Random walks and local cuts in graphs

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Abstract

For a specified subset S of vertices in a graph G we consider local cuts that separate a subset of S. We consider the local Cheeger constant which is the minimum Cheeger ratio over all subsets of S, and we examine the relationship between the local Cheeger constant and the Dirichlet eigenvalue of the induced subgraph on S. These relationships are summarized in a local Cheeger inequality. The proofs are based on the methods of establishing isoperimetric inequalities using random walks and the spectral methods for eigenvalues with Dirichlet boundary conditions.

Key words: Laplacian, local Cheeger constant, local random walk, Dirichlet eigenvalues, isoperimetric inequality.

Mathematics Subject Classification: 05C50, 15A18, 60J10, 68R10

1 Introduction

In the study of large information networks such as Internet graphs, social networks or biological networks [1, 2, 4, 9], it is essential to have local perspectives since global sweeps could be extremely expensive or logistically impossible. In our (global) graph G, we are only concerned about a given subset S of the vertices and their incident edges. In this paper we examine the problem of finding 'good' local cuts restricted to S (detailed definitions to be given in the next section). Of interest is the local cut which has the minimum *local Cheeger ratio* h_S. (The local Cheeger ratio of $T \subset S$ is the ratio of the size of the edge boundary of T and the volume of T.)

One of the ways for controlling the Cheeger ratio of cuts in a graph G is by using spectral methods [3]. The Cheeger constant $h(G)$ of G is defined by

$$h(G) = \min_{X \subset V(G)} \frac{|\partial(X)|}{\min\{\text{vol }(X), \text{vol }(\bar{X})\}} \qquad (1)$$

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where vol (X) denotes the sum of the degrees in X, $\partial(X)$ denote the set of edges leaving X and $\bar{X}$ is the complement of X. Let λ denote the first nontrivial eigenvalue of the (normalized) Laplacian of G. A relationship between λ and $h(G)$ is described by the Cheeger inequality:

$$2h(G) \geq \lambda \geq \frac{h(G)^2}{2}.$$

We will establish a local version of the Cheeger inequality involving a local notion of the eigenvalues — the Dirichlet eigenvalues with Dirichlet boundary condition (as described in Section 2). If the first Dirichlet eigenvalue with Dirichlet boundary condition on the boundary of S is called λ_S, then a local Cheeger inequality is

$$h_S \geq \lambda_S \geq \frac{h_S^2}{2}. \tag{2}$$

We will give two proofs. The first one is to establish (2) using spectral methods. The second proof for a local Cheeger ineqality with slightly weaker lower bound (off by a factor of 2) is by using random walks. Both proofs lead to fast algorithms for finding good local cuts that have local Cheeger ratios within a quadratic bound of the optimum. The first algorithm takes advantage of the corresponding eigenvector and has computational complexity of the same order as computing the eigenvector of a matrix of size $|S| \times |S|$. The second algorithm uses random walks with a similar flavor as the work of Lovász and Simonovits [5]. The algorithm using random walks is quite robust and has a fast approximation algorithm [8].

2 Preliminaries

Suppose a graph G has a vertex set $V(G)$ and edge set $E(G)$. For a subset S of $V(G)$, there are two types of boundary of S — the *vertex boundary* $\delta(S)$ and *edge boundary* $\partial(S)$.

$$\begin{aligned} \delta(S) &= \{u \in V(G) \setminus S \ : \ u \sim v \text{ for some } v \in S\}, \\ \partial(S) &= \{\{u,v\} \in E(G) \ : \ u \in S, v \notin S\}. \end{aligned}$$

For a single vertex v, the *degree* of v, denoted by d_v is equal to $|\delta(v)|$ (which is short for $|\delta(\{v\})|$.

For a subset T, the *local Cheeger ratio* is defined by

$$H(T) = \frac{|\partial(T)|}{\text{vol }(T)}.$$

Note that $H(T)$ is simpler than the way the (usual) Cheeger ratio is defined in (1). These two definitions are equivalent when vol $(T) \leq$ vol $(\bar{T})$.

We are concerned with a *specified* subset $S \subset V(G)$. In the remainder of the paper, we mainly consider *local* vertices (i.e., vertices in S) and *local* edges (i.e., edges incident to S). The *local Cheeger constant* h_S is defined by

$$h_S = \min_{T \subset S} H(T).$$

For a function $f : V \to \mathbb{R}$, suppose that we order the vertices so that

$$f(v_1) \geq f(v_2) \geq \cdots \geq f(v_n).$$

Let $V_f^{(i)}$ denote the set consisting of the i vertices with largest f-values, i.e., $v_1, v_2, \ldots, v_i$. We define the Cheeger ratio with respect to f by

$$h_f = \min_i H(V_f^{(i)}).$$

It is of interest to efficiently find functions f so that the associated cuts using f will be 'good'. In the remainder of the paper we will consider two kinds of functions — the Dirichlet eigenvector and the diffusion determined by random walks. Note that the problem of finding the optimum cut achieving the local Cheeger constant is a NP-hard problem and therefore it is desirable to have efficient ways to find suitable functions f that can lead to local cuts with good local Cheeger ratios as asserted by the local Cheeger inequality.

The *closure* of S, denoted by S^*, is the union of S and δS. For a function $f : S^* \to \mathbb{R}$, we say f satisfies the *Dirichlet boundary condition* if $f(u) = 0$ for all $u \in \delta(S)$. We use the notation $f \in \mathbf{D}_S^*$ to denote that f satisfies the Dirichlet boundary condition and we require that $f \neq 0$.

For $f \in \mathbf{D}_S^*$, we define a Rayleigh quotient:

$$R(f) = \frac{\sum_{x \sim y} (f(x) - f(y))^2}{\sum_{x \in S} |f(x)|^2 d_x}. \quad (3)$$

where the sum is to be taken over all unordered pairs of vertices $x, y \in S^*$ such that $s \sim y$.

The Dirichlet eigenvalue of an induced subgraph on S of a graph G can be defined as follows:

$$\begin{aligned}
\lambda_S &= \inf_{f \in \mathbf{D}_S^*} R(f) \\
&= \inf_{f \in \mathbf{D}_S^*} \frac{\langle f, (D_S - A_S) f \rangle}{\langle f, Df \rangle} \\
&= \inf_{g \in \mathbf{D}_S^*} \frac{\langle g, \mathcal{L} g \rangle}{\langle g, g \rangle} \\
&= \inf_{g \in \mathbf{D}_S^*} \frac{\langle g, \mathcal{L}_S g \rangle}{\langle g, g \rangle},
\end{aligned}$$

where D is the diagonal degree matrix, A is the adjacency matrix, and the Laplacian $\mathcal{L} = D^{-1/2}(D-A)D^{-1/2}$. In addition, $\mathcal{L}_S$ denotes the submatrix of $\mathcal{L}$ with rows and columns restricted to those indexed by vertices in S. The Dirichlet eigenvalues are the eigenvalues of $\mathcal{L}_S$ and λ_S denotes the smallest eigenvalue of $\mathcal{L}_S$. If the induced subgraph on S is connected, then the eigenvector of $\mathcal{L}_S$ associated with λ_S is all positive (using the Perron-Frobenius Theorem [7] on $I - \mathcal{L}_S$). In the remainder of the paper, we are mainly interested in the case that the induced subgraph on S is connected.

3 Finding a good local cut using eigenvectors

First we prove the easy half of the local Cheeger inequality.

Lemma 1 *For a subset S of the vertex set of a graph G, the local Cheeger constant h_S satisfies*

$$h_S \geq \lambda_S$$

where λ_S is the first Dirichlet eigenvalue.

Proof: Suppose that T is a subset which achieves the local Cheeger ratio with respect to S. Let χ_T be defined by

$$\chi_T(v) = \begin{cases} 1 & \text{if } v \in T, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, χ_T satisfies the Dirichlet boundary condition. It is not difficult to verify that

$$\begin{aligned} h_S &= h(T) \\ &= \frac{\langle \chi_T, (D-A)\chi_T \rangle}{\langle \chi_T, D\chi_T \rangle} \\ &\geq \lambda_S \end{aligned}$$

as desired. □

The proof for the local Cheeger inequality is quite similar to and simpler than that for the (usual) Cheeger inequality. For completeness, we include the proof here.

Theorem 1 *In a graph G and a subset S of the vertex set of G, suppose the induced subgraph on S is connected. Then the local Cheeger constant h_S and the Dirichlet eigenvalue λ_S are related by:*

$$h_S \geq \lambda_S \geq \frac{h_f^2}{2} \geq \frac{h_S^2}{2},$$

where $f = gD^{-1/2}$ and g is the eigenfunction of $\mathcal{L}_S$ with eigenvalue λ_S.

Proof: From the definition of Dirichlet eigenvalues, we see that the Rayleigh quotient of f achieves λ_S. We have

$$\begin{aligned}
\lambda_S &= R(f) \\
&= \frac{\sum_{v\sim u}(f(v)-f(u))^2}{\sum_{v\in S} f^2(v)d_v} \\
&= \frac{\sum_{v\sim u}(f(v)-f(u))^2 \sum_{v\sim u}(f(v)+f(u))^2}{\sum_{v\in S} f^2(v)d_v \sum_{v\sim u}(f(v)+f(u))^2} \\
&\geq \frac{\big(\sum_{v\sim u}|f^2(v)-f^2(u)|\big)^2}{2\big(\sum_{v\in S} f^2(v)d_v\big)^2} \\
&\geq \frac{\big(\sum_i (f^2(v_i)-f^2(v_{i+1}))|\partial(V_f^{(i)})|\big)^2}{2\big(\sum_{v\in S} f^2(v)d_v\big)^2} \\
&\geq \frac{\big(\sum_i (f^2(v_i)-f^2(v_{i+1}))h_f \sum_{j\leq i} d_j\big)^2}{2\big(\sum_{v\in S} f^2(v)d_v\big)^2} \\
&\geq \frac{h_f^2}{2} \\
i &\geq \frac{h_S^2}{2}.
\end{aligned}$$

□

Theorem 1 immediately leads to an algorithm for a local cut: First, compute the eigenvector g of the Laplacian restricted to S associated with λ_S. Among the cuts with respect to $f = gD^{-1/2}$, choose the one with the least local Cheeger ratio. Theorem 1 guarantees that this cut is within a quadratic of the optimum.

4 Finding a good local cut using random walks

In a graph G, a typical random walk is determined by the transition probability matrix P, defined by

$$P(u,v) = \begin{cases} \frac{1}{d_u} & \text{if } u \sim v, \\ 0 & \text{otherwise.} \end{cases}$$

The Perron-Frobenius Theorem implies that the random walk with respect to P converges to the stationary distribution if G is connected and not bipartite. Suppose we consider the lazy walk which is the random walk with transition probability matrix $\mathcal{P} = (I+P)/2$. Then the lazy walk converges to its stationary distribution $\pi(v) = d_v/\text{vol }(G)$ if G is connected.

Lovász and Simonovits [5, 6] proved a strong isoperimetric inequality in their work on computing the volume of a convex body. In particular, they considered cuts of the following type:

For a fixed integer k and a fixed vertex v, order the vertices u so that the ratios $\mathcal{P}^k(v,u)/d_u$ are non-increasing. An LS-cut is the boundary of the set, denoted by $S_{j,k,v}$, consisting of the j vertices with the largest j such ratios.

The result of Lovász and Simonivits can be described as follows[1]:

Theorem 2 *In a connected graph G, the lazy walk after t steps satisfies the following:*

$$\mathcal{P}^t(u,v) - \pi(v) \leq (1 - \frac{\beta_{t,v}^2}{8})^t \sqrt{\frac{d_v}{d_u}}$$

where

$$\beta_{t,v} = \inf_{t' \leq t} \inf_j \frac{|\partial(S_{j,t',v})|}{\min\{\text{vol }(S_{j,t',v}), \text{vol }(\bar{S}_{j,t',v})\}}. \tag{4}$$

The theorem of Lovász and Simonovits has strong algorithmic implications. The efficient algorithms using LS-cuts are the basis of numerous works on sparse approximations of graphs [8] and various nearly-linear time algorithms.

The above theorem immediately implies the following:

Theorem 3 *For a connected graph G, the Cheeger constant and the first non-trivial Laplacian eigenvalue λ are related by the following:*

$$2h_G \quad \geq \quad \lambda \quad \geq \quad 1 - \lim_{t\to\infty} (\Delta(t))^{1/t} \quad \geq \quad \frac{\beta_G^2}{8} \quad \geq \quad \frac{h_G^2}{8}$$

where $\beta_G = \inf_{t,v} \beta_{t,v}$, as defined in (4) and $\Delta(t)$ is the relative pointwise distance

$$\Delta(t) = \max_{u,v} \frac{\mathcal{P}^t(u,v) - \pi(v)}{\pi(v)}.$$

For a subset S of a graph G, we define the local random walk by the transition probability matrix P_S:

$$P_S(u,v) \quad = \quad \begin{cases} P(u,v) & \text{if } u,v \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $P_S = D_S^{-1} A_S = D_S^{-1/2}(I - \mathcal{L}_S) D_S^{1/2}$. The function $P_S(u,v)$ satisfies the Dirichlet boundary condition for the set S.

For a fixed integer k and a fixed vertex v, order the vertices u so that the ratios ${\mathcal{P}_S}^k(v,u)/d_u$ are non-increasing. A local LS-cut is the boundary of the

[1] In the paper of Lovász and Simonovits [5], the conductance is defined for a lazy walk and is off by a factor 2 in the definition of β.

set, denoted by $T_{j,k,v,S}$, consisting of the j vertices in S with the largest j such ratios.

We are now ready to state three theorems that relate local random walks, local cuts, and Dirichlet eigenvalues through local Cheeger inequalities. The proofs will be given in the next section.

Theorem 4 *In a connected induced subgraph on S in a graph G, the lazy local walk starting from a fixed vertex v satisfies, for any integer t,*

$$\mathcal{P}_S^t(v,u) \leq (1 - \frac{\beta_{t,v,S}^2}{8})^t \sqrt{\frac{d_u}{d_v}}$$

where $\mathcal{P} = (I+P)/2$ and

$$\beta_{t,v,S} = \inf_{t' \leq t} \inf_j \frac{|\partial(T_{j,t',v,S})|}{\text{vol}\ (T_{j,t',v,S})}.$$

As an immediate consequence, We have the following:

Corollary 1 *In a connected induced subgraph on S in a graph G, the Laplacian $\mathcal{L}_S$ satisfies, for all u, v in S:*

$$(I_S - \frac{\mathcal{L}_S}{2})^t(u,v) \leq (1 - \frac{\beta_{t,S}^2}{8})^t$$

for all integers t where

$$\beta_{t,S} = \inf_{\substack{t' \leq t \\ v \in S}} \beta_{t',v,S}.$$

Theorem 5 *In a connected induced graph G_S, the local Cheeger constant h_S and the Dirichlet eigenvalue λ_S are related as follows:*

$$h_S \ \geq \ \lambda_S \ \geq \ \frac{\beta_{t,S}^2}{4} - \frac{\log|S|}{t} + O\big((\frac{\log|S|}{t})^2\big)$$

where for any integer t, we let $\beta_{t,S}$ denote the smallest Cheeger ratio of the LS-cuts for lazy random walks $\mathcal{P}^t(v,\cdot)$, for $v \in S$ of no more than t steps.

5 Proving local Cheeger inequalities

We first state several facts that can be used to prove Theorems 3, 4 and 5. These facts are based on the ideas in [5]. For completeness, we give the proofs below.

In this section, a function $f : V \to \mathbb{R}$ is represented by a row vector. For any square matrix M with rows and columns indexed by V and $v \in V$, we use the notation $fM(v)$ to denote the entry of the row vector fM with index v.

Fact 1 *In a connected graph G with vertex set V, for a function $f : V \to \mathbb{R}$ and any subset T of vertices and for $\mathcal{P} = (I + P)/2$, we can write*

$$\sum_{v \in T} f D \mathcal{P}(v) = \frac{f \cdot g_1 + f \cdot g_2}{2}$$

where g_1 and g_2 are both column vectors with non-negative entries satisfying

$$\begin{aligned} 0 \ &\leq \ g_1(v) \ \leq \ d_v, \\ 0 \ &\leq \ g_2(v) \ \leq \ d_v, \\ \sum_v g_1(v) \ &= \ \text{vol }(T) - |\partial(T)|, \\ \sum_v g_2(v) \ &= \ \text{vol }(T) + |\partial(T)|. \end{aligned}$$

Proof:

$$\begin{aligned} \sum_{v \in T} f D \mathcal{P}(v) &= \sum_{v \in T} \sum_u f(u) d_u \mathcal{P}(u, v) \\ &= \sum_{u \in T} f(u) d_u \sum_{v \in T} \mathcal{P}(u, v) + \sum_{u \notin T} f(u) d_u \sum_{v \in T} \mathcal{P}(u, v) \\ &= \sum_{u \in T} f(u) d_u \frac{1 + \sum_{v \in T} P(u, v)}{2} + \sum_{u \notin T} f(u) d_u \frac{\sum_{v \in T} P(u, v)}{2} \\ &= \frac{1}{2} \sum_{u \in T} f(u) d_u (1 - \sum_{v \notin T} P(u, v)) \\ &\quad + \frac{1}{2} \sum_{u \in T} f(u) d_u + \frac{1}{2} \sum_{u \notin T} f(u) d_u \sum_{v \in T} P(u, v) \\ &= \frac{f \cdot g_1 + f \cdot g_2}{2}. \end{aligned}$$

Note that

$$\begin{aligned} \sum_u g_1(u) &= \sum_{u \in T} d_u (1 - \sum_{v \notin T} P(u, v)) = \text{vol }(T) - |\partial(T)| \\ \text{and} \quad \sum_u g_2(u) &= \sum_{u \in T} d_u + \sum_{u \notin T} \sum_{v \in T} d_u P(u, v) = \text{vol }(T) + |\partial(T)|. \end{aligned}$$

□

The following is a consequence of Fact 1.

Fact 2 *In a connected graph G with vertex set V, for a function $f : V \to$*

$\mathbb{R}^+ \cup \{0\}$ and subsets $T \subset S$ of vertices and for $\mathcal{P} = (I+P)/2$, we can write

$$\begin{aligned} \sum_{v \in T} f_S D_S \mathcal{P}_S(v) &\leq \sum_{v \in T} f D \mathcal{P}(v) \\ &= \frac{f \cdot g_1 + f \cdot g_2}{2} \end{aligned}$$

where g_1 and g_2 are both column vectors with non-negative entries satisfying

$$\begin{aligned} 0 &\leq g_1(v) \leq d_v, \\ 0 &\leq g_2(v) \leq d_v, \\ \sum_v g_1(v) &= \text{vol } (T) - |\partial(T)|, \\ \sum_v g_2(v) &= \text{vol } (T) + |\partial(T)|. \end{aligned}$$

For $f : V \to \mathbb{R}$ and any positive real $x \leq \sum_v d_v$, let $\hat{f}(x)$ denote

$$\hat{f}(x) = \max\{f \cdot g : 0 \leq g(v) \leq d_v \text{ for all } v \text{ and } \sum_v g(v) = x\}.$$

It is not hard to check that if $f(v_1) \geq f(v_2) \geq \cdots \geq f(v_n)$ and $\sum_{i=1}^{k} d_{v_i} \leq x < \sum_{i=1}^{k+1} d_{v_i}$, then

$$\hat{f}(x) = \sum_{i=1}^{k} f(v_i) d_{v_i} + f(v_{k+1})(x - \sum_{i=1}^{k} d_{v_i}).$$

Proof of Theorem 4:
We now consider, for a fixed integer k and a vertex u,

$$f_k(v) = \frac{\mathcal{P}_S^k(u,v)}{d_v}.$$

Clearly,

$$f_k D_S \mathcal{P}_S(v) = \mathcal{P}_S^{k+1}(u,v) = f_{k+1} d_v.$$

Fact 3 *$\hat{f}_k$, as defined above, satisfies*

$$\hat{f}_{k+1}(x) \leq \frac{\hat{f}_k(x(1-\beta_{k,u})) + \hat{f}_k(x(1+\beta_{k,u}))}{2}.$$

Proof: For any subset T of vertices, we can apply Fact 2 and obtain

$$\begin{aligned} \sum_{v \in T} \mathcal{P}_S^{k+1}(u,v) &= \sum_{v \in T} f_k D_S \mathcal{P}_S(v) \\ &\leq \frac{f_k \cdot g_1 + f_k \cdot g_2}{2} \\ &\leq \frac{\hat{f}_k(\text{vol } (T) - |\partial(T)|) + \hat{f}_k(\text{vol } (T) + |\partial(T)|)}{2}. \end{aligned}$$

In particular, by taking T to be the set of vertices with the j largest values of f_{k+1} and $y = \text{vol}\,(T)$, we have

$$\begin{aligned}
\hat{f}_{k+1}(y) = \hat{f}_{k+1}(\text{vol}\,(T)) &\leq \frac{\hat{f}_k(\text{vol}\,(T) - |\partial T|) + \hat{f}_k(\text{vol}\,(T) + |\partial T|)}{2} \\
&\leq \frac{\hat{f}_k(y(1-\beta_{k,u})) + \hat{f}_k(y(1+\beta_{k,u}))}{2}. \qquad (5)
\end{aligned}$$

This allows us to extend the inequality (5) to all other values, say, $x = \text{vol}\,(T) + \alpha d_{v_{j+1}}$, for some $\alpha \leq 1$, by considering the following linear combination:

$$\begin{aligned}
\hat{f}_k(x) &\leq \alpha \hat{f}_k(\text{vol}\,(T')) + (1-\alpha)\hat{f}_k(\text{vol}\,(T)) \\
&\leq \frac{\hat{f}_k(x(1-\beta_{k,u})) + \hat{f}_k(x(1+\beta_{k,u}))}{2}
\end{aligned}$$

where $T' = T \cup \{v_{j+1}\}$. □

Fact 4 *For nonnegative integers k,*

$$\hat{f}_k(x) \leq (1 - \frac{\beta_{k,u}^2}{8})^k \frac{\sqrt{x}}{\sqrt{d_u}}.$$

Proof: The proof is by induction. For $k = 0$,

$$\hat{f}_0(d_u) \leq 1 \qquad \text{and} \qquad \hat{f}_0(x) \leq \frac{\min\{x, d_u\}}{d_u} \leq \frac{\sqrt{x}}{\sqrt{d_u}}.$$

By using the induction hypothesis and Fact 3, we have

$$\begin{aligned}
\hat{f}_{k+1}(x) &\leq (1 - \frac{\beta_{k,u}^2}{8})^k \sqrt{\frac{x}{d_u}} (\frac{\sqrt{1-\beta_{k+1,u}} + \sqrt{1+\beta_{k+1,u}}}{2}) \\
&\leq (1 - \frac{\beta_{k,u}^2}{8})^k \sqrt{\frac{x}{d_u}} (1 - \frac{\beta_{k+1,u}^2}{8}) \\
&\leq (1 - \frac{\beta_{k+1,u}^2}{8})^{k+1} \sqrt{\frac{x}{d_u}}
\end{aligned}$$

since $\beta_{k,u} \geq \beta_{k+1,u}$ and $\frac{1}{2}(\sqrt{1-z} + \sqrt{1+z}) \leq 1 - z^2/8$ for $z \in (0,1)$. Fact 4 is proved. □

Since $\mathcal{P}_S^k(u,v) \leq \hat{f}_k(d_v)$, Theorem 4 follows from Fact 4.

Proof of Theorem 2:
We use Fact 1 but with a different function F_k (instead of f_k):

$$F_k(v) = \frac{\mathcal{P}^k(u,v) - \pi(v)}{\pi(v)},$$

where $\pi(v) = d_v/\text{vol }(G)$. Clearly,

$$F_k \Pi \mathcal{P}(v) = \mathcal{P}^{k+1}(u,v) - \pi(v) = F_{k+1}\pi(v)$$

where Π is the diagonal matrix with entries $\Pi(v,v) = \pi(v)$.

In a similar way as in the proof of Theorem 4 using Fact 1, it can be checked that $\hat{F}_k$ satisfies, for $x \le 1/2$,

$$\hat{F}_{k+1}(x) \quad \le \quad \frac{\hat{F}_k(x(1-\beta_{k,u})) + \hat{F}_k(x(1+\beta_{k,u}))}{2}$$

and for $1 \ge x > 1/2$,

$$\hat{F}_{k+1}(x) \quad \le \quad \frac{\hat{F}_k(x-\beta_{k,u}(1-x)) + \hat{F}_k(x+\beta_{k,u}(1-x))}{2}.$$

We can then prove again by induction that

$$\hat{F}_k(x) \le (1 - \frac{\beta_{k,u}^2}{8})^k \frac{\min\{\sqrt{x}, \sqrt{1-x}\}}{\sqrt{d_u}}.$$

This implies that

$$\mathcal{P}^k(u,v) - \pi(v) \le \hat{f}_k(\pi(v)) \le (1 - \frac{\beta_{k,u}^2}{8})^k \sqrt{\frac{d_v}{d_u}}.$$

Theorem 2 is proved. □

Proof of Theorem 5:
We consider a connected induced subgraph on S. Let T denote a subset of S with the least local Cheeger ratio. Let λ_T denote the first Dirichlet eigenvalue for the induced subgraph on T.

Let χ_T denote the characteristic function with $\chi_T(x) = 1$ if $x \in T$ and 0 otherwise. From the definitions, the local Cheeger ratio $H(T)$ can be written as the Rayleigh quotient and therefore satisfies

$$h_S = H(T) = R(\chi_T) \ge \lambda_T.$$

From the definition, we also have

$$\lambda_T \ge \lambda_S$$

since $T \subseteq S$.

Let φ denote the eigenvector of $\mathcal{L}_S$ with all positive entries.

We consider

$$\langle \varphi, (I_S - \frac{\mathcal{L}_S}{2})^t \varphi \rangle = \big(1 - \frac{\lambda_S}{2}\big)^t \langle \varphi, \varphi \rangle.$$

On the other hand, we have

$$
\begin{aligned}
\langle \varphi, (I_S - \frac{\mathcal{L}_S}{2})^t \varphi \rangle &= \sum_{x,y \in S} \varphi(x)(I_S - \frac{\mathcal{L}_S}{2})^t(x,y)\varphi(y) \\
&\leq \big(\sum_{x \in S} \varphi(x)\big)^2 (1 - \frac{\beta_{t,S}^2}{8})^t \\
&\leq (1 - \frac{\beta_{t,S}^2}{8})^t |S| \sum_{x \in S} \varphi^2(x)
\end{aligned}
$$

by using Corollary 1.

Therefore we have

$$
\big(1 - \frac{\lambda_S}{2}\big)^t \leq |S|(1 - \frac{\beta_{t,S}^2}{8})^t.
$$

Therefore

$$
\begin{aligned}
\frac{\lambda_S}{2} &\geq 1 - (1 - \frac{\beta_{t,S}^2}{8})|S|^{1/t} \\
&= \frac{\beta_{t,S}^2}{8} - (1 - \frac{\beta_{t,S}^2}{8})(|S|^{1/t} - 1) \\
&\geq \frac{\beta_{t,S}^2}{8} - (|S|^{1/t} - 1) \\
&= \frac{\beta_{t,S}^2}{8} - (e^{(\log |S|)/t} - 1) \\
&\geq \frac{\beta_{t,S}^2}{8} - \frac{\log |S|}{t} + O((\frac{\log |S|}{t})^2)
\end{aligned}
$$

as claimed. Theorem 5 is proved.

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