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The Laplacian of a Hypergraph

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1. Introduction

Suppose G is a graph with node set N and edge set E consisting of unordered pairs of N. The Laplacian of G, denoted by $L(G)$, is defined to be $D - A$ where A is the adjacency matrix of G (i.e., $A_{ij} = 1$ if $\{i, j\}$ is in E and 0 otherwise), and D is a diagonal matrix with $(D)_{ii} = d(i)$, the degree of the i-th node. Laplacians and the distribution of their eigenvalues imply many important properties of graphs [7,14,15,18,22,29], and lead to many applications in a variety of areas [2,6,13,27,28,32,34,35]. A natural generalization of graphs are so-called hypergraphs. In particular, a k-uniform hypergraph (or, a k-graph for short) has a node set N and edges consisting of k-subsets of N. (Thus, ordinary graphs are 2-graphs.) Many attempts have been made to define the analogue of the Laplacian for hypergraphs and/or some notion of eigenvalues of k-graphs [3,16]. However, various obstructions seem to make the generalization to k-graphs difficult.

In this paper, we will define the Laplacian of a k-graph by considering various homological aspects of hypergraphs. The eigenvalues of the Laplacians will be examined and relations to the other graph properties will be derived. In particular, the eigenvalues of some specified hypergraphs will be evaluated.

In an earlier paper [10], the cohomological aspects of hypergraphs over the finite field Z_2 (and, in general, Z_p) were investigated. The Laplacians for the case of Z_2 have quite different properties from the Laplacians considered here. In this paper, since the operations are over C, the field of complex numbers, the eigenvalues of the Laplacian can be considered. This paper is organized as follows. The definition of the Laplacian will be given in Section 2. The homological

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setting will be described in Sections 3 and 4. Various properties of the eigenvalues of the Laplacian are discussed in Section 5. Some special hypergraphs will be discussed in Section 6. In Section 7 we consider the Laplacian of random graphs. In Section 8, relations of the Laplacian to other graph invariants are discussed and further problems are raised.

2. The definition of the Laplacian

Suppose G is a k-graph with node set N and edge set E which is a subset of $\binom{N}{k}$, the set of all k-sets of N. For a $(k-1)$-subset x of N, its degree, denoted by $d(x)$, is $|\ \{y \in E : x \subset y\}\ |$. The average degree d is $\frac{1}{\binom{n}{k-1}} \sum_{x \in \binom{N}{k-1}} d(x)$ where the cardinality of N is n. We say G is d-regular if $d(x) = d$ for all x. The Laplacian of G involves the following matrices whose columns and rows are indexed by $\left[{N \atop k-1}\right]$ the set of $(k-1)$-tuples of distinct elements in N:

(i): $D = D(G)$, the diagonal matrix. For $x \in \left[{N \atop k-1}\right]$, $D(x,x)$ is defined to be $d(x)$. Also, $D(x,y) = 0$ if $x \neq y$.

(ii): $A = A(G)$, the adjacency matrix. For $x, y \in \left[{N \atop k-1}\right]$, $A(x,y)$ is 1 if $x = x_1x_2 \ldots x_{k-1}$, $y = y_1x_2 \ldots x_{k-1}$, where $x_i, y_j \in N$, and $y_1, x_1, \ldots, x_{k-1}$ is in E and 0 otherwise. Therefore, $A(x,x)$ is zero.

(iii): $K = K(G)$, the complete graph. That is, for $x, y \in \left[{N \atop k-1}\right]$, $K(x,y)$ is 1 if $x = x_1x_2 \ldots x_{k-1}$ and $y = y_1x_2 \ldots x_{k-1}$ for $x_1 \neq y_1$ and 0 otherwise.

(iv): $I = I_{k-1}$, the identity matrix.

Now we are ready to define the Laplacian $L(G)$ of a k-graph G, where $k \geq 3$, as follows:

$$L(G) = D - A + \rho(K + (k-1)I).$$

where $\rho = d/n$ is called the density of G.

Sometimes we write $\hat{K} = K + (k-1)I$ and $L(G) = D - A + \rho\hat{K}$ for a k-graph G, $k \geq 3$. When $k = 2$, we have $L(G) = D - A$. As we shall see, the Laplacian corresponds in a natural way to a self-adjoint operator $\partial\delta + \rho\delta\partial$ for some simplicial complex where ρ is a positive constant (uniquely determined by the graph G), as we will discuss in Section 3 and 4. Furthermore, the eigenvalues of the Laplacian defined above play an important role in capturing the essential properties of the graphs.

Lemma 2.1. If $k \geq 3$, $L(G)$ has an eigenvalue ρn with the corresponding eigenvector having all coordinates 1's.

Let f_1 denote the vector with all coordinates 1's. Since we have

$$\begin{aligned} (D - A)f_1 &= 0, \\ (K + (k-1)I)f_1 &= nf_1, \end{aligned}$$

we conclude that $Lf_1 = \rho n f_1$, and f_1 is a eigenvector of $L(G)$ with eigenvalue ρn.

Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\left[{n \atop k-1}\right]}$ denote the eigenvalues of the adjacency matrix A of a d-regular k-graph G. The well-known results of Perron-Frobenius [17,21,31] state that $\lambda_1 = d$ and for $i \neq 1$, $|\lambda_i| \leq d$ (except when A is reducible). Since the eigenvectors associated with $\lambda_i, i \neq 1$, are orthogonal to the all 1's vector, it is easy to see that the Laplacian $L(G)$ has eigenvalues $d, d - \lambda_2, \ldots, d - \lambda_n$.

For a general k-graph G, we define λ_i so that $L(G)$ has eigenvalues $d - \lambda_i$ where d is the average degree. Let us define the spectral value of $L(G)$ to be $\lambda = \lambda(G) = \max_{i \neq 1} |\lambda_i|$. If we can find a good upper bound for λ, then several isoperimetric properties of the k-graphs can be derived. For example, for 2-graphs, the "smallness" of λ implies various properties of the graphs such as: the expansion property (each subset of the node has "many" neighbors), the discrepancy property (each subset induces about the average number of edges), among others. The reader is referred to [7,11] for more details on 2-graphs.

3. The Laplacian of 2-graphs

The Laplacian of a 2-graph is quite simple in comparison to the Laplacian for general k-graphs. Still, it is helpful to review the homological setting for the case of $k = 2$. Throughout this section G is a 2-graph with node set $N = N(G)$ and edge set $E = E(G)$ consisting of unordered pairs of N. We can define the 1-simplicial complex C_1 to be a vector space over R generated by all ordered pairs of N. In addition, we require $(u, v) = -(v, u)$. (Namely, C_1 can be viewed as an exterior algebra with $u \wedge v = -v \wedge u = (u, v)$. We will not use "$\wedge$" notation here.) The boundary operator ∂ is defined by $\partial(u, v) = u - v$ and

$$C_1 \underset{\delta}{\overset{\partial}{\rightleftarrows}} C_0$$

In other words, $\partial : C_1 \to C_0$ can also be interpreted as a matrix W of size $n(n-1) \times n$ where the rows are indexed by $\left[{N \atop 2}\right]$ and the columns are indexed by $\left[{N \atop 1}\right]$. For $x \in \left[{N \atop 1}\right]$, $y \in \left[{N \atop 2}\right]$,

$$W(x, y) = \begin{cases} 1 & \text{if } x = (y, u) \text{ for some } u \\ -1 & \text{if } x = (v, y) \text{ for some } v, \\ 0 & \text{otherwise.} \end{cases}$$

The coboundary operator δ is just the transpose of W.

For a 2-graph G, let $\hat{G}$ denote an orientation of G. That is, each edge (i.e., unordered pair) is assigned a direction (i.e., an ordered pair) and together the directed edges form $E(\hat{G})$. $C_1(\hat{G})$ is defined to be $C_1 \cap E(\hat{G})$ and $C_0(\hat{G}) = \{u \in N : u \text{ is in some edge of } E(G)\}$. As we can see, the Laplacian is independent of the choice of the orientation of G.

The boundary operator ∂ and the coboundary operator δ are just the restriction of the above operators to $C_1(G)$ and $C_0(G)$,

$$C_1(\hat{G}) \underset{\delta}{\overset{\partial}{\rightleftarrows}} C_0(\hat{G}),$$

∂ corresponds to the matrix $W_{(G)}$ with

$$W^{(G)}(x,y) = \begin{cases} W(x,y) & \text{if } x \in C_1(\hat{G}) \text{ and } y \in C_0(\hat{G}) \\ 0 & \text{otherwise} \end{cases}$$

and δ corresponds to the transpose of $W^{(G)}$.

We have $L(G) = \delta\partial = W^{(G)}\ (W^{(G)})^T = D - A$. Therefore $L(G)$ is semi-positive definite and has one eigenvalue 0 with corresponding eigenvector being the all 1's vector. In general, for a function $f : N \to C$, we have $Lf(v) = \delta\partial(v) = \sum_{\substack{u \\ u,v \in E(G)}} (f(v) - f(u))$. The Laplacians $L(G)$ can be viewed as a discrete analog of the (continuous) Laplace operator on a manifold which maps a function to the corresponding function involving the differences from neighboring points.

4. A homology theory for hypergraphs

Suppose $k > 1$. The simplicial complex C_k is a vector space over R generated by $[{N \atop k+1}]$ satisfying the property that for f in C_k and $x, y \in [{N \atop k+1}]$, $f(x) = (-1)^t f(y) = sign(x,y)f(y)$ where t is the number of transpositions in xy^{-1} if x is a permutation of y. For $x \in [{N \atop t}]$ and $y \in [{N \atop t-1}], sign(x,y)$ is defined to be $sign(x, uy)$ where u is in x but not in y. The boundary operator ∂ can be described as follows:

$$\cdots \underset{\delta}{\overset{\partial}{\rightleftarrows}} C_{k-1} \underset{\delta}{\overset{\partial}{\rightleftarrows}} C_{k-2} \underset{\delta}{\overset{\partial}{\rightleftarrows}} \cdots$$

For x in $C_{k-1}, \partial x = \sum_y sign(x,y)y$ where y is a permutation of a $(k-1)$-subset of x.

The coboundary operator δ is the dual of ∂. It is easy to verify the following:

Lemma 4.1. $\partial\partial = 0$ and $\delta\delta = 0$.

Let G be a k-graph. For each edge $x = \{x_1, \dots, x_k\}$ of G, we can choose a k-tuple $\hat{x} = x_1 \dots x_k$. These $\hat{x}$ consequently form $C_{k-1}(\hat{G})$ (as it turns out, the choice of the permutations $\hat{x}$ does not affect the Laplacian of G). For $r < k$, $C_r(\hat{G})$ consists of all r-tuples y so that y is contained in ∂x for some x in $C_{r+1}(\hat{G})$.

We consider the Laplacian $\partial\delta + \rho\delta\partial$ restricted to $C_{k-1}(\hat{G}), C_{k-2}(\hat{G})$ and $C_{k-3}(\hat{G})$. It is not difficult to verify the following.

Lemma 4.2 For $y, z \in C_{k-2}(\hat{G})$ $k \geq 3$,

$$\partial\delta z(y) = \begin{cases} \operatorname{sign}(x,y)\operatorname{sign}(x,z) & \text{if the union of the elements of } y \text{ and } z \text{ forms} \\ & \quad \text{an edge } x \text{ of } G, \\ d(y)\cdot \operatorname{sign}(y,z) & \text{if } z \text{ is a permutation of } y, \\ 0 & \text{otherwise,} \end{cases}$$

where $\partial\delta z(y)$ denotes the coefficient of y in $\partial\delta z$. In other words, $\partial\delta z = \sum_y \partial\delta z(y) y$. In a similar way, we have the following:

Lemma 4.3. For $y, z \in C_{k-2}(\hat{G})$,

$$\partial\delta z(y) = \begin{cases} \operatorname{sign}(y,x')\operatorname{sign}(z,x') & \text{if the intersection of elements of } y \text{ and } z \\ & \quad \text{is a } (k-2)\text{-set } x', \\ (k-1)\operatorname{sign}(y,z) & \text{if } z \text{ is a permutation of } y, \\ 0 & \text{otherwise.} \end{cases}$$

Now, let f denote a cochain in $C_{k-2}(G)$. That is, $f = \sum_{x\in\left[{N \atop k-1}\right]} f(x)\cdot x$ and f satisfies $f(x) = \operatorname{sign}(x,y)f(y)$ if y is a permutation of x.

Here we use the following notation: For a $(k-1)$-tuple $x = x_1 \dots x_{k-1}$ in $\left[{N \atop k-1}\right]$, we define $\bar{x} = \sum_{y,\tilde{y}=\tilde{x}} \operatorname{sign}(y,x)\cdot y$ where $\tilde{x} = x_1, \dots, x_{k-1}$. Therefore

$$\begin{aligned} f(\bar{x}) &= \sum_{\substack{y \\ \tilde{y}=\tilde{x}}} \operatorname{sign}(y,x) f(y) \\ &= (k-1)! f(x). \end{aligned}$$

We consider the following:

$$\begin{aligned} \partial\delta f(y) &= \sum_{\substack{z \\ zw=\tilde{y}}} d(y)\operatorname{sign}(y,z)f(z) + \sum_{\substack{z \\ \tilde{y}\cup zw=\tilde{x}\in E}} \operatorname{sign}(x,y)\operatorname{sign}(x,z)f(z) \\ &= d(y)f(\bar{y}) + \sum_{\substack{w\in\left[{N \atop k-2}\right] \\ \tilde{w}\subset\tilde{y}}} \sum_{\substack{u\in N-\tilde{y} \\ \tilde{x}=\overline{uy}\in E}} \operatorname{sign}(x,y)\operatorname{sign}(x,uw)f(\overline{uw}) \\ &= Df(\bar{y}) - \sum_{\substack{w\in\left[{N \atop k-2}\right] \\ \tilde{w}\cup v=\tilde{y}}} \operatorname{sign}(y,vw)Af(\overline{vw}) \\ &= Df(\bar{y}) - Af\Big(\sum_{\substack{w\in\left[{N \atop k-2}\right] \\ \tilde{y}=\tilde{w}\cup v}} \operatorname{sign}(y,vw)\overline{vw}\Big) \\ &= Df(\bar{y}) - Af(\bar{y}) \end{aligned}$$

In a similar way, we have

$$\begin{aligned}
\delta\partial f(y) &= \sum_{\substack{z\\ zw=\tilde{y}}} (k-1)\,\mathrm{sign}(y,z)f(z) + \sum_{\substack{z\\ \tilde{y}\cap zw=\tilde{w}\in\binom{N}{k-2}}} \mathrm{sign}(y,w)\,\mathrm{sign}(z,w)f(z)\\
&= (k-1)f(\bar{y}) + \sum_{\substack{w\in\left[{N \atop k-2}\right]\\ \tilde{w}\subset\tilde{y}}} \mathrm{sign}(y,w) \sum_{\substack{u\in N-\tilde{y}\\ zw=\tilde{w}\cup u}} \mathrm{sign}(z,uw)f(z)\\
&= (k-1)f(\bar{y}) + \sum_{\substack{w\in\left[{N \atop k-2}\right]\\ \tilde{w}\subset\tilde{y}}} \mathrm{sign}(y,w) \sum_{u\in N-\tilde{y}} f(\overline{uw})\\
&= (k-1)f(\bar{y}) + \sum_{\substack{w\in\left[{N \atop k-2}\right]\\ \tilde{y}=\tilde{w}\cup v}} \mathrm{sign}(y,w)Kf(\overline{vw})\\
&= (k-1)f(\bar{y}) + Kf(\bar{y})
\end{aligned}$$

Hence $\delta\partial f(y) = ((k-1)I + K)(\bar{y})$.

Therefore we can view $L(G)$ as $\partial\delta + \rho\delta\partial$ where ρ is the density of G.

In particular, d-regular graphs have edge density $\rho = \frac{d}{n}$ and therefore we have

$$L(G) = dI - A + \frac{d}{n}\hat{K}.$$

We remark that one of the main reasons for the different formulation of the Laplacians for 2-graphs and for k-graphs, $k \geq 3$, is that $C_{k-2}, k \geq 3$ is generated by an oriented basis while the opposite is true for $k = 2$. In fact for $k \geq 3$, the cochains form a vector space of dimension $\binom{n}{k-1}$. Although the Laplacian L has as an eigenvector the all 1's vector (as seen in Lemma 2.1), it is easy to see that all the cochains are orthogonal to the all 1's vector when $k \geq 3$, while this is not so for $k = 2$.

5. Spectral values of the Laplacian

Throughout this section, we again assume G is a d-regular k-graph. We will prove a number of isoperimetric inequalities in terms of the spectral values. First, we prove a lower bound for the spectral value $\lambda(G)$.

THEOREM 5.1. $\lambda(G) \geq \sqrt{d(1-d/n)}$

Proof: We consider $M = dI - L = A - \frac{d}{n}\hat{K}$. It is easy to check that M has eigenvalues $\lambda_i, 1 \leq i \leq \left[{n \atop k-1}\right]$ where $\left[{n \atop k-1}\right] = n(n-1)\ldots(n-k-2)$, satisfying $\lambda_1 = 0$ and the spectral value $\lambda = max_{i\neq 1}\lambda_i$. We consider the trace of MM^T. We have

$$\begin{aligned}([{n \atop k-1}]-1)\lambda^2 &\geq \sum_i \lambda_i^2\\ &= TrMM^T\\ &= d\,[{n \atop k-1}]-\frac{d^2}{n}\,[{n \atop k-1}]\end{aligned}$$

Therefore

$$\lambda^2 \geq d(1-d/n)$$

and Theorem 5.1 is proved.

As we will see in Section 7, a random d-regular graph has the spectral value of size $O(\sqrt{d})$. Furthermore several explicit constructions in Section 6 also achieve a spectral value of size $O(\sqrt{d})$, which is within a constant factor of the least possible value.

We now consider using the spectral value to derive isoperimetric inequalities for a graph G.

THEOREM 5.2. *Let S be a subset of the node set N of a d-regular k-graph G. The number $e(S)$ of edges x of G with $x \subseteq S$ satisfies the following:*

$$\left|e(S)-\frac{d}{n}\binom{|S|}{k}\right| \leq \frac{\lambda}{k}\binom{|S|}{k-1}+\frac{d(k-1)}{kn}\binom{|S|}{k-1}$$

Proof: We consider a vector f indexed by $[{N \atop k-1}]$, satisfying $f(x)=1$ if $\tilde{x}\subseteq S$ and $f(x)=0$ otherwise. We now consider the bilinear product $\langle f, Lf\rangle$ where $L=L(G)$. It is easily seen that

$$\langle f, Af\rangle = k!e(S).$$

Since $\langle f, \frac{d}{n}\hat{K}\ f\ \rangle = \frac{d}{n}k!\ \binom{|S|}{k}+\ \frac{d}{n}(k-1)[{|S| \atop k-1}]$, we conclude

$$\begin{aligned}k!\left(e(S)-\frac{d}{n}\binom{|S|}{k}\right) &- \frac{d(k-1)}{n}[{|S| \atop k-1}]\\ &= \langle f,(L-D)f\rangle\\ &\leq \lambda\langle f,f\rangle\\ &= \lambda[{|S| \atop k-1}].\end{aligned}$$

Theorem 5.2 is proved.

We note that $\frac{d}{n}\ \binom{|S|}{k}$ is the expected number of edges contained in $|S|$. The quantity $|e(S)-\frac{d}{n}\ \binom{|S|}{k}|$ is often called the discrepancy of S, and the discrepancy of G is defined to be the maximum discrepancy over all subsets S, (see [5,7,11] for more discussion on this). The above theorem provides an upper bound for discrepancy in terms of the spectral value.

The inequality in (1) implies that if the spectral value is small in comparison with d, then the number of edges in S is close to the expected quantity. As an immediate consequence, the number of edges involving nodes in S but not entirely contained in S is also close to the expected value (which is almost all the edges involving S, when $|S|$ is small in comparison with n.)

The following isoperimetric equality is slightly stronger than that in Theorem 5.2.

THEOREM 5.3. *Let F denote a subset of $\binom{N}{k-1}$ where N is the node set of a d-regular k-graph G. We define $e_G(F) = |\{(f_1, f_2, x) : f_1, f_2 \in F, f_1 \cup f_2 = x \in E\}|$. Then we have*

$$|e_G(F) - \frac{d}{n}e_K(F)| \leq \frac{\lambda}{k}|F| + \frac{d(k-1)}{kn}|F|$$

where K denotes the complete graph with edge set $\binom{N}{k}$.

Proof: We consider a vector f indexed by $\left[{N \atop k-1}\right]$ with $f(x) = 1$ if $\tilde{x} \in F$ and 0 otherwise. We consider

$$\langle f, Af\rangle = k!e_G(F).$$

Also,

$$\langle f, \frac{d}{n}\hat{K}f\rangle = \frac{d}{n}k!e_K(F) + \frac{d}{n}(k-1)|F|(k-1)!$$

Therefore

$$\begin{aligned} k!(e_G(F) - \frac{d}{n}e_K(F) & - & \frac{d}{kn}(k-1)|F|) \\ & = & \langle f, (L-D)f\rangle \\ & \leq & \lambda\langle f, f\rangle \\ & = & \lambda|F|(k-1)! \end{aligned}$$

It would be useful to prove the reverse directions of the above theorem by bounding the spectral value in terms of the discrepancy. However, the following problem for 2-graphs still remains unresolved [7].

Conjecture: Suppose G is a 2-graph satisfying

$$\left|e(S) - \rho\binom{|S|}{2}\right| \leq \alpha|S|$$

where ρ is the density. Is it true that $\lambda(G) \leq c\alpha$ for some absolute constant c?

Now, we consider isoperimetric inequalities of a somewhat different flavor. For a subset S of the node set N of a k-graph G, we define the neighborhood of S as follows:

$\Gamma(S) = \{u \in N : \{u\} \cup w \in E(G)$ for some $(k-1)$-subset w of $S\}$.

THEOREM 5.4. *Let S be a subset of the node set N of a d-regular k-graph G where $k \geq 3$. Then for any $S \subseteq N$, we have*

$$|\Gamma(S)| \geq \frac{d^2\left[{|S| \atop k-1}\right]}{\lambda^2\left(1-\frac{\left[{|S| \atop k-1}\right]}{\left[{n \atop k-1}\right]}\right)+\frac{d^2|S|}{kn}}.$$

Proof: We define a vector f indexed by $\left[{N \atop k-1}\right]$, so that $f(x) = 1$ if $\tilde{x} \subseteq S$ and 0 otherwise. Suppose that the eigenvalues of the Laplacian $L(G)$ are $d - \lambda_i$ where $\lambda_1 = 0$ and the orthonormal eigenvectors are denoted by v_i.

Suppose $f = \sum a_i v_i$ and therefore $\sum a_i^2 = ||f||^2 = \left[{|S| \atop k-1}\right]$. We consider the following inner product:

$$\begin{aligned}
&\langle \quad f(A-\frac{d}{n}\hat{K}),(A-\frac{d}{n}\hat{K})f\rangle \\
&= \quad \langle fA, Af\rangle - 2\langle fA, \frac{d}{n}\hat{K}f\rangle + \frac{d^2}{n^2}\langle f\hat{K}, \hat{K}f\rangle \\
&= \quad \langle fA, Af\rangle - \frac{d^2}{n^2}\langle f\hat{K}, \hat{K}f\rangle \\
&= \quad \langle fA, Af\rangle - \frac{d^2}{n}\left[{|S| \atop k}\right].
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\langle f(dI-L),(dI-L)f\rangle &= \langle f(A-\frac{d}{n}\hat{K}),(A-\frac{d}{n}\hat{K})f\rangle \\
&= \sum a_i^2\lambda_i^2 \\
&= \leq \lambda^2\left(\left[{|S| \atop k-1}\right] - a_1^2\right) \\
&= \lambda^2\left(\left[{|S| \atop k-1}\right] - \frac{\left[{|S| \atop k-1}\right]^2}{\left[{n \atop k-1}\right]}\right)
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\langle fA, Af\rangle &= \sum_{w,w' \in \left[{S \atop k-1}\right]} |\{u \in N : w = aw'', w' = bw'' \text{ and } \overline{abw''} \in E\}| \\
&= \sum_{u \in N} |\{w \in \left[{S \atop k-1}\right] : \overline{uw} \in E\}|^2 \\
&\geq \frac{\left(\sum_{u\in N} |\{w \in \left[{S \atop k-1}\right] : \overline{uw} \in E\}|\right)^2}{|\Gamma(S)|} \\
&\geq \frac{\left(\sum_{w \in \left[{S \atop k-1}\right]} d(w)\right)^2}{|\Gamma(S)|} \\
&= \frac{d^2\left[{|S| \atop k-1}\right]^2}{|\Gamma(S)|}
\end{aligned}$$

All together, we get

$$|\Gamma(S)| \geq \frac{d^2 \left[{|S| \atop k-1} \right]^2}{\lambda^2 \left(\left[{|S| \atop k-1} \right] - \frac{\left[{|S| \atop k-1} \right]^2}{\left[{n \atop k-1} \right]} \right) + \frac{d^2}{n} \left[{|S| \atop k} \right]}$$

$$\geq \frac{d^2 \left[{|S| \atop k-1} \right]}{\lambda^2 \left(1 - \frac{\left[{|S| \atop k-1} \right]}{\left[{n \atop k-1} \right]} \right) + \frac{d^2}{n} \frac{|S|}{k}}$$

We note that for the case of $k = 2$, the statement of Theorem 6.6 is also true. It is shown in [1] and [36] that for a d-regular 2-graph G with node set N, for any $S \subseteq N$ we have

$$|\Gamma(S)| \geq \frac{d^2 |S|}{\lambda^2 \left(1 - \frac{|S|}{n} \right) + d^2 \frac{|S|}{n}}$$

where λ is the spectral value of the Laplacian $L(G) = D - A$.

For dense 2-graphs (e.g. having $d > n^{31/32}$) small discrepancy implies certain bounds for the spectral value by using some recent work on "quasi-randomness". The reader is referred to [5,8,11] for the interrelationships of many random-like properties for dense graphs. Here we sketch the connection of spectral value with several invariants related to quasi-randomness.

Let ρ denote the density of a k-graph G. We define a function f: $N \times \ldots \times N \to R$ such that $f(v_1, \ldots, v_k) = 1 - \rho$ if $\{v_1, \ldots, v_k\}$ is an edge of G; and $f(v_1, \ldots, v_k) = -\rho$ otherwise. The 2-deviation of G, denoted by $dev_2(G)$, is defined by

$$(5.1) \quad n^{k+2} dev_2(G) = \sum_{\substack{u_1, v_1, u_2, v_2 \\ u_3, \ldots, u_k \in N}} f(u_1, u_2, v_3, \ldots, v_k) f(u_1, v_2, v_3, \ldots, v_k) \\ f(v_1, u_2, v_3, \ldots, v_k) f(v_1, v_2, v_3, \ldots, v_k)$$

THEOREM 5.5.

$$n^{k+2}(dev_2(G) + O(\frac{1}{n})) = \sum_{i=1}^{\left[{n \atop k-1} \right]} \lambda_i^4$$

where $d - \lambda$'s are the eigenvalues by the Laplacian.

Proof: This follows from the fact that

$$\sum_i \lambda_i^4 = Tr(dI - L)^4 = Tr(A - \rho \hat{K})^4 = n^k + 2(dev_2(G) + O(\frac{1}{n}))$$

since for $x, y \in \binom{N}{k}$, $(L(G))(x,y) = f(u_1, \ldots, u_k)$ where the union of the elements of X and Y is $u_1, \ldots, u_k$.

In [5] it was shown that the discrepancy of G is at least $cn^k(dev_2G)^4$ and therefore for a 2-graph G, $disc_2G \geq c\lambda^{16}n^{-14}$ for some constant c. The same approach does not seem to yield nontrivial upper bounds for λ for general $k \geq 3$.

6. Laplacians of Cayley graphs and its generalizations

In this section we consider Laplacians of k-graphs and their generalizations with density ranging from 1/2 to $n^{-1+1/t}$ for a constant t. As we will see, the spectral value of these explicitly constructed graphs are quite close to the optimum.

First we consider a special type of matrices whose eigenvalues can be easily determined. Let g be a real-valued function defined on Z_p, integers modulo p. The matrix $M_{n,g}^{(k)}$ is of size $n^{k-1} \times n^{k-1}$ with (x,y)-entries to be $g(x_1 + \ldots + x_k)$ for $x = (x_1, \ldots, x_{k-1})$ and $y = (y_1, x_2, \ldots, x_{k-1})$.

THEOREM 6.1. *The matrix $M_{n,g}^{(3)}$ has eigenvalues $\sum_x g(x)$ and $\epsilon|\sum_x g(x)\theta^x|$ where θ is an nth root of unity $\neq 1$ and $\epsilon = 1$ or -1.*

Proof:

Proof: We will first construct vectors using θ and ψ and subsequently show these are indeed eigenvectors of M. We define the vector $f = f_{\theta\psi}$ to be

$$f(x,y) = \theta^x\psi^{y+x} + \theta^{x+y}\psi^x + (\theta^{-x}\psi^{-y-x} + \theta^{-x-y}\psi^{-x}) \cdot \frac{\epsilon\sum_t g(t)(\theta\psi)^t}{|\sum_t g(t)(\theta\psi)^t|}$$

where $\epsilon = 1$ or -1.

Now $(Mf)(y,z) = \sum_x g(x+y+z)f(x,z)$

We note that

$$\sum_x g(x+y+z)\theta^x\psi^{x+z} = \theta^{-y-z}\psi^{-y}\sum_x g(x+y+z)(\theta\psi)^{x+y+z}$$

Therefore

$$Mf_{\theta\psi} = \epsilon|\sum_x g(x)(\theta\psi)^x|f_{\theta\psi}$$

Therefore $f_{\theta\psi}$ are eigenvectors. In fact, $f_{\theta\psi}$ are all the eigenvectors (by considering the rank of M). Theorem 6.1 is proved.

Now we can consider Cayley k-graphs $C_{n,S}$ with node set Z_n and edges $\{x_1, \ldots, x_k\}$ if and only if $x_1 + \ldots + x_k$ is in S for some fixed set S.

THEOREM 6.2. *The matrix C_{ng}^k has eigenvalues $\sum_x g(x)$ and $\epsilon|\sum_x g(x)\theta^x|$ where θ is an nth root of unity $\neq 1$ and $\epsilon = 1$ or -1.*

We note that the above Cayley graph is, in fact, defined with edge set N^k. We remark that hypergraph properties discussed in Section 5 still hold with this slight modification.

THEOREM 6.3. *The Laplacian of the Cayley graph $C^{(k)}_{n,S}$ has spectral value*

$$\max_{\theta\neq 1}|\sum_x \psi_s(x)\theta^x| \text{ where } \psi_S(x) \text{ is } 1-\rho \text{ if } x\in S, \text{ and is } -\rho \text{ if } x\notin S$$

Proof: Let $\tilde{\psi}_S$ denote the function $\psi_S(x)=1$ if x is in S and 0 otherwise. Since the Laplacian of $C_{n,S}$ satisfies $L-D=M_{n,\tilde{\psi}_S}-\frac{|S|}{n}M^{(k)}_{n,\tilde{\psi}_N}$, using the proof of Theorem 6.1, the eigenvector with the eigenvalue $|S|$ for M^k_{n,ψ_s} also has eigenvalue $|S|$ for $\frac{|S|}{n}M^{(k)}_{n,\tilde{\psi}_N}$. Therefore $L-D$ has eigenvalues 0 and $\epsilon|\sum_\psi \psi_S(x)\theta^x|$ for $\theta\neq 1$ since $\tilde{\psi}_S(x)-\frac{\rho}{n}\ \tilde{\psi}_N(x)=\psi_S(x)$.

The Paley graph $P^{(k)}_p$ is just a special case of the Cayley graph $C^{(k)}_{n,S}$ taking n to be some prime congruent to $1 \mod p$ and S to be the set of quadratic residues of p. Therefore we have the following:

THEOREM 6.4. *The Laplacian of the Paley graph $P^{(k)}_p$ has spectral value at most $\sqrt{p}/2$.*

Proof: For $\theta\neq 1$, it is well known [23] that

$$|\sum_x \chi(x)\theta^x|\leq\sqrt{p}.$$

where χ is the usual non-principal quadratic character given by $\chi(x)=1$ if x is a quadratic residue and -1 otherwise. Therefore, $\lambda\leq\sqrt{p}/2$ since $\tilde{\psi}(x)=\chi(x)/2+\frac{1}{p}$ where Q denotes the set of quadratic residues.

There are several ways to obtain generalization of Paley k-graphs. Suppose H is a subgroup of $GF(p)^\star$ with index α^{-1} (i.e., $\alpha=|H|/(p-1)$). We can construct $P^{(k)}_{p,\alpha}$ to be the Cayley graph $C_{p,H}$ with edge density α.

THEOREM 6.5. *The Laplacian of the generalized Paley graph $P^{(k)}_{p,\alpha}$ has spectral value at most $\sqrt{p}/2$.*

Proof: By Theorem 6.3, we want to bound $\sum_x \psi_S(x)\zeta^{jx}$ for $j\neq 0$ and $\zeta=e^{\frac{2pii}{p}}$. Let Φ_H denote the set of all nontrivial characters χ from $GF(p)^\star$ to $C^\star$ such that $\chi|H=1$ and $\chi(0)=0$. It is not difficult (see [23,33]) to check that

$$\psi_H=\alpha\sum_{\chi\in\Phi_H}\chi$$

and $|\Phi_H| = \alpha^{-1} - 1$. Therefore we have

$$\begin{aligned} &\left| \sum_{x \in GF(p)} \psi_S(x)\zeta^{jx} \right| \\ = &\ \alpha \left| \sum_{x \in GF(p)} \sum_{\chi \in \Phi_H} \chi(x)\zeta^{jx} \right| \\ \leq &\ \alpha \sum_{\chi \in \Phi_H} \left| \sum_{x \in GF(p)} \chi(x)\zeta^{jx} \right| \\ \leq &\ \alpha|\Phi_H|\sqrt{p} \\ \leq &\ (1-\alpha)\sqrt{p}. \end{aligned}$$

The proof for Theorem 6.5 is completed.

Another family of constructions are the so-called coset graphs (see [6]). We consider $GF(p^t)$ and let X denote a coset $x + GF(p)$ where $GF(p^t) = GF(p)(x)$ and the coset k-graph $C_{\psi,t}^{(t)}$ is the Cayley graph $C_{p^t,X}^{(k)}$ with edge density $n^{1-\frac{1}{t}}$ where $n = p^t - 1$.

THEOREM 6.6. *The coset k-graph $C_{p,t}^{(k)}$ has $n = p^t - 1$ nodes, degree p and spectral value at most $(t-1)\sqrt{p}$.*

Proof: The proof follows from the following generalization of the character sum inequality which was conjectured in [6] and proved by N. Katz [25]:

$$\left| \sum_{a \in GF(p)} \chi(x+a) \right| \leq (t-1)\sqrt{p}$$

where χ is a nontrivial multiplicative character of $GF(p^t)$.

7. The spectral value of the Laplacian of a random graph

When we say a random graph with density ρ has spectral value $c\sqrt{\rho n}$, we mean that with probability approaching 1 almost all graphs on n nodes with density ρ have spectral value $c\sqrt{\rho n}$ where c denotes some absolute constant. The proofs for determining the spectral value of a random dense graph (ρ being a positive constant) are considerably easier than the proofs for sparse graphs ($\rho = \frac{d}{n}$ for constant degree d).

For sparse graphs, there are basically two different approaches for estimating the spectral value. One usual method is to examine the trace of A^t for some large $t << \log n$ (see [4,16,24]) along the line given in Friedman [16]. The second method, used by Kahn and Szemerédi, is to reduce the problem so that the adjacency matrix of the random graph operates on selected finitely many vectors with expected behaviors. Both methods use elaborated techniques and careful analysis. Although similar approaches can be carried on for hypergraphs to obtain an upper bound of $c\sqrt{\rho n}$, we will not give the arguments here. Instead, here we give a short proof of a (weak) upper bound $O(n^{1/2+\epsilon})$ for dense random graphs with constant edge density ρ. We consider A^t for some large constant $t > k$. For $x, y \in \binom{N}{k-1}$, the (x, y)-entry of A^t can be estimated depending on the set intersection of x and y. If x and y share w common elements, the (x, y)-entry of A^t differs from the expected value of $\rho^{t-k+2-w}n^{t-k+2-w}$ by at most $ct\rho^{(t-k+2-w)/2}n^{(t-k+2-w)/2}$. In a similar way, the (x, y)-entry of $\hat{K}^t$ differs from the expected value of $n^{t-k+2-w}$ by at most $c+n^{(t-k+2-w)/2}$. So, let us consider $\frac{\langle f, Mf\rangle}{\langle f, f\rangle}$ for f orthogonal to the all 1's vector where $M = (L - D)^t = (A - \rho\hat{K})^t$. Therefore,

$$\begin{aligned}
\langle f, A^t f\rangle &= \langle f, Mf\rangle. \\
&\leq |\sum_{x,y} M(x,y)f(x)f(y)| \\
&\leq \sum_{x,y} |M(x,y)|(f^2(x) + f^2(y))/2 \\
&\leq ct\left(\sum_{w=0}^{k-1}\binom{n-k+1}{k-1-w}\binom{k-1}{w}\rho^{(t-k+2-w)/2}n^{(t-k+1)/2}\right)\sum_x f^2(x) \\
&\leq c't\left(\rho^{(t-k+2)/2}n^{(t-k+2-w)/2}\right)\langle f, f\rangle.
\end{aligned}$$

Therefore $\lambda^t \leq c't\rho^{(t-k+2)/2}\, n^{(t-2+w)/2}$. By taking large t, we obtain,

$$\lambda \leq c(\rho n)^{1/2+\epsilon}$$

for any $\epsilon > 0$.

8. Concluding remarks

In previous sections, we have examined relations between the spectral value of the Laplacian and several graph invariants. Numerous questions remains unresolved several of which we mention here.

One natural question is the following: Is it true that small spectral value implies "quasi-randomness" (its definition is given in [5,11,12]). The answer is however negative. It is not difficult to check that the following k-graph with edge density 1/2 has spectral value at most $csqrtn$ but is not quasi-random. Let $G_{(k-1)}$ denote a random $(k-1)$-graph on n nodes. Construct a k-graph H so that $x = \{x_1, \ldots, x_k\}$ is an edge of H if and only if $|\binom{x}{k-1} \cap E(G_{k-1})|$ is odd. It is shown in [8,12] that H is not quasi-random but it is not difficult to check H has spectral value of the same order as a random k-graph.

One possible definition for a "strong" Laplacian (which can be related to "quasi-randomness") is as follows: Here we will only describe the definition for 3-graphs G while it can be easily generalized to k-graphs. We define $L^\star = D - T_G$ where for $p_1, p_2, p_3 \in [{N \atop 2}]$, $T_G(p_1, p_2, p_3)$ is defined to be $1-\rho$ if the union of p_1, p_2 and p_3 is an edge and ρ otherwise. As usual, here ρ denotes the edge density of G. This "strong" Laplacian seems to have intriguing potential but appears difficult to deal with.

The Laplacian and the eigenvalues of (2-) graphs have numerous applications ranging from extremal graph theory to randomized algorithms and approximation algorithms. Hypergraphs are general structures with rich properties. The Laplacian of a hypergraph is not only interesting on its own right but is also related to various applications such as amplifying random bits [35], communication complexity [13] and computational complexity. Basically, a boolean function can be viewed as an (ordered) hypergraph and a hypergraph can be viewed as a symmetric boolean function. Therefore, problems in various areas of computational complexity can perhaps be examined by using the Laplacian to capture the underlying structural properties.

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