988) 334–343.

[17] R. J. Kimble and A. J. Schwenk, On universal caterpillars. *The Theory and Applications of Graphs*. John Wiley & Sons, New York (1981) 437–447.
[18] R. J. Lipton and R. E. Tarjan, A separator theorm for planar graphs. *SIAM J. Appl. Math.* **36** (1979) 177–189.
[19] J. W. Moon, On minimal *n*-universal graphs. *Proc. Glasgow Math. Soc.* **7** (1965) 32–33.
[20] C. St. J. A. Nash-Williams, Edge-disjoint spanning trees of finite graphs. *J. London Math. Soc.* **36** (1961) 445–450.
[21] R. Rado, Universal graphs and universal functions. *Acta Arith.* **9** (1964) 331–340.
[22] A. L. Rosenberg, L. J. Stockmeyer, and L. Snyder, Uniform data encodings. *Theore. Comput. Sci.* **II** (1980) 145–165.
[23] L. G. Valiant, Universality considerations in VLSI circuits. *IEEE Trans. Comput. C-30* (1981) 135–140.