Generalized Eulerian Sums

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Dedicated to Adriano Garsia on the occasion of his 84^{th} birthday.

Abstract

In this note, we derive a number of symmetrical sums involving Eulerian numbers and some of their generalizations. These extend earlier identities of Don Knuth and the authors, and also include several q-nomial sums inspired by recent work of Shareshian and Wachs on the joint distribution of various permutation statistics, such as the number of excedances, the major index and the number of fixed points of a permutation. We also produce symmetrical sums involving "restricted" Eulerian numbers which count permutations π on $\{1, 2, \ldots, n\}$ with a given number of descents and which, in addition, have the value of $\pi^{-1}(n)$ specified.

1 Introduction

The classical Eulerian numbers, which we will denote by $\left\langle {n \atop k} \right\rangle$, occur in a variety of contexts in combinatorics, number theory and computer science (see [4]). These numbers were introduced by Euler [3] in 1736 and have many interesting properties. We list some small values in Table 1.

		k = 0	1	2	3	4	5
	1	1	0	0	0	0	0
	2	1	1	0	0	0	0
n	3	1	4	1	0	0	0
	4	1	11	11	1	0	0
	5	1	26	66	26	1	0
	6	1	57	302	302	57	1

Some Eulerian numbers $\left\langle {n \atop k} \right\rangle$
Table 1

In particular, $\left\langle {n \atop k} \right\rangle$ enumerates the number of permutations π on $[n] := \{1, 2, \ldots, n\}$ which have k *descents* (i.e., $i \leq n-1$ with $\pi(i) > \pi(i+1)$) as well as the number

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of permutations π on $[n]$ which have k *excedances* (i.e., $i < n$ with $\pi(i) > i$). Eulerian numbers satisfy the reflection property

$$\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-k-1} \right\rangle, \quad n > 0, \tag{1}$$

and the recurrence

$$\left\langle {n \atop k} \right\rangle = (k+1)\left\langle {n-1 \atop k} \right\rangle + (n-k)\left\langle {n-1 \atop k-1} \right\rangle.$$

They also have the explicit representation

$$\left\langle {n \atop k} \right\rangle = \sum_{i=0}^{k} (-1)^i \binom{n+1}{i} (k+1-i)^n$$

and have the following generating function (see [4]):

$$E_n(t) = \sum_i \left\langle {n \atop i} \right\rangle t^i, \quad n > 0, \tag{2}$$

$$E(t,x) = \frac{(1-t)e^x}{e^{tx} - te^x} = 1 + \sum_{n>0, i\geq 0} \left\langle {n \atop i} \right\rangle t^i \frac{x^n}{n!} = 1 + \sum_{n>0} E_n(t) \frac{x^n}{n!}. \tag{3}$$

Eulerian numbers grow rapidly in size with increasing n, and one doesn't expect as many combinatorial identities to exist for them compared with other sequences such as binomial coefficients or Stirling numbers. Nevertheless, it was shown in [1] that the following symmetrical identity holds:

$$\sum_{k\geq 0} \binom{a+b}{k} \left\langle {k \atop a-1} \right\rangle = \sum_{k\geq 0} \binom{a+b}{k} \left\langle {k \atop b-1} \right\rangle \tag{4}$$

for $a, b > 0$ (where unless stated otherwise, we use the convention that $\left\langle {0 \atop 0} \right\rangle = 0$).

In Section 2, we prove several natural generalizations of this identity. In Section 3, we consider a variant of the usual Eulerian numbers in which we count the number of permutations π on $[n]$ having a given number of descents with the additional constraint that $\pi(n)^{-1}$ is specified. In Sections 4 and 5, we deal with q-coefficients arising in the joint distribution of several permutation statistics studied by Shareshian and Wachs [7, 8]. Finally, in Section 7, we close with a number of open problems.

2 An alternating Eulerian sum identity

Theorem 1.

$$\sum_{k\geq 0} (-1)^k \binom{a+b}{k} \left\langle {k \atop a} \right\rangle = \sum_{k\geq 0} (-1)^k \binom{a+b}{k} \left\langle {k \atop b} \right\rangle$$

for $a, b > 0$.

Proof. The generating function for our "modified" Eulerian numbers (i.e., with $\left\langle {0 \atop 0} \right\rangle = 0$) is obtained by subtracting 1 from $\frac{(1-t)e^x}{e^{tx}-te^x}$ in (3). This gives

$$\frac{e^x - e^{tx}}{e^{tx} - te^x} = \sum_{n,i\geq 0} \left\langle {n \atop i} \right\rangle t^i \frac{x^n}{n!}. \tag{5}$$

Thus,

$$\begin{aligned} \left(\frac{1}{e^x} - \frac{t}{e^{tx}}\right) \sum_{n,i} \left\langle {n \atop i} \right\rangle t^i \frac{x^n}{n!} &= \left(\frac{1}{e^x} - \frac{t}{e^{tx}}\right)\left(\frac{e^x - e^{tx}}{e^{tx} - te^x}\right) \\ &= \frac{1}{e^{tx}} - \frac{1}{e^x}. \end{aligned}$$

Expressing the exponentials as sums implies

$$\begin{aligned} &\left(\sum_{k\geq 0}(-1)^k \frac{x^k}{k!} - t\sum_{k\geq 0}(-1)^k \frac{t^k x^k}{k!}\right) \sum_{n,i} \left\langle {n \atop i} \right\rangle \frac{t^i x^n}{n!} \\ &\qquad = \sum_{n\geq 0}(-1)^n \frac{t^n x^n}{n!} - \sum_{n\geq 0}(-1)^n \frac{x^n}{n!}, \\ &\sum_{k,n,i}(-1)^k \left\langle {n \atop i} \right\rangle \frac{t^i x^{n+k}}{k!n!} - \sum_{k,n,i}(-1)^k \left\langle {n \atop i} \right\rangle \frac{t^{i+k+1} x^{n+k}}{k!n!} \\ &\qquad = \sum_{n\geq 0}(-1)^n \frac{(t^n - 1)x^n}{n!}. \end{aligned}$$

Shifting the indices of summation yields

$$\begin{aligned} \sum_{k,n,i}(-1)^k \left\langle {n-k \atop i} \right\rangle \binom{n}{k} \frac{t^i x^n}{n!} &- \sum_{k,n,i}(-1)^k \left\langle {n-k \atop i-k-1} \right\rangle \binom{n}{k} \frac{t^i x^n}{n!} \\ &= \sum_{n\geq 0}(-1)^n \frac{(t^n-1)x^n}{n!}. \end{aligned} \tag{6}$$

Identifying the coefficients of x^n gives

$$\sum_{k,i}(-1)^k \left\langle {n-k \atop i} \right\rangle \binom{n}{k} t^i - \sum_{k,i}(-1)^k \left\langle {n-k \atop i-k-1} \right\rangle \binom{n}{k} t^i = (-1)^n (t^n - 1).$$

Assuming $i \neq 0, n$ and identifying the coefficients of t^i gives

$$\sum_{k}(-1)^k \binom{n}{k} \left\langle {n-k \atop i} \right\rangle - \sum_{k}(-1)^k \binom{n}{k} \left\langle {n-k \atop i-k-1} \right\rangle = 0.$$

Finally, using the reflection properties of the Eulerian numbers in (1), replacing k by $n-k$, and using the reflection properties of the binomial coefficients, we obtain

$$\sum_k (-1)^k \binom{n}{k} \left\langle {k \atop i} \right\rangle = \sum_k (-1)^k \binom{n}{k} \left\langle {k \atop n-i} \right\rangle,$$

for $0 < i < n$. Therefore, if we set $n = a+b, i = a$, then we have

$$\sum_{k\geq 0} (-1)^k \binom{a+b}{k} \left\langle {k \atop a} \right\rangle = \sum_{k\geq 0} (-1)^k \binom{a+b}{k} \left\langle {k \atop b} \right\rangle \tag{7}$$

for $a, b > 0$. This proves the theorem.

□

Note that (7) can be considered as a companion to (4). We can rewrite (4) in the form

$$\sum_k \binom{n}{k} \left\langle {n-k \atop i-k} \right\rangle = \sum_k \binom{n}{k} \left\langle {n-k \atop i-1} \right\rangle, \tag{8}$$

for $0 < i < n$. As pointed out in [1], this is the first in an infinite sequence of (increasingly complex) sums of this type. The next two are:

$$\sum_k 2^k \binom{n}{k} \left\langle {n-k \atop i-k} \right\rangle - \sum_k 2^k \binom{n}{k} \left\langle {n-k \atop i-2} \right\rangle = \binom{n}{i} - \binom{n}{i-1}, \quad \text{for } 1 < i < n. \tag{9}$$

$$\begin{aligned}\sum_k 3^k \binom{n}{k} \left\langle {n-k \atop i-k} \right\rangle - \sum_k 3^k \binom{n}{k} \left\langle {n-k \atop i-3} \right\rangle &= 2^i \binom{n}{i} + 2^{n-i+1} \binom{n}{i-1} \\ &\quad -2^{i-1} \binom{n}{i-1} - 2^{n-i+2} \binom{n}{i-2} \\ &\quad \text{for } 2 < i < n.\end{aligned} \tag{10}$$

The companion sum (7) is also the first of an infinite sequence of similar sums, the next few being:

$$\begin{aligned}\sum_k (-1)^k 2^{n-k}\binom{n}{k}\left\langle {k \atop i} \right\rangle &- \sum_k (-1)^k 2^{n-k}\binom{n}{k}\left\langle {k \atop n-i+1} \right\rangle \\ &= \binom{n}{i} - \binom{n}{i-1}, \text{ for } 1 \le i \le n. \qquad (11)\\ \sum_k (-1)^k 3^{n-k}\binom{n}{k}\left\langle {k \atop i} \right\rangle &- \sum_k (-1)^k 3^{n-k}\binom{n}{k}\left\langle {k \atop n-i+2} \right\rangle \\ &= 2^{n-i}\binom{n}{i} - 2^{n-i+1}\binom{n}{i-1} \\ &\quad +2^{i-1}\binom{n}{i-1} - 2^{i-2}\binom{n}{i-2} \\ &\quad \text{for } 1 \le i \le n. \qquad (12)\end{aligned}$$

The proofs of these follow the same lines as that of (7) and are omitted.

3 Generalized Eulerian identities for restricted descent polynomials

In this section we consider a restricted version of Eulerian numbers. We define the restricted Eulerian number $b(n,k,i)$, for $1 \le k \le n$, $0 \le i < n$, to be the number of $\pi \in S_n$ with $\mathrm{des}(\pi) = i$ and with $\pi(k) = n$. This restriction is similar to one used by the authors for investigating the joint statistics of permutations having a given number of descents and a given bound on drop size [2]. (A permutation π has a drop at i if $\pi(i) < i$ and the drop size is $i - \pi(i)$).

From the definition of $b(n,k,i)$, it follows that

$$\begin{aligned}\sum_k b(n,k,i) &= \left\langle {n \atop i} \right\rangle, \\ b(k,k,i) &= \left\langle {k-1 \atop i} \right\rangle, \\ b(n,k,i) &= 0, \quad \text{if } n < k \text{ or } i \ge n \text{ or } i < 0.\end{aligned}$$

The quantities $b(n,k,i)$ satisfy the following reflection property which will be needed later.

Lemma 1. *For $1 < k < n$,, we have*

$$b(n,k,i) = b(n,k,n-i-1). \qquad (13)$$

Proof. The proof follows by considering the bijection which maps $\pi \in S_n$ with $\pi(k) = n$ to $\pi' \in S_n$ with $\pi'(k) = n$ defined by $\pi' = (\pi(k-1), \dots, \pi(1), n, \pi(n), \dots, \pi(k+1))$. □

We consider the following descent polynomial $B_{n,k}(t)$ defined by:

$$B_{n,k}(t) = \sum_{i \geq 0} b(n,k,i)t^i.$$

We will need the generating function for this polynomial.

Lemma 2. *The descent polynomial $B_{n,k}(t)$ has the following generating function:*

$$\mathbf{B}_k(t,x) = \sum_{n \geq k} B_{n,k}(t) \frac{x^{n-1}}{(n-1)!} = E_{k-1}(t) \frac{x^{k-1}\ (e^{tx} - te^{tx})}{(k-1)!(e^{tx} - te^{x})}. \tag{14}$$

Proof. We first consider $B_{n,k}(t)$ for the case that $k < n$. Let $S_{m,l}$ denote the set of permutations $\pi \in S_m$ which have l descents. For a fixed $(k-1)$-subset of $\{1, \dots, n\}$, there is a straightforward bijection from the set of permutations $\pi \in S_n$ with $\pi(k) = n$ and having i descents to the following set:

$$\bigcup_{0 \leq j < i} \left(S_{k-1,j} \times S_{n-k,i-j-1}\right)$$

Thus, for $k < n$, we have

$$\begin{aligned} B_{k,n}(t) &= \sum_{i} b(n,k,i)t^i \\ &= t\binom{n-1}{k-1} \sum_{j_1} \left\langle {k-1 \atop j_1} \right\rangle t^{j_1} \sum_{j_2} \left\langle {n-k \atop j_2} \right\rangle t^{j_2}. \end{aligned}$$

By multiplying both sides by $\frac{x^{n-1}}{(n-1)!}$ and summing over all $n > k$, we have for fixed k,

$$\begin{aligned} \sum_{n>k} \sum_{i} b(n,k,i)t^i \frac{x^{n-1}}{(n-1)!} &= \frac{t}{(k-1)!} E_{k-1}(t) \sum_{n>k} \frac{E_{n-k}(t)x^{n-1}}{(n-k)!} \\ &= \frac{tx^{k-1}E_{k-1}(t)}{(k-1)!} \sum_{n>k} \frac{E_{n-k}(t)x^{n-k}}{(n-k)!} \\ &= \frac{tx^{k-1}E_{k-1}(t)}{(k-1)!} \left(\frac{(1-t)e^x}{e^{tx} - te^x} - 1 \right) \\ &= \frac{tx^{k-1}E_{k-1}(t)}{(k-1)!} \left(\frac{e^x - e^{tx}}{e^{tx} - te^x} \right). \end{aligned}$$

Adding to this sum a term for the case $n = k$, we obtain

$$\begin{aligned}
\mathbf{B}_k(t,x) &= \sum_{n\geq k}\sum_i b(n,k,i)t^i \frac{x^{n-1}}{(n-1)!} \\
&= \frac{tx^{k-1}E_{k-1}(t)}{(k-1)!}\left(\frac{e^x - e^{tx}}{e^{tx} - te^x}\right) + \sum_i b(k,k,i)t^i \frac{x^{k-1}}{(k-1)!} \\
&= \frac{tx^{k-1}E_{k-1}(t)}{(k-1)!}\left(\frac{e^x - e^{tx}}{e^{tx} - te^x}\right) + \frac{x^{k-1}E_{k-1}(t)}{(k-1)!} \\
&= \frac{x^{k-1}E_{k-1}(t)}{(k-1)!}\left(\frac{t(e^x - e^{tx})}{e^{tx} - te^x} + 1\right) \\
&= \frac{x^{k-1}E_{k-1}(t)}{(k-1)!}\left(\frac{e^{tx} - te^{tx}}{e^{tx} - te^x}\right).
\end{aligned}$$

□

Now we are ready to prove the following generalized equality for our restricted descent polynomials.

Theorem 2. *For $k > 1$, $r, s > 0$, the restricted Eulerian numbers $b(n,k,i)$ satisfy*

$$\sum_v b(v,k,r)\binom{r+s+1}{v-1} = \sum_v b(v,k,s)\binom{r+s+1}{v-1}. \tag{15}$$

Proof. We start with the generating function given in Lemma 2. By cross-multiplying, we have

$$\left(\sum_{n,i} b(n,k,i)t^i \frac{x^{n-1}}{(n-1)!}\right)(e^{tx} - te^x) = \frac{x^{k-1}E_{k-1}(t)}{(k-1)!}(e^{tx} - te^{tx})$$

Expanding the exponential functions, we have

$$\begin{aligned}
\sum_{n,i,j} b(n,k,i)\frac{t^{i+j}x^{n+j-1}}{j!(n-1)!} &- \sum_{n,i,j} b(n,k,i)\frac{t^{i+1}x^{n+j-1}}{j!(n-1)!} \\
&= \frac{x^{k-1}E_{k-1}(t)}{(k-1)!}\sum_{n\geq 0}\frac{(1-t)t^n x^n}{n!}.
\end{aligned}$$

Identifying the coefficient of x^{n-1} gives

$$\begin{aligned}
\sum_{i,j} b(n-j,k,i)\frac{t^{i+j}}{j!(n-j-1)!} &- \sum_{i,j} b(n-j,k,i)\frac{t^{i+1}}{j!(n-j-1)!} \\
&= \frac{E_{k-1}(t)}{(k-1)!}\left(\frac{t^{n-k}}{(n-k)!} - \frac{t^{n-k+1}}{(n-k)!}\right).
\end{aligned}$$

We further identify the coefficients of t^l to get

$$\begin{aligned}\sum_i \frac{b(n-l+i,k,i)}{(l-i)!(n-l+i-1)!} &- \sum_j \frac{b(n-j,k,l-1)}{j!(n-j-1)!} \\ &= \frac{1}{(k-1)!(n-k)!}\left(\left\langle {k-1 \atop l+k-n} \right\rangle - \left\langle {k-1 \atop l-n+k-1} \right\rangle\right) \\ &= \frac{1}{(k-1)!(n-k)!}\left(\left\langle {k-1 \atop n-l-2} \right\rangle - \left\langle {k-1 \atop n-l-1} \right\rangle\right).\end{aligned}$$

Multiplying both sides by $(n-1)!$, we obtain

$$\begin{aligned}\sum_i b(n-l+i,k,i)\binom{n-1}{l-i} &- \sum_j b(n-j,k,l-1)\binom{n-1}{j} \\ &= \binom{n-1}{k-1}\left(\left\langle {k-1 \atop n-l-2} \right\rangle - \left\langle {k-1 \atop n-l-1} \right\rangle\right).\end{aligned} \tag{16}$$

We will deal with each term in (16) separately. For the first sum on the left side of (16), we first change variables by setting $l-i=u$ and then $n-u=v$ as follows:

$$\begin{aligned}\sum_i b(n-l+i,k,i)\binom{n-1}{l-i} &= \sum_u b(n-u,k,l-u)\binom{n-1}{u} \\ &= \sum_v b(v,k,l-n+v)\binom{n-1}{n-v}.\end{aligned}$$

Now, using the reflection property in Lemma 1, we have

$$\begin{aligned}&\sum_v b(v,k,l-n+v)\binom{n-1}{n-v} \\ =\ &\sum_{v\neq k} b(v,k,n-l-1)\binom{n-1}{v-1} + \left\langle {k-1 \atop l-n+k} \right\rangle \binom{n-1}{k-1} \\ =\ &\sum_{v\neq k} b(v,k,n-l-1)\binom{n-1}{v-1} + \left\langle {k-1 \atop n-l-2} \right\rangle \binom{n-1}{k-1} \\ =\ &\sum_{v\neq k} b(v,k,r)\binom{r+s+1}{v-1} + \left\langle {k-1 \atop r-1} \right\rangle \binom{n-1}{k-1}\end{aligned} \tag{17}$$

by setting $l=s+1$ and $n=r+s+2$.

The second sum of the left side of (16) can be treated as follows:

$$\begin{aligned}\sum_j b(n-j,k,l-1)\binom{n-1}{j} &= \sum_v b(v,k,l-1)\binom{n-1}{n-v}\\ &= \sum_v b(v,k,l-1)\binom{n-1}{v-1}\\ &= \sum_v b(v,k,s)\binom{r+s+1}{v-1}. \qquad (18)\end{aligned}$$

The right side of (16) gives

$$\begin{aligned}&\left(\left\langle {k-1 \atop n-l-2}\right\rangle - \left\langle {k-1 \atop n-l-1}\right\rangle\right)\binom{n-1}{k-1}\\ =\ &\left(\left\langle {k-1 \atop r-1}\right\rangle - \left\langle {k-1 \atop r}\right\rangle\right)\binom{n-1}{k-1}\\ =\ &\left\langle {k-1 \atop r-1}\right\rangle\binom{n-1}{k-1} - b(k,k,r)\binom{n-1}{k-1}. \qquad (19)\end{aligned}$$

Substituting (17), (18) and (19) into (16), we have

$$\sum_v b(v,k,r)\binom{r+s+1}{v-1} = \sum_v b(v,k,s)\binom{r+s+1}{v-1}$$

as claimed. The proof of Theorem 2 is complete. □

The only case that is left out in Theorem 2 is the case of $k=1$. However, for this case, $b(n,1,i) = \left\langle {n-1 \atop i-1}\right\rangle$, and this case reduces to the original Eulerian equality (4).

4 Some q-nomial generalizations

Recently, Shareshian and Wachs [7, 8] found an elegant expression for the joint distribution of the excedance and the major index of a permutation in S_n. First, we give some standard definitions (see [8]).

For $\pi \in S_n$, define

$$\begin{aligned}\mathrm{EXC}(\pi) &= \{i : \pi(i) > i\}, & \mathrm{exc}(\pi) = |\mathrm{EXC}(\pi)|,\\ \mathrm{DES}(\pi) &= \{i : \pi(i) > \pi(i+1)\}, & \mathrm{des}(\pi) = |\mathrm{DES}(\pi)|,\\ \mathrm{maj}(\pi) &= \sum_{i \in \mathrm{DES}(\pi)} i.\end{aligned}$$

Also set

$$\begin{aligned}A_n^{\mathrm{maj},\mathrm{exc}}(q,t) &= \sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} t^{\mathrm{exc}(\pi)}\\ &= \sum_{i,j \ge 0} a_n(i,j) q^i t^j. \qquad (20)\end{aligned}$$

We will define the polynomial $C_n(j) = C_n(j)(q)$ by

$$C_n(j) := \sum_i a_n(i,j) q^{i-j} \tag{21}$$

so that we can write

$$A_n^{\mathrm{maj,exc}}(q,t) = \sum_j C_n(j)(qt)^j.$$

As usual, define

$$\begin{aligned} \left[n\right]_q &= 1 + q + \ldots + q^{n-1}, \\ \left[n\right]_q! &= [n]_q[n-1]_q \ldots [1]_q, \\ \begin{bmatrix} a \\ b \end{bmatrix}_q &= \frac{[a]_q!}{[b]_q![a-b]_q!}. \end{aligned}$$

Theorem 3 ([7, 8])**.**

$$\sum_{n\geq 0} A_n^{\mathrm{maj,exc}}(q,t)\frac{z^n}{[n]_q!} = \frac{(1-qt)\exp_q(z)}{\exp_q(qtz) - qt\exp_q(z)} \tag{22}$$

where $A_n^{\mathrm{maj,exc}}(q,0) = 1$ *and*

$$\exp_q(z) = \sum_{n\geq 0} \frac{z^n}{[n]_q!}.$$

From this result, we will show how to derive q-nomial generalizations of (4) and (7).

Theorem 4. *For $a, b > 0$, the polynomials C_k defined in (21) and (20) satisfy:*

$$\sum_k \begin{bmatrix} a+b \\ k \end{bmatrix}_q C_k(a-1) = \sum_k \begin{bmatrix} a+b \\ k \end{bmatrix}_q C_k(b-1). \tag{23}$$

Note that (21) generalizes (4) since

$$|C_n(j)|_{q=1} = \sum_{i\geq 0} a_n(i,j) = \left\langle \begin{matrix} n \\ j \end{matrix} \right\rangle.$$

Proof. We will use the convention that $A_0^{\mathrm{maj,exc}}(q,t) = 0$ (rather than 1). In this case, subtracting 1 from the expression on (22), we have

$$\frac{\exp_q(z) - \exp_q(qtz)}{\exp_q(qtz) - qt\exp_q(z)} = \sum_{n,j\geq 0} C_n(j)(qt)^j \frac{z^n}{[n]_q!}. \tag{24}$$

Multiplying by $\exp_q(tgz) - qt\exp_q(z)$ and expanding, we obtain

$$\left(\sum_{k\geq 0}\frac{(qtz)^k}{[k]_q!} - qt\sum_{k\geq 0}\frac{z^k}{[k]_q!}\right)\sum_{n,j\geq 0} C_n(j)(qt)^j\frac{z^n}{[n]_q!} = \sum_{n\geq 0}\frac{z^n}{[n]_q!} - \sum_{n\geq 0}\frac{(qtz)^n}{[n]_q!}. \quad (25)$$

Thus,

$$\sum_{k,n,j\geq 0}\frac{C_n(j)(qt)^{j+k}z^{n+k}}{[k]_q![n]_q!} - \sum_{k,n,j\geq 0}\frac{C_n(j)(qt)^{j+1}z^{n+k}}{[k]_q![n]_q!} = \sum_{n\geq 0}\frac{(1-(qt)^n)z^n}{[n]_q!}.$$

Shifting the summation index n gives

$$\sum_{k,n,j\geq 0} C_{n-k}(j)(qt)^{j+k}\begin{bmatrix} n \\ k\end{bmatrix}_q\frac{z^n}{[n]_q} - \sum_{k,n,j\geq 0} C_{n-k}(j)(qt)^{j+1}\begin{bmatrix} n \\ k\end{bmatrix}_q\frac{z^n}{[n]_q}$$
$$= \sum_{n\geq 0}\frac{(1-(qt)^n)z^n}{[n]_q!}. \quad (26)$$

Identifying the coefficients of z^n gives

$$\sum_{k,j\geq 0} C_{n-k}(j)(qt)^{j+k}\begin{bmatrix} n \\ k\end{bmatrix}_q - \sum_{k,j\geq 0} C_{n-k}(j)(qt)^{j+1}\begin{bmatrix} n \\ k\end{bmatrix}_q = 1-(qt)^n.$$

Now, shifting the summation index j, we have

$$\sum_{k,j\geq 0} C_{n-k}(j-k)(qt)^{j}\begin{bmatrix} n \\ k\end{bmatrix}_q - \sum_{k,j\geq 0} C_{n-k}(j-1)(qt)^{j}\begin{bmatrix} n \\ k\end{bmatrix}_q = 1-(qt)^n.$$

Identifying coefficients of t^j then gives

$$\sum_{k} C_{n-k}(j-k)\begin{bmatrix} n \\ k\end{bmatrix}_q - \sum_{k} C_{n-k}(j-1)\begin{bmatrix} n \\ k\end{bmatrix}_q = \begin{cases} 1 & \text{if } j=0,\\ -1 & \text{if } j=n,\\ 0 & \text{if } 0<j<n.\end{cases} \quad (27)$$

It was shown in [8] that the $C_n(j)$ enjoy the symmetry property:

$$C_n(r) = C_n(n-1-r). \quad (28)$$

Hence, we can rewrite (27) as

$$\sum_{k} C_{n-k}(n-k-1-j+k)\begin{bmatrix} n \\ k\end{bmatrix}_q - \sum_{k} C_{n-k}(j-1)\begin{bmatrix} n \\ k\end{bmatrix}_q$$
$$= \sum_{k} C_{k}(n-1-j)\begin{bmatrix} n \\ k\end{bmatrix}_q - \sum_{k} C_{k}(j-1)\begin{bmatrix} n \\ k\end{bmatrix}_q = \begin{cases} 1 & \text{if } j=0,\\ -1 & \text{if } j=n,\\ 0 & \text{if } 0<j<n.\end{cases}$$

Setting $n = a + b$ and $j = b$ yields

$$\sum_k \begin{bmatrix} a+b \\ k \end{bmatrix}_q C_k(a-1) = \sum_k \begin{bmatrix} a+b \\ k \end{bmatrix}_q C_k(b-1),$$

for $a, b > 0$. This proves the theorem. $\square$

The same techniques can be used to prove the following companion sum for (22).

Theorem 5. *For $a, b > 0$,*

$$\sum_k (-1)^k \begin{bmatrix} a+b \\ k \end{bmatrix}_q q^{\binom{a+b-k}{2}} C_k(a) = \sum_k (-1)^k \begin{bmatrix} a+b \\ k \end{bmatrix}_q q^{\binom{a+b-k}{2}} C_k(b). \tag{29}$$

This is a q-nomial generalization of (7).

5 Further q-nomial generalizations

In [7, 8], Shareshian and Wachs prove the following more general version of (22):

Theorem 6 ([7, 8])**.**

$$\sum_{n\geq 0} A_n^{\mathrm{maj,exc,fix}}(q,t,r)\frac{z^n}{[n]_q!} = \frac{(1-qt)\exp_q(rz)}{\exp_q(qtz) - qt\exp_q(z)} \tag{30}$$

where $A_n^{\mathrm{maj,exc,fix}}(q,0,0) = 1$ and fix*(π) denotes the number of fixed points of $\pi \in S_n$, i.e., the number of i such that $\pi(i) = i$.*

Let us write

$$\frac{(1-qt)\exp_q(rz)}{\exp_q(zqt) - qt\exp_q(z)} = \sum_{n,i,j,k} a_n(i,j,k) q^i t^j r^k \frac{z^n}{[n]_q!}.$$

and define the polynomials $C_n(j,k) = C_n(j,k)(q)$ by

$$C_n(j,k) = \sum_i a_n(i,j,k) q^{i-j}.$$

Theorem 7. *For $a, b, k \geq 0$, the polynomials $C_u(q)$ as defined in (31) satisfy*

$$\sum_u C_u(a,k) \begin{bmatrix} a+b+k+1 \\ u \end{bmatrix}_q = \sum_u C_u(b,k) \begin{bmatrix} a+b+k+1 \\ u \end{bmatrix}_q. \tag{31}$$

Proof. Cross-multiplying, substituting and expanding the exponentials in (30), we obtain

$$\sum_{u\geq 0} \frac{(qtz)^u}{[u]_q!} \sum_{n,j,k} C_n(j,k)(qt)^j r^k \frac{z^n}{[n]_q!} - qt \sum_{u\geq 0} \frac{z^u}{[u]_q!} \sum_{n,j,k} C_n(i,j,k)(qt)^j r^k \frac{z^n}{[n]_q!}$$
$$= (1-qt) \sum_{n\geq 0} \frac{(rz)^n}{[n]_q!}.$$

Now, shifting indices and identifying the coefficients of z^n (as before) yields

$$\sum_{j,k,u} C_{n-u}(j-u,k)(qt)^j r^k \begin{bmatrix} n \\ u \end{bmatrix}_q - \sum_{j,k,u} C_{n-u}(j-1,k)(qt)^j r^k \begin{bmatrix} n \\ u \end{bmatrix}_q = (1-qt)r^n.$$

Hence, for $k < n$, we can identify the coefficients of r and t and obtain

$$\sum_u C_{n-u}(j-u,k) \begin{bmatrix} n \\ u \end{bmatrix}_q = \sum_u C_{n-u}(j-1,k) \begin{bmatrix} n \\ u \end{bmatrix}_q . \tag{32}$$

We next note the following nice symmetry property of the $C_n(j,k)$.

Fact.

$$C_n(j,k) = C_n(n-j-k,k) \tag{33}$$

for $n, j, k \geq 0$.

Proof. First, a straightforward computation from (30) verifies that

$$\sum_{n\geq 0} A_n^{maj,exc,fix}(q,t,r)\frac{z^n}{[n]_q!} = \sum_{n\geq 0} A_n^{maj,exc,fix}\left(q, \frac{1}{q^2 t}, \frac{r}{qt}\right)\frac{(qtz)^n}{[n]_q!}. \tag{34}$$

This implies

$$\sum_{n,i,j,k} a_n(i,j,k)q^i t^j r^k \frac{z^n}{[n]_q!} = \sum_{n,i,j,k} a_n(i,j,k)q^{i-2j-k+n}t^{-j-k+n}r^k\frac{z^n}{[n]_q!}. \tag{35}$$

Identifying coefficients of z^n and r^k then gives

$$\sum_{i,j} a_n(i,j,k)q^i t^j = \sum_{i,j} a_n(i,j,k)q^{i-2j-k+n}t^{-j-k+n}. \tag{36}$$

Shifting the indices of summation in the second sum by $i' = i-2j-k+n$ and $j' = -j-k+n$ yields

$$\sum_{i,j} a_n(i,j,k)q^i t^j = \sum_{i',j'} a_n(i'-2j'-k+n, -j'-k+n, k)q^{i'}t^{j'}. \tag{37}$$

Now, identifying the coefficients of q^i and t^j gives us

$$a_n(i,j,k) = a_n(n+i-2j-k, n-j-k, k) \tag{38}$$

for $i, j, k, n \geq 0$. Finally, summing over i gives

$$\begin{aligned}\sum_i a_n(i,j,k)q^{i-j} &= \sum_i a_n(n+i-2j-k, n-j-k, k)q^{i-j} \\ &= \sum_i a_n(n+i-2j-k, n-j-k, k)q^{(n+i-2j-k)-(n-j-k)} \\ &= \sum_i a_n(i, n-j-k, k)q^{i-n+j+k}\end{aligned}$$

i.e.,

$$C_n(j,k) = C_n(n-j-k,k),$$

and the Fact is proved.
Now, continuing the proof of (31), we will apply equation (33) to (32). Thus,

$$\begin{aligned}\sum_u C_{n-u}(j-u,k)\begin{bmatrix} n \\ u \end{bmatrix}_q &= \sum_u C_{n-u}(n-j-k,k)\begin{bmatrix} n \\ u \end{bmatrix}_q \\ &= \sum_u C_u(j-1,k)\begin{bmatrix} n \\ u \end{bmatrix}_q .\end{aligned}$$

Finally, letting $n = a+b+k+1, j = b+1$, we get

$$\sum_u C_u(a,k)\begin{bmatrix} a+b+k+1 \\ u \end{bmatrix}_q = \sum_u C_u(b,k)\begin{bmatrix} a+b+k+1 \\ u \end{bmatrix}_q$$

as required. □

Observe that setting $q = 1$ in Theorem 6 gives us symmetrical sums for the quantities $c_n(j,k) = C_n(j,k)|_{q=1}$, which count the number of π in S_n with $\mathrm{exc}(\pi) = j$ and $\mathrm{fix}(\pi) = k$. These sums can be written as follows:

Theorem 8. *For $a, b, k > 0$, the c_u defined above satisfy*

$$\sum_u c_u(a,k)\binom{a+b+k+1}{u} = \sum_u c_u(b,k)\binom{a+b+k+1}{u}. \tag{39}$$

6 Concluding remarks

In general, we would like to find symmetrical identities similar in form to (4), (15), or (23). which depend on coefficients arising in the enumeration of various joint statistics on permutations. For example, we could try to prove identities of the form

$$\sum_k \begin{bmatrix} a+b \\ u \end{bmatrix}_q P(k,a-1) = \sum_k \begin{bmatrix} a+b \\ u \end{bmatrix}_q P(k,b-1) \tag{40}$$

for some appropriate polynomials $P(i,j)(q)$. A natural place to look is at the polynomials $S(n,j) = \sum_i s(n,i,j)q^i$ where $s(n,i,j)$ is defined to be the number of $\pi \in S_n$ such that $\mathrm{inv}(\pi) = i$ and $\mathrm{des}(\pi) = j$. Unfortunately, we didn't find any variation of (40) which was valid. We find this to be slightly surprising since a classic result of Stanley [6], namely,

$$\sum_n A^{\mathrm{inv}(\pi),\mathrm{des}(\pi)}(q,t)\frac{z^n}{[n]_q!} = \frac{1-t}{\exp_q(tz)-t} = \sum_n S(n,j)t^j\frac{z^n}{[n]_q!},$$

would seem to be in the appropriate form for our analysis. However, we suspect that such symmetrical sums do exist.

In another direction, the preceding techniques can be applied to a variety of other expressions which have an appropriate generating function. For example, for $\pi \in S_n$, we can define

$$D_n(q,t) := \sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} t^{\mathrm{des}(\pi)} = \sum_{i,j} d(n,i,j) q^i t^j,$$

$$D_n^k(q,t) := \sum_{\pi \in S_n, \pi(k)=n} q^{\mathrm{maj}(\pi)} t^{\mathrm{des}(\pi)} = \sum_{i,j} d(n,i,k,j) q^i t^j$$

We can then apply the preceding arguments to obtain sums similar to (15) and (29) for those coefficients.

It would be interesting to find bijective proofs for some of these identities such as (7), (9), (10), (11), (15), (23) or (31). A beautiful (and non-trivial) bijective proof of (4) was discovered by Don Knuth [1] but no one has yet found a corresponding bijective proof for its companion (7).

7 Acknowledgements

The authors wish to thank Adriano Garsia, Jeff Remmel and an anonymous referee for their helpful comments while we were preparing this note.

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