Quasi-random graphs with given degree sequences

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Abstract

It is now known that many properties of the objects in certain combinatorial structures are equivalent, in the sense that any object possessing *any* of the properties must of necessity possess them all. These properties, termed *quasirandom*, have been described for a variety of structures such as graphs, hypergraphs, tournaments, Boolean functions, and subsets of $\mathbf{Z}_n$, and most recently, sparse graphs. In this paper, we extend these ideas to the more complex case of graphs which have a given degree sequence.

1 Introduction

During recent years there has been increasing interest in investigating the following phenomenon. For a given finite collection C of "objects", suppose we have some probability distribution given on C. Typically, there are many properties which are satisfied by most (or almost all) of the objects in C as seen in [4]. It turns out, however, that in many cases there is a large subclass Q of these properties which are strongly correlated, in the sense that *any* object in C which satisfies *any* of the properties in Q must in fact necessarily satisfy *all* the properties in Q. Such properties are called "quasi-random". Specific cases where this behavior is investigated can be found in [15, 16, 21] (for graphs), [12, 13, 17, 19] (for hypergraphs), [14] (for tournaments), [18] (for sequences), [25] (for permutations) and [20] (for sparse graphs), for example.

In this paper we will take C to be the class $\mathcal{G}_n(\mathbf{d})$ of all graphs on n vertices having some given degree sequence $\mathbf{d}$. This is rather different from the classical model of a random graph, in which all vertices have the same expected degree. Special cases of such graph families include the so-called *power law graphs* in which the number of vertices of degree k is proportional to $k^{-\beta}$ for some positive

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real β. Such graphs arise in a variety of applications such as Web connectivity [5, 6, 9, 24, 26, 28, 29], communication networks [1, 3], biological networks [22], collaboration graphs [27], etc.

In this paper, we will introduce a class of quasi-random properties for $\mathcal{G}_n(\mathbf{d})$ and establish quantitative bounds on the strength of correlation between these properties. In particular, these results generalize and strengthen those in [20, 21].

2 Notation

We will consider graphs $G = (V, E)$ where V denotes the set of vertices of G and E denotes the set of edges of G. (For undefined graph theory terminology, see [33].) Our graphs will be undirected, having no loops or multiple edges. We will let $|V|$, the cardinality of V, be denoted by n.

If $\{x, y\} \in E$ is an edge of G, we say that x and y are adjacent, and write this as $x \sim y$. The *neighborhood* $nd(x)$ of a vertex $x \in V$ is defined by

$$nd(x) := \{y \in V : y \sim x \text{ in } G\}.$$

For $x \in V$, the *degree* d_x of x, denotes $|nd(x)|$. The degree sequence $\mathbf{d} = \mathbf{d}_G$ of G is given by

$$\mathbf{d} = (d_x : x \in V),$$

or equivalently, $\mathbf{d}$ can be viewed as a mapping $\mathbf{d} : V \to \mathbb{Z}^+ \cup \{0\}$. For $X, Y \subseteq V$, define

$$e(X, Y) :=| \{(x, y) : x \in X, y \in Y \text{ and } x \sim y\} | .$$

For $X \subseteq V$, define $\mathrm{vol}(X)$, the *volume* of X, by

$$\mathrm{vol}(X) = \sum_{x \in X} d_x.$$

A *walk* $P = P_t(x, y)$ from x to y is a sequence $P = (x_0, x_1, \ldots, x_t)$, where $x_0 = x, x_t = y$ and $x_i \sim x_{i+1}$ for $0 \leq i < t$. Such a walk is said to have length t. Here we do not require all x_i's to be distinct. If all x_i's are different, we say the walk is a *path*.

In this paper, we consider graphs with every vertex having positive degree. The *weight* $w(P)$ of such a walk P is defined to be

$$w(P) = \prod_{0<i<t} \frac{1}{d_{x_i}}$$

(thus, both endpoints are excluded in the product). If P has length 1 (and therefore is an edge of G), then $w(P)$ is defined to be 1.

A *circuit* C of length t is a sequence of t vertices $(x_1, x_2, \ldots, x_t)$ where $x_i \sim x_{i+1}, 1 \le i < t$, and $x_t \sim x_1$. (We remark that in this definition, a circuit can be viewed as a rooted closed walk.) The weight $w(C)$ of such a circuit is defined by

$$w(C) = \prod_{1 \le i \le t} \frac{1}{d_{x_i}}.$$

The *weighted adjacency matrix* $M = M(G)$ is an $n \times n$ matrix with rows and columns indexed by V, defined by

$$M(x,y) = \begin{cases} \frac{1}{\sqrt{d_x d_y}} & \text{if } x \sim y, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Note that M can be written as $M = I - \mathcal{L}$ where I is the identity matrix and $\mathcal{L}$ denotes the (normalized) Laplacian (see [11]). The eigenvalues of M are denoted by $\rho_i, 0 \le i \le n-1$, indexed so that

$$1 = \rho_0 \ge |\rho_1| \ge |\rho_2| \ge \ldots \ge |\rho_{n-1}|$$

using the Perron-Frobenius theorem. Note that $\rho_0 = 1$ has as its eigenvector $(\sqrt{d_x})_{x \in V}$.

Finally, define for $X, Y \subseteq V$, and $t \ge 1$,

$$e_t(X,Y) = \sum_{P \in P_t(X,Y)} w(P)$$

where $P_t(X,Y)$ denotes the set of all walks of length t between $x \in X$ and $y \in Y$. This is a weighted version of the number of walks of length t between X and Y. Note that $e_1(X,Y) = e(X,Y)$. In particular, $e_1(V,V) = \sum_x d_x = \text{vol}(G)$. It is not difficult to check that for $t \ge 1$, we have $e_t(V,V) = \text{vol}(G)$.

3 The quasi-random properties

In this section we will state various properties that the $G \in \mathcal{G}_n(\mathbf{d})$ might satisfy. Each of these properties will depend on a parameter ϵ, which we will always assume to satisfy $0 < \epsilon < 1$. The closer ϵ is to 0, the more the graph in question behaves like a random graph with respect to the property in question, that is, the more the value of the corresponding parameter is closer to its expected value for a random graph in $\mathcal{G}_n(\mathbf{d})$.

DISC(ϵ):

For all $X, Y \subseteq V$,

$$|e(X,Y) - \frac{\text{vol}(X)\text{vol}(Y)}{\text{vol}(G)}| \le \epsilon \ \text{vol}(G).$$

DISC$_t(\epsilon)$:

For all $X, Y \subseteq V$,

$$|e_t(X,Y) - \frac{\text{vol}(X)\text{vol}(Y)}{\text{vol}(G)}| \leq \epsilon \ \text{vol}(G).$$

Note that $\text{DISC}_1(\epsilon)$ is just $\text{DISC}(\epsilon)$.

EIG(ϵ):

With the matrix $M = M(G) = (M(x,y))_{x,y\in V}$ as defined in (1) and with eigenvalues satisfying

$$1 = \rho_0 \geq |\rho_1| \geq |\rho_2| \geq \ldots \geq |\rho_{n-1}|,$$

we have

$$|\rho_i| < \epsilon \quad \text{for all} \ \ i \geq 1.$$

TRACE$_{2t}(\epsilon)$:

The eigenvalues of M satisfy

$$\sum_{i\geq 1} \rho_i^{2t} \leq \epsilon.$$

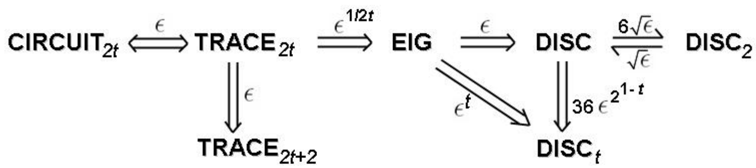
CIRCUIT$_t(\epsilon)$:

The weighted sum of the t-circuits C_t in G satisfies

$$|\sum_{C_t:t-\text{circuit}} w(C_t) - 1 | \leq \epsilon.$$

We will prove the following implications in Section 4:

Theorem 1 *For $t \geq 2$, the following implications hold.*



Here the notation $A \overset{\delta}{\Rightarrow} B$ is shorthand for $A(\epsilon) \Rightarrow B(\delta)$. We say A implies B, denoted by $A \Rightarrow B$, if for every $\beta > 0$, there exists $\alpha > 0$ such that $A(\alpha) \Rightarrow B(\beta)$.

There are several one-way implications in the above figure. A natural question is which, if any, of the reverse directions hold for any of these implications. In Section 5, we will give counterexamples which show that EIG $\not\Rightarrow$ Trace_{2t} for any $t \geq 1$.

In Section 6, we introduce an additional property $\mathbf{U}_t$. Then we show that if a graph satisfies $\mathbf{U}_{t-1}$ for some $t \geq 2$, then DISC $\Rightarrow$ $\mathrm{CIRCUIT}_{2t}$. Using property $\mathbf{U}_t$, we will prove the following result.

Theorem 2 *If G satisfies $\mathbf{U}_{t-1}$ for some $t \geq 2$, then* $\mathrm{CIRCUIT}_{2t}$*,* TRACE_{2t}*,* EIG, DISC, DISC_2*,* DISC_t *are all equivalent.*

4 The implications

Lemma 1 $\mathrm{EIG}(\epsilon) \Longrightarrow \mathrm{DISC}(\epsilon)$.

Proof: For $S \subseteq V$, define

$$f_S(x) = \begin{cases} \sqrt{d_x} & \text{if } x \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for $X, Y \subseteq V$,

$$e(X, Y) = \langle f_X, M f_Y \rangle$$

where $\langle f, g \rangle = \sum_{x \in V} f(x)g(x)$ denotes the usual inner product.

Now, write

$$f_X = \sum_i a_i \phi_i$$

where the ϕ_i's form an orthonormal basis of eigenvectors with

$$\phi_0(v) = \sqrt{\frac{d_v}{\mathrm{vol}(G)}},$$

for all $v \in V$. Hence,

$$\begin{aligned} a_0 &= \langle f_X, \phi_0 \rangle \\ &= \sum_{x \in X} \frac{d_x}{\sqrt{\mathrm{vol}(G)}} \\ &= \frac{\mathrm{vol}(X)}{\sqrt{\mathrm{vol}(G)}}. \end{aligned}$$

Similarly, we write

$$f_Y = \sum_i b_i \phi_i.$$

Thus,

$$\begin{aligned} \langle f_X, M f_Y \rangle &= a_0 b_0 + \sum_{i \geq 1} \rho_i a_i b_i \\ &= \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(G)} + \sum_{i \geq 1} \rho_i a_i b_i \end{aligned}$$

Therefore,

$$\begin{aligned} |e(X,Y) - \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(G)}| &= |\sum_{i \geq 1} \rho_i a_i b_i| \\ &\leq \max_{i \geq 1} |\rho_i| (\sum_{i \geq 1} |a_i|^2)^{1/2} (\sum_{i \geq 1} |b_i|^2)^{1/2} \\ &\leq \epsilon \|f_X\| \, \|f_Y\| \\ &= \epsilon \sqrt{\mathrm{vol}(X)\mathrm{vol}(Y)} \\ &\leq \epsilon \ \mathrm{vol}(G) \end{aligned}$$

by using EIG and the Cauchy-Schwarz inequality where $\| \cdot \|$ denotes the L_2-norm. Therefore, the proof is complete. □

In a similar way, we prove

Lemma 2 EIG(ϵ) $\Longrightarrow$ DISC$_t$(ϵ^t) *for any* $t \geq 1$.

Proof: In this case we observe that for $X, Y \subseteq V$,

$$e_t(X,Y) = \langle f_X, M^t f_Y \rangle$$

(using the notation of Lemma 1). Thus, writing

$$f_X = \sum_i a_i \phi_i, f_Y = \sum_i b_i \phi_i,$$

we find

$$
\begin{aligned}
|\langle f_X, M^t f_Y\rangle - \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(G)}| &= |\langle f_X, M^t f_Y\rangle - \rho_0^t a_0 b_0| \\
&\leq \max_{i\geq 1} |\rho_i|^t \sum_{i\geq 1} |a_i b_i| \\
&\leq \max_{i\geq 1} |\rho_i|^t \|f_X\| \, \|f_Y\| \\
&\leq \epsilon^t \sqrt{\mathrm{vol}(X)\mathrm{vol}(Y)} \\
&\leq \epsilon^t \ \mathrm{vol}(G)
\end{aligned}
$$

and Lemma 2 is proved. □

Lemma 3 $\mathrm{CIRCUIT}_{2t}(\epsilon) \Longleftrightarrow \mathrm{TRACE}_{2t}(\epsilon)$.

Proof: Let $C^*_{2t}(u)$ denote a rooted $2t$-circuit with starting and ending point u. Then,

$$M^{2t}(u,u) = \sum_{C^*_{2t}(u)} w(C^*_{2t}(u)).$$

Thus, the trace of the matrix M^{2t} can be expressed as:

$$
\begin{aligned}
\mathrm{Tr}(M^{2t}) &= \sum_u \sum_{C^*_{2t}(u)} w(C^*_{2t}(u)) \\
&= \sum_{C_{2t}} w(C_{2t}).
\end{aligned}
$$

On the other hand, the same trace can be evaluated using eigenvalues:

$$
\begin{aligned}
\mathrm{Tr}(M^{2t}) &= \sum_i \rho_i^{2t} \\
&= 1 + \sum_{i\geq 1} \rho_i^{2t}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
|\sum_{C_{2t}} w(C_{2t}) \ - 1 \ | &= | \ \mathrm{Tr}(M^{2t}) \ - 1 \ | \\
&= \sum_{i\geq 1} \rho_i^{2t}
\end{aligned}
$$

and Lemma 3 is proved. □

Lemma 4 $\mathrm{TRACE}_{2t}(\epsilon) \Longrightarrow \mathrm{EIG}(\epsilon^{1/2t})$, *for any* $t \geq 1$.

Proof: By hypothesis, we have

$$|\sum_i \rho_i^{2t} - 1| = \sum_{i\geq 1} \rho_i^{2t} \leq \epsilon.$$

Therefore

$$\max_{i\geq 1} |\rho_i| \leq \epsilon^{1/2t}.$$

□

Lemma 5 $\mathrm{TRACE}_{2t}(\epsilon) \Longrightarrow \mathrm{TRACE}_{2t+2}(\epsilon)$.

Proof: Since $|\rho_i| \leq 1$ for all i, we have

$$\sum_{i\geq 1} \rho_i^{2t+2} \leq \sum_{i\geq 1} \rho_i^{2t} \leq \epsilon$$

by hypothesis. □

Lemma 6 *For* $t \geq 1$, $\mathrm{DISC}_{2t}(\epsilon) \Longrightarrow \mathrm{DISC}_t(\sqrt{\epsilon})$.

Proof: For $X \subseteq V$,

$$\begin{aligned} e_{2t}(X,X) &= \sum_{x,x'\in X} \sum_y \frac{e_t(x,y)e_t(y,x')}{d_y} \qquad (2) \\ &= \sum_y \frac{e_t(y,X)^2}{d_y}. \end{aligned}$$

By applying $\mathrm{DISC}_{2t}(\epsilon)$ to $e_{2t}(X,X)$, we have

$$\sum_y \frac{e_t(y,X)^2}{d_y} \leq \frac{\mathrm{vol}(X)^2}{\mathrm{vol}(G)} + \epsilon \ \mathrm{vol}(G).$$

Note that

$$\sum_y e_t(y,X) = e_t(V,X) = \sum_{x\in V} d_x = \mathrm{vol}(X).$$

Therefore,

$$\begin{aligned} &\sum_y (e_t(y,X) - \frac{d_y \mathrm{vol}(X)}{\mathrm{vol}(G)})^2 \frac{1}{d_y} \\ = \quad &\sum_y \frac{e_t(y,X)^2}{d_y} - 2e_t(V,X)\frac{\mathrm{vol}(X)}{\mathrm{vol}(G)} + \frac{\mathrm{vol}(X)^2}{\mathrm{vol}(G)} \\ = \quad &\sum_y \frac{e_t(y,X)^2}{d_y} - \frac{\mathrm{vol}(X)^2}{\mathrm{vol}(G)} \\ \leq \quad &\epsilon \ \mathrm{vol}(G). \end{aligned}$$

by (2) and $\mathrm{DISC}_{2t}(\epsilon)$. But

$$\begin{aligned}
& \sum_y (e_t(y,X) - \frac{d_y \mathrm{vol}(X)}{\mathrm{vol}(G)})^2 \frac{1}{d_y} \\
\geq \; & \sum_{y \in Y} (e_t(y,X) - \frac{d_y \mathrm{vol}(X)}{\mathrm{vol}(G)})^2 \frac{1}{d_y} \\
\geq \; & \left(\sum_{y \in Y} (e_t(y,X) - d_y \frac{\mathrm{vol}(X)}{\mathrm{vol}(G)}) \right)^2 \frac{1}{\sum_{y \in Y} d_y} \\
= \; & \left(e_t(Y,X) - \frac{\mathrm{vol}(Y)\mathrm{vol}(X)}{\mathrm{vol}(G)} \right)^2 / \mathrm{vol}(Y)
\end{aligned}$$

by applying the Cauchy-Schwarz inequality. Thus,

$$\begin{aligned}
\mid e_t(Y,X) - \frac{\mathrm{vol}(Y)\mathrm{vol}(X)}{\mathrm{vol}(G)} \mid \; & \leq \; \sqrt{\epsilon \ \mathrm{vol}(Y)\mathrm{vol}(G)} \\
& \leq \; \sqrt{\epsilon} \ \mathrm{vol}(G).
\end{aligned}$$

This is exactly $\mathrm{DISC}_t(\sqrt{\epsilon})$. □

Lemma 7 *For any* $t \geq 1$, $\mathrm{DISC}_t(\epsilon) \Longrightarrow \mathrm{DISC}_{t+1}(6\sqrt{\epsilon})$.

Proof:

For $X, Y \subseteq V$,

$$\mid e_t(X,Y) - \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(G)} \mid \leq \epsilon \ \mathrm{vol}(G). \tag{3}$$

Consider

$$e_{t+1}(X,Y) = \sum_v \frac{e(X,v)e_t(v,Y)}{d_v}.$$

Define

$$S_1 := \{z \in V : e_t(z,Y) > \frac{d_z}{\mathrm{vol}(G)}(\mathrm{vol}(Y) + \sqrt{\epsilon} \ \mathrm{vol}(G))\}.$$

Thus,

$$\sum_{z \in S_1} e_t(z,Y) = e_t(S_1,Y) > \frac{\mathrm{vol}(S_1)\mathrm{vol}(Y)}{\mathrm{vol}(G)} + \sqrt{\epsilon} \ \mathrm{vol}(S_1).$$

Hence, by (3) applied to $X = S_1$ and Y,

$$\mathrm{vol}(S_1) < \sqrt{\epsilon} \ \mathrm{vol}(G). \tag{4}$$

In the same way, if we define $S_2 := \{z \in V : e_t(z,Y) < \frac{d_z}{\mathrm{vol}(G)}(\mathrm{vol}(Y) - \sqrt{\epsilon} \ \mathrm{vol}(G))\}$, then

$$\mathrm{vol}(S_2) < \sqrt{\epsilon} \ \mathrm{vol}(G). \tag{5}$$

Now,

$$e_{t+1}(X,Y) = \left(\sum_{v \notin S_1 \cup S_2} + \sum_{v \in S_1 \cup S_2} \right) \frac{e(X,v)e_t(v,Y)}{d_v}.$$

For the first sum, we have

$$\begin{aligned}
\sum_{v \notin S_1 \cup S_2} \frac{e(X,v)e_t(v,Y)}{d_v} &\leq \sum_{v \notin S_1 \cup S_2} \frac{e(X,v)}{d_v} \frac{d_v}{\mathrm{vol}(G)} (\mathrm{vol}(Y) + \sqrt{\epsilon}\ \mathrm{vol}(G)) \\
&\leq \sum_{v} \frac{e(X,v)}{\mathrm{vol}(G)} (\mathrm{vol}(Y) + \sqrt{\epsilon}\ \mathrm{vol}(G)) \\
&\leq \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(G)} + \sqrt{\epsilon}\ \mathrm{vol}(G)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{v \notin S_1 \cup S_2} \frac{e(X,v)e_t(v,Y)}{d_v} &\geq \sum_{v \notin S_1 \cup S_2} \frac{e(X,v)}{d_v} \frac{d_v}{\mathrm{vol}(G)} (\mathrm{vol}(Y) - \sqrt{\epsilon}\ \mathrm{vol}(G)) \\
&\geq \sum_{v \notin S_1 \cup S_2} \frac{e(X,v)}{\mathrm{vol}(G)} (\mathrm{vol}(Y) - \sqrt{\epsilon}\ \mathrm{vol}(G)) \\
&\geq \frac{(\mathrm{vol}(X) - \mathrm{vol}(S_1) - \mathrm{vol}(S_2))}{\mathrm{vol}(G)} (\mathrm{vol}(Y) - \sqrt{\epsilon}\ \mathrm{vol}(G)) \\
&\geq \frac{(\mathrm{vol}(X) - 2\sqrt{\epsilon}\ \mathrm{vol}(G))}{\mathrm{vol}(G)} (\mathrm{vol}(Y) - \sqrt{\epsilon}\ \mathrm{vol}(G)) \\
&\geq \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(G)} - 3\sqrt{\epsilon}\ \mathrm{vol}(G),
\end{aligned}$$

by using (4) and (5). Thus,

$$\left| \sum_{v \notin S_1 \cup S_2} \frac{e(X,v)e_t(v,Y)}{d_v} - \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(G)} \right| \leq 3\sqrt{\epsilon}\ \mathrm{vol}(G).$$

For the second sum, we have

$$\begin{aligned}
\sum_{v \in S_1 \cup S_2} \frac{e(X,v)e_t(v,Y)}{d_v} &\leq \sum_{v \in S_1 \cup S_2} \frac{d_v e_t(v,Y)}{d_v} \\
&= e_t(S_1 \cup S_2, Y) \\
&\leq \frac{(\mathrm{vol}(S_1) + \mathrm{vol}(S_2))\mathrm{vol}(Y)}{\mathrm{vol}(G)} + \epsilon\ \mathrm{vol}(G) \qquad \text{by DISC}_t(\epsilon) \\
&\leq \frac{2\sqrt{\epsilon}\ \mathrm{vol}(G)\mathrm{vol}(Y)}{\mathrm{vol}(G)} + \epsilon\ \mathrm{vol}(G) \\
&= 2\sqrt{\epsilon}\ \mathrm{vol}(Y) + \epsilon\ \mathrm{vol}(G) \\
&\leq 3\sqrt{\epsilon}\ \mathrm{vol}(G).
\end{aligned}$$

Putting these two estimates together, we obtain

$$| \; e_{t+1}(X,Y) - \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(G)} \; | \quad \le \quad 6\sqrt{\epsilon}\ \mathrm{vol}(G)$$

which is $\mathrm{DISC}_{t+1}(6\sqrt{\epsilon})$. □

Lemma 8 *For any integers s and t, $DISC_s$ and $DISC_t$ are related as follows:*

(i) *If $s < t$, then* $\mathrm{DISC}_s(\epsilon) \Longrightarrow \mathrm{DISC}_t(36\epsilon^{1/2^{t-s}})$.
As a special case, $\mathrm{DISC}(\epsilon) \Longrightarrow \mathrm{DISC}_t(36\epsilon^{1/2^{t-1}})$.

(ii) *If $t < s \le 2^k t$ for some k, then* $\mathrm{DISC}_s(\epsilon) \Longrightarrow \mathrm{DISC}_t(36^{1/2^k}\epsilon^{1/2^{2^k t+k-s}})$.

Proof: (i) follows from Lemma 7, i.e.,

$$\begin{aligned} \mathrm{DISC}_s(\epsilon) &\Rightarrow \mathrm{DISC}_s(36\epsilon) \\ &\Rightarrow \mathrm{DISC}_{s+1}(36\epsilon^{1/2}) \\ &\Rightarrow \ldots \\ &\Rightarrow \mathrm{DISC}_t(36\epsilon^{1/2^{t-s}}) \end{aligned}$$

To prove (ii), we have, from (i) that

$$\mathrm{DISC}_s(\epsilon) \quad \Rightarrow \quad \mathrm{DISC}_{2^k t}(36\epsilon^{1/2^{2^k t-s}})$$

Now apply Lemma 6 k times to get the desired implication. □

By combining Lemmas 1 to 8, we have proved all the implications in Theorem 1.

5 Separation of properties

In this section we give an example showing that at least one of the implications in Theorem 1 cannot be reversed. Whether this is true of the others is not known at this point.

Fact 1 *For any $t \ge 1$,*

$$\mathrm{EIG}(\epsilon) \not\Longrightarrow \mathrm{TRACE}_{2t}(\delta)$$

for any $\delta = \delta(\epsilon)$.

Proof: Choose $t \geq 1$ and let $G = G(n)$ be a random regular graph with n vertices and vertex degree $n^{1/t}$. Thus, $M = M(G)$ has

$$M(u,v) \quad = \quad \begin{cases} 1/n^{1/t} & \text{if } u \sim v, \\ 0 & \text{otherwise.} \end{cases}$$

It was shown in [23] that the eigenvalue distribution of $M(G)$ for a random graph G with a given expected degree distribution satisfies the semi-circle law if the minimum degree is greater than a power of $\log n$. As a consequence, if $1 = \rho_0 \geq |\rho_1| \geq |\rho_2| \geq \ldots \geq |\rho_{n-1}|$ are the eigenvalues of M, then

(1) $\rho_1 = (1 + o(1))2/n^{1/2t}$,

(2) If $N(x)$ denotes the number of ρ_i with $\rho_i \leq 2x/n^{1/2t}$, then

$$\frac{N(x)}{n} = (1 + o(1))\frac{2}{\pi}\int_{-1}^{x} \sqrt{1-u^2}du.$$

In particular, for $x = 1/2$, we have

$$\frac{N(1/2)}{n} = (1 + o(1))(\frac{2}{3} + \frac{\sqrt{3}}{4\pi}) \approx 0.8045\ldots.$$

Thus,

$$\sum_{i \geq 1} \rho_i^{2t} \geq 2(n - N(1/2))(\frac{1}{n^{1/2t}})^{2t} \geq 0.391.$$

Hence, for any $\epsilon > 0$, G satisfies EIG(ϵ), provided $n \geq n_0$, but G does *not* satisfy $\text{TRACE}_{2t}(0.39)$.

It would be interesting to know if some of the other possible implications hold. For example, does DISC $\Rightarrow$ EIG ?

Recently, Bilu and Linial [7] proved the following partial implication for regular graphs:

For a d-regular graph G on n vertices, if for all $X, Y \subset V$,

$$\left| e(X,Y) - \frac{d}{n}|X|\ |Y| \right| \leq \alpha d\sqrt{|X|\ |Y|}, \tag{6}$$

then $|\rho_1| = O(\alpha(\log(1/\alpha) + 1))$.

The above property in (6) was introduced by Thomason [31] in the context of what he called (p, α)-jumbled graphs. Of course, this property is quite a bit stronger than DISC. Properties of random graphs based on this concept (without the equivalence relations) are often referred as the pseudo-random properties. The reader is referred to [30, 32] for discussions on pseudo-random graphs.

Butler [10] combines the methods in [7] and [8] to prove the following :

For a graph G with no isolated vertices, if for all $X, Y \subset V$,

$$\left| e_t(X,Y) - \frac{\mathrm{vol}(X)\,\mathrm{vol}(Y)}{\mathrm{vol}(G)} \right| \le \alpha \sqrt{\mathrm{vol}(X)\mathrm{vol}(Y)},$$

then $|\rho_1|^t \le 18\alpha(1 - \frac{5}{2}\log\alpha)$.

For $t = 1$, this is the best possible (up to a constant) by considering a class of regular graphs constructed by Bollobás and Nikiforov [8]. In their example, the graphs have $\alpha = Cn^{-1/6}$ and $|\rho_1| \ge c\alpha\log n$ for some constants c and C.

6 Reversing the implications

It is clear from the examples in the preceding section that in order to establish some of the reverse implications, e.g., DISC $\Rightarrow$ CIRCUIT$_{2t}$, we will have to make further assumptions for the $G \in \mathcal{G}_n(\mathbf{d})$. One such condition is the following:

For $t \ge 1$, a graph satisfies $U_t(C)$ if for all $x, y \in V, e_t(x,y) \le C\frac{d_x d_y}{\mathrm{vol}(G)}$.

We will think of C as a large positive real. We note that for $t = 1$ and for G with minimum degree αn, the property $U_1(C)$ is automatically satisfied for $C \ge 1/\alpha^2$.

Note that for a d-regular graph, U_t implies that $n \le Cd^t$ or, equivalently, the volume of the graph is of order at least $n^{1+1/t}$.

Lemma 9 *For any $t \ge 1$,*

$$U_t(C) \Longrightarrow U_{t+1}(C).$$

Proof: Observe that

$$\begin{aligned} e_{t+1}(x,y) &= \sum_z \frac{e(x,z)e_t(z,y)}{d_z} \\ &\le \sum_z \frac{e(x,z)}{d_z} \cdot C\frac{d_z d_y}{\mathrm{vol}(G)} \\ &= C\frac{d_y}{\mathrm{vol}(G)} \sum_z e(x,z) \\ &= C\frac{d_x d_y}{\mathrm{vol}(G)}. \end{aligned}$$

The lemma is proved. □

Theorem 3 *If G satisfies $U_{t-1}(C)$ for some $t \geq 2$, then*

$$\mathrm{DISC}(\epsilon) \Longrightarrow \mathrm{CIRCUIT}_{2t}(\eta)$$

where $\eta = 2C'C^2\epsilon/\delta + 2C^2(C'+1)^2\delta + 20\sqrt{\delta} + 12\delta + 16C'^4\delta^{3/2} + 8C'^2\delta^{3/2}$, with $C' = \lceil C/\delta^{1/4} \rceil$, and $\delta = \max\{\sqrt{\epsilon}, 36\epsilon^{1/2^{t-2}}\}$. (Note that $\eta \to 0$ as $\epsilon \to 0$.)

Proof: We are going to consider the sum

$$\sum_{u,v \in V} \frac{1}{d_u d_v}(e_t(u,v) - \frac{d_u d_v}{\mathrm{vol}(G)})^2$$

where, as usual, $V = V(G)$.

Since G satisfies $\mathrm{DISC}(\epsilon)$ by hypothesis, then by Lemma 8, G also satisfies $\mathrm{DISC}_{t-1}(\delta)$ where $\delta \geq 36\epsilon^{1/2^{t-2}}$, i.e.,

$$|\ e_{t-1}(X,Y) - \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(G)}\ | \quad \leq \delta\ \mathrm{vol}(G)$$

for all $X, Y \subseteq V$.

We here choose $\delta = \max\{\sqrt{\epsilon}, 36\epsilon^{1/2^{t-2}}\}$. For a fixed vertex u, we partition the vertex set V into the sets $W_i = W_i(u), 0 \leq i < C'$, as follows. (To simplify the notation, we use W_i instead of $W_i(u)$ below.)

$$\begin{aligned}
W_0 &= \{v : 0 \leq e_{t-1}(u,v) < \delta^{1/4}\frac{d_u d_v}{\mathrm{vol}(G)}\}, \\
W_1 &= \{v : \delta^{1/4}\frac{d_u d_v}{\mathrm{vol}(G)} \leq e_{t-1}(u,v) < 2\delta^{1/4}\frac{d_u d_v}{\mathrm{vol}(G)}\}, \\
W_2 &= \{v : 2\delta^{1/4}\frac{d_u d_v}{\mathrm{vol}(G)} \leq e_{t-1}(u,v) < 3\delta^{1/4}\frac{d_u d_v}{\mathrm{vol}(G)}\},
\end{aligned}$$

and, in general,

$$W_i = \{v : i\delta^{1/4}\frac{d_u d_v}{\mathrm{vol}(G)} \leq e_{t-1}(u,v) < (i+1)\delta^{1/4}\frac{d_u d_v}{\mathrm{vol}(G)}\}$$

for $0 \leq i < C' = \lceil C/\delta^{1/4} \rceil$. Since $e_{t-1}(u,v) \leq C d_u d_v/\mathrm{vol}(G)$ by $\mathrm{U}_{t-1}(C)$, the W_i form a partition of V.

Since

$$W_i \subseteq \{v : |e_{t-1}(u,v) - i\delta^{1/4}\frac{d_u d_v}{\mathrm{vol}(G)}| < \delta^{1/4}\frac{d_u d_v}{\mathrm{vol}(G)}\},$$

then

$$\begin{aligned}
|\sum_{v \in W_i}(e_{t-1}(u,v) - i\delta^{1/4}\frac{d_u d_v}{\mathrm{vol}(G)})\ | &< \sum_{v \in W_i}\delta^{1/4}\frac{d_u d_v}{\mathrm{vol}(G)} \\
\text{and} \quad |\ e_{t-1}(u,W_i) - i\delta^{1/4}\frac{d_u \mathrm{vol}(W_i)}{\mathrm{vol}(G)}\ | &< \delta^{1/4}\frac{d_u \mathrm{vol}(W_i)}{\mathrm{vol}(G)}. \qquad (7)
\end{aligned}$$

Since $\sum_i e_{t-1}(u,W_i) = e_{t-1}(u,V) = d_u$, then

$$\begin{aligned} & \mid \sum_i i\delta^{1/4}\frac{d_u \mathrm{vol}(W_i)}{\mathrm{vol}(G)} - d_u \mid \\ = & \mid \sum_i i\delta^{1/4}\frac{d_u \mathrm{vol}(W_i)}{\mathrm{vol}(G)} - \sum_i e_{t-1}(u,W_i) \mid \\ \leq & \delta^{1/4}\sum_i \frac{d_u \mathrm{vol}(W_i)}{\mathrm{vol}(G)} \\ = & \delta^{1/4} d_u. \end{aligned} \qquad (8)$$

Now, for each i, if $\mathrm{vol}(W_i) \geq \sqrt{\delta}\mathrm{vol}(G)$, then define $X_i = X_i(u)$ and $X'_i = X'_i(u)$ as follows:

$$\begin{aligned} X_i &= \{v : e(v,W_i) > \frac{d_v \mathrm{vol}(W_i)}{\mathrm{vol}(G)}(1+\sqrt{\delta})\}, \\ X'_i &= \{v : e(v,W_i) < \frac{d_v \mathrm{vol}(W_i)}{\mathrm{vol}(G)}(1-\sqrt{\delta})\}. \end{aligned}$$

If $\mathrm{vol}(W_i) < \sqrt{\delta}\mathrm{vol}(G)$ then define $X_i = X'_i = \emptyset$. Also define

$$W_u^* = \bigcup\{W_i : \mathrm{vol}(W_i) < \sqrt{\delta}\mathrm{vol}(G)\}.$$

Thus,

$$\mathrm{vol}(W_u^*) \leq C'\sqrt{\delta}\mathrm{vol}(G)$$

since there are just C' possible values of i.

By DISC(ϵ), we have

$$\mid e(W_i,X_i) - \frac{\mathrm{vol}(W_i)\mathrm{vol}(X_i)}{\mathrm{vol}(G)} \mid \quad \leq \epsilon \ \mathrm{vol}(G),$$

but from the definition of X_i, we have

$$\begin{aligned} \mid e(W_i,X_i) - \frac{\mathrm{vol}(W_i)\mathrm{vol}(X_i)}{\mathrm{vol}(G)} \mid \quad & \geq \sqrt{\delta}\frac{\mathrm{vol}(X_i)\mathrm{vol}(W_i)}{\mathrm{vol}(G)} \\ & \geq \sqrt{\delta}\frac{\sqrt{\delta}\mathrm{vol}(G)\mathrm{vol}(X_i)}{\mathrm{vol}(G)} \\ & = \delta \ \mathrm{vol}(X_i). \end{aligned}$$

Therefore,

$$\mathrm{vol}(X_i) \leq \epsilon/\delta \ \mathrm{vol}(G).$$

A similar argument shows that

$$\mathrm{vol}(X'_i) \leq \epsilon/\delta \ \mathrm{vol}(G)$$

as well. Consequently, for each u, we consider

$$X_u := \bigcup \{X_i \cup X_i' : W_i \not\subseteq W_u^*\}$$

and we have

$$\mathrm{vol}(X_u) \leq 2C'\epsilon/\delta \ \mathrm{vol}(G). \tag{9}$$

For $v \notin X_u$, we have, from the definition of X_u,

$$\begin{aligned}
e(W_u^*, v) &= d_v - \sum_{W_i \not\subseteq W_u^*} e(W_i, v) \\
&\leq d_v - \sum_{W_i \not\subseteq W_u^*} (1-\sqrt{\delta}) \frac{d_v \mathrm{vol}(W_i)}{\mathrm{vol}(G)} \\
&= d_v - (1-\sqrt{\delta}) \frac{d_v(\mathrm{vol}(G) - \mathrm{vol}(W_u^*))}{\mathrm{vol}(G)} \\
&= \sqrt{\delta} d_v + (1-\sqrt{\delta}) \frac{\mathrm{vol}(W_u^*)}{\mathrm{vol}(G)} d_v \\
&\leq \sqrt{\delta} d_v + (1-\sqrt{\delta}) C' \frac{\sqrt{\delta}\mathrm{vol}(G)}{\mathrm{vol}(G)} d_v \\
&\leq (C'+1)\sqrt{\delta} d_v. \qquad (10)
\end{aligned}$$

We now begin considering the sum,

$$\sum_u \sum_v \frac{1}{d_u d_v}(e_t(u,v) - \frac{d_u d_v}{\mathrm{vol}(G)})^2 = \\
\sum_u \left(\sum_{v \in X_u} + \sum_{v \notin X_u} \right) \frac{1}{d_u d_v}(e_t(u,v) - \frac{d_u d_v}{\mathrm{vol}(G)})^2.$$

For the first sum, we use property $\mathbf{U}_t(C)$ and Lemma 9 to obtain the following estimate:

$$\begin{aligned}
\sum_u \sum_{v \in X_u} \frac{1}{d_u d_v}(e_t(u,v) - \frac{d_u d_v}{\mathrm{vol}(G)})^2 &\leq \sum_u \sum_{v \in X_u} \frac{1}{d_u d_v} C^2 (\frac{d_u d_v}{\mathrm{vol}(G)})^2 \\
&= C^2 \frac{\mathrm{vol}(X_u)}{\mathrm{vol}(G)} \\
&\leq 2C'C^2\epsilon/\delta \qquad (11)
\end{aligned}$$

by (9). For the second sum we have

$$
\begin{aligned}
& \sum_u \sum_{v \notin X_u} \frac{1}{d_u d_v} (e_t(u,v) - \frac{d_u d_v}{\mathrm{vol}(G)})^2 \\
= & \sum_u \sum_{v \notin X_u} \frac{1}{d_u d_v} (\sum_z \frac{e_{t-1}(u,z)e(z,v)}{d_z} - \frac{d_u d_v}{\mathrm{vol}(G)})^2 \\
= & \sum_u \sum_{v \notin X_u} \frac{1}{d_u d_v} ((\sum_{z \in W_u^*} + \sum_{z \notin W_u^*}) \frac{e_{t-1}(u,z)e(z,v)}{d_z} - \frac{d_u d_v}{\mathrm{vol}(G)})^2 \\
\leq & \sum_u \sum_{v \notin X_u} \frac{2}{d_u d_v} \left((\sum_{z \in W_u^*} \frac{e_{t-1}(u,z)e(z,v)}{d_z})^2 + (\sum_{z \notin W_u^*} \frac{e_{t-1}(u,z)e(z,v)}{d_z} - \frac{d_u d_v}{\mathrm{vol}(G)})^2 \right).
\end{aligned}
$$

For the first sum just above (without a factor of 2), we have

$$
\begin{aligned}
& \sum_u \sum_{v \notin X_u} \frac{1}{d_u d_v} \left(\sum_{z \in W_u^*} \frac{e_{t-1}(u,z)e(z,v)}{d_z} \right)^2 \\
\leq & \sum_u \sum_{v \notin X_u} \frac{1}{d_u d_v} \left(\sum_{z \in W_u^*} \frac{C d_u d_z e(z,v)}{d_z \mathrm{vol}(G)} \right)^2 \\
& \qquad\qquad\qquad\qquad \text{by } \mathrm{U}_{t-1}(C), \\
= & \sum_u \sum_{v \notin X_u} \frac{C^2 d_u^2 (e(W_u^*, v))^2}{d_u d_v \mathrm{vol}(G)^2} \\
\leq & \sum_u \sum_{v \notin X_u} \frac{C^2 (C'+1)^2 \delta d_u d_v}{\mathrm{vol}(G)^2} \qquad \text{by (10)} \\
\leq & C^2 (C'+1)^2 \delta. \qquad (12)
\end{aligned}
$$

For the second sum above, we have

$$
\begin{aligned}
& \sum_u \sum_{v \notin X_u} \frac{1}{d_u d_v} \left(\sum_{z \notin W_u^*} \frac{e_{t-1}(u,z)e(z,v)}{d_z} - \frac{d_u d_v}{\mathrm{vol}(G)} \right)^2 \\
= & \sum_u \sum_{v \notin X_u} \frac{1}{d_u d_v} \left(\sum_{W_i \not\subseteq W_u^*} \sum_{z \in W_i} \frac{e_{t-1}(u,z)e(z,v)}{d_z} - \frac{d_u d_v}{\mathrm{vol}(G)} \right)^2 \\
\leq & \sum_u \sum_{v \notin X_u} \frac{2}{d_u d_v} \Bigg((\sum_{W_i \not\subseteq W_u^*} \sum_{z \in W_i} \frac{i \delta^{1/4} d_u d_z e(z,v)}{d_z \mathrm{vol}(G)} - \frac{d_u d_v}{\mathrm{vol}(G)})^2 \\
& \qquad + (\sum_{W_i \not\subseteq W_u^*} \sum_{z \in W_i} \frac{\delta^{1/4} d_u d_z e(z,v)}{d_z \mathrm{vol}(G)})^2 \Bigg)
\end{aligned}
$$

since $|A - a| \leq b \Rightarrow (A-B)^2 \leq 2((a-B)^2 + b^2)$ and inequalities in (7).

For the second sum we have

$$
\begin{aligned}
&\sum_u \sum_{v \notin X_u} \frac{1}{d_u d_v} \Big(\sum_{W_i \not\subseteq W_u^*} \sum_{z \in W_i} \frac{\delta^{1/4} d_u e(z,v)}{\mathrm{vol}(G)} \Big)^2 \\
= \quad & \sqrt{\delta} \sum_u \sum_{v \notin X_u} \frac{1}{d_u d_v} \Big(\sum_{W_i \not\subseteq W_u^*} \frac{d_u e(W_i, v)}{\mathrm{vol}(G)} \Big)^2 \\
\leq \quad & \sqrt{\delta} \sum_u \sum_{v \notin X_u} \frac{1}{d_u d_v} \Big(\sum_{W_i \not\subseteq W_u^*} \frac{d_u}{\mathrm{vol}(G)} (1+\sqrt{\delta}) \frac{d_v \mathrm{vol}(W_i)}{\mathrm{vol}(G)} \Big)^2 \qquad \text{by def. of } X_u \\
\leq \quad & \sqrt{\delta} \sum_u \sum_{v \notin X_u} \frac{1}{d_u d_v} \left((1+\sqrt{\delta}) \frac{d_u d_v}{\mathrm{vol}(G)} \right)^2 \\
\leq \quad & \sqrt{\delta} + 3\delta \qquad (13)
\end{aligned}
$$

(upper bounded by the sum over all u and v).

Finally, for the first sum we have

$$
\begin{aligned}
&\sum_u \sum_{v \notin X_u} \frac{1}{d_u d_v} \left(\sum_{W_i \not\subseteq W_u^*} \sum_{z \in W_i} \frac{i\delta^{1/4} d_u e(z,v)}{\mathrm{vol}(G)} - \frac{d_u d_v}{\mathrm{vol}(G)} \right)^2 \\
= \quad & \sum_u \sum_{v \notin X_u} \frac{1}{d_u d_v} \left(\sum_{W_i \not\subseteq W_u^*} \frac{i\delta^{1/4} d_u e(W_i, v)}{\mathrm{vol}(G)} - \frac{d_u d_v}{\mathrm{vol}(G)} \right)^2 \\
\leq \quad & \sum_u \sum_{v \notin X_u} \frac{2}{d_u d_v} \left(\left(\sum_{W_i \not\subseteq W_u^*} \frac{i\delta^{1/4} d_u d_v \mathrm{vol}(W_i)}{\mathrm{vol}(G)^2} - \frac{d_u d_v}{\mathrm{vol}(G)} \right)^2 \right. \\
& \qquad \left. + \left(\sum_{W_i \not\subseteq W_u^*} \frac{i\delta^{1/4} d_u d_v \mathrm{vol}(W_i)}{\mathrm{vol}(G)^2} \sqrt{\delta} \right)^2 \right) \qquad \text{by the def. of } X_u, \\
\leq \quad & \sum_u \sum_{v \notin X_u} \frac{2}{d_u d_v} \left(2 \left(\sum_i \frac{i\delta^{1/4} d_u d_v \mathrm{vol}(W_i)}{\mathrm{vol}(G)^2} - \frac{d_u d_v}{\mathrm{vol}(G)} \right)^2 \right. \\
& \qquad \left. + 2 \left(\frac{\sum_{W_i \subseteq W_u^*} i\delta^{1/4} d_u d_v \mathrm{vol}(W_i)}{\mathrm{vol}(G)^2} \right)^2 + \left(\sum_{W_i \not\subseteq W_u^*} \frac{i\delta^{1/4} d_u d_v \mathrm{vol}(W_i)}{\mathrm{vol}(G)^2} \sqrt{\delta} \right)^2 \right) \\
\leq \quad & \sum_{u,v} \frac{1}{d_u d_v} \left(4 \left(\frac{d_u d_v \delta^{1/4}}{\mathrm{vol}(G)} \right)^2 + 4 \left(\frac{C'^2 \delta^{1/4} d_u d_v \delta^{1/2}}{\mathrm{vol}(G)} \right)^2 + 2 \left(\frac{C' d_u d_v \delta^{3/4}}{\mathrm{vol}(G)} \right)^2 \right) \\
& \qquad \text{by (8), def. of } W_u^* \text{ and the fact that } i < C', \\
\leq \quad & 4\sqrt{\delta} + 4C'^4 \delta^{3/2} + 2C'^2 \delta^{3/2}. \qquad (14)
\end{aligned}
$$

Now, we have to put everything together.

First observe that

$$
\begin{aligned}
& \sum_{u,v\in V} \frac{1}{d_u d_v}(e_t(u,v) - \frac{d_u d_v}{\mathrm{vol}(G)})^2 \\
= & \sum_{u,v\in V} \frac{1}{d_u d_v} e_t(u,v)^2 - 2\sum_{u,v\in V} \frac{e_t(u,v)}{\mathrm{vol}(G)} + \sum_{u,v\in V} \frac{d_u d_v}{\mathrm{vol}(G)^2} \\
= & \sum_{C_{2t}\, 2t-\mathrm{circuit}} w(C_{2t}) - 2\frac{e_t(V,V)}{\mathrm{vol}(G)} + 1 \\
= & \sum_{C_{2t}\, 2t-\mathrm{circuit}} w(C_{2t}) - 1
\end{aligned}
$$

(using $e_t(V,V) = \mathrm{vol}(G)$) so that the preceding results, including inequalities (11), (12), (13) and (14), give

$$
\begin{aligned}
& |\sum_{C_{2t}\, 2t-\mathrm{circuit}} w(C_{2t}) - 1| \\
= & \sum_{u,v\in V} \frac{1}{d_u d_v}(e_t(u,v) - \frac{d_u d_v}{\mathrm{vol}(G)})^2 \\
\leq & 2C'C^2\epsilon/\delta + 2C^2(C'+1)^2\delta + 4(\sqrt{\delta} + 3\delta) + 4(4\sqrt{\delta} + 4C'^4\delta^{3/2} + 2C'^2\delta^{3/2}) \\
\leq & 2C'C^2\epsilon/\delta + 2C^2(C'+1)^2\delta + 20\sqrt{\delta} + 12\delta + 16C'^4\delta^{3/2} + 8C'^2\delta^{3/2}.
\end{aligned}
$$

This proves Theorem 3. □

Corollary 1 *If G has minimum degree αn, then*

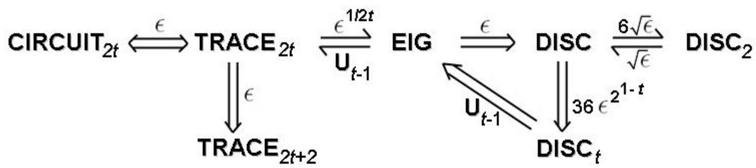
$$\mathrm{DISC}(\epsilon) \Longrightarrow \mathrm{CIRCUIT}_{2t}(\eta)$$

where η depends only on ϵ, α and t.

Theorem 4 *If G has minimum degree αn for some constant α, then* $\mathrm{CIRCUIT}_{2t}$, TRACE_{2t}, EIG, DISC, DISC_2, DISC_t *are all equivalent for $t \geq 2$.*

7 Concluding remarks

We can summarize the main theorems in the following:



We should note that if for our degree sequence $\mathbf{d}$, we choose all d_i to be (approximately) equal, so that the $G \in \mathcal{G}(\mathbf{d})$ are (approximately) regular, then our results specialize to the case of sparse random graphs considered in [20], except that here we get explicit functions of ϵ (as opposed to the expressions with $o(1)$ terms occurring in [20]). What are other properties which might be included in Theorem 1? Can condition U_{t-1} be replaced by a weaker condition to allow DISC $\Rightarrow$ CIRCUIT$_{2t}$ to be proved? We hope to return to this in the future.

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