

A generalized Alon-Boppana bound and weak Ramanujan graphs

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Abstract

A basic eigenvalue bound due to Alon and Boppana holds only for regular graphs. In this paper we give a generalized Alon-Boppana bound for eigenvalues of graphs that are not required to be regular. We show that a graph G with diameter k and vertex set V , the smallest nontrivial eigenvalue λ_1 of the normalized Laplacian \mathcal{L} satisfies

$$\lambda_1 \leq 1 - \sigma \left(1 - \frac{5}{k}\right)$$

provided $\sigma = 2 \sum_v d_v \sqrt{d_v - 1} / \sum_v d_v^2 \leq 1/2$ and $k(1.5)^k \geq \sigma^{-1}$ where d_v denotes the degree of the vertex v with minimum degree at least 2.

We consider weak Ramanujan graphs defined as graphs satisfying $\lambda_1 \geq 1 - \sigma$. We examine the vertex expansion and edge expansion of weak Ramanujan graphs and then use the expansion properties among other methods to derive the above Alon-Boppana bound.

1 Introduction

The well-known Alon-Boppana bound [8] states that for any d -regular graph with diameter k , the second largest eigenvalue ρ of the adjacency matrix satisfies

$$\rho \geq 2\sqrt{d-1} \left(1 - \frac{2}{k}\right) - \frac{2}{k}. \quad (1)$$

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A natural question is to extend Alon-Boppana bounds for graphs that are irregular. Hoory [6] showed that for an irregular graph, the second largest eigenvalue ρ of the adjacency matrix satisfies

$$\rho \geq 2\sqrt{d-1} \left(1 - \frac{c \log r}{r}\right)$$

if the average degree of the graph after deleting a ball of radius r is at least d where $r, d > 2$.

For irregular graphs, it is often advantageous to consider eigenvalues of the normalized Laplacian for deriving various graph properties. For a graph G , the normalized Laplacian \mathcal{L} , defined by

$$\mathcal{L} = I - D^{-1/2} A D^{-1/2}$$

where D is the diagonal degree matrix and A denotes the adjacency matrix of G . One of the main tools for dealing with general graphs is the Cheeger inequality which relates the least nontrivial eigenvalue λ_1 to the Cheeger constant h_G :

$$2h_G \geq \lambda_1 \geq \frac{h_G^2}{2} \tag{2}$$

where $h_G = \min_S |\partial(S)|/\text{vol}(S)$ for S ranging over all vertex subsets with volume $\text{vol}(S) = \sum_{u \in S} d_u$ no more than half of $\sum_{u \in V} d_u$ and $\partial(S)$ denotes the set of edges leaving S . For k -regular graphs, we have $\lambda_1 = 1 - \rho/k$ where ρ denotes the second largest eigenvalue of the adjacency matrix. In general,

$$\frac{\rho}{\max_v d_v} \leq 1 - \lambda_1 \leq \frac{\rho}{\min_v d_v}$$

which can be used to derive a version of the Cheeger inequality involving ρ which is less effective than (2) for irregular graphs.

In this paper, we will show that for a connected graph G with diameter k , λ_1 is upper bounded by

$$\lambda_1 \leq 1 - \sigma \left(1 - \frac{c}{k}\right) \tag{3}$$

for a constant c where $\sigma = 2 \sum_v d_v \sqrt{d_v - 1} / \sum_v d_v^2$. The above inequality will be proved in Section 6.

The above bound of Alon-Boppana type improves a result of Young [10] who derived a similar eigenvalue bound using a different method. In [10] the notion of (r, d, δ) -robust graphs was considered and it was shown that for a (r, d, δ) -robust graph, the least non-trivial eigenvalue λ_1 satisfies

$$\lambda_1 \leq 1 - \frac{2d\sqrt{d-1}}{\delta} \left(1 - \frac{c}{r}\right). \tag{4}$$

Here (r, d, δ) -robustness means for every vertex v and the ball $B_r(v)$ consisting of all vertices with distance at most r , the induced subgraph on the complement of $B_r(v)$ has

average degree at least d and $\sum_{v \notin B_r(v)} d_v^2 / |V \setminus B_r(v)| \leq \delta$. We remark that our result in (3) does not require the condition of robustness.

We define *weak Ramanujan* graphs to be graphs with eigenvalue λ_1 satisfying

$$\lambda_1 \geq 1 - \sigma \geq \frac{1}{2} \tag{5}$$

where $\sigma = 2 \sum_v d_v \sqrt{d_v - 1} / \sum_v d_v^2$.

To prove the Alon-Boppana bound in (3), it suffices to consider only weak Ramanujan graphs. Weak Ramanujan graphs satisfy various expansion properties. We will describe several vertex-expansion and edge-expansion properties involving λ_1 in Section 3, which will be needed later for proving a diameter bound for weak Ramanujan graphs in Section 4. The diameter bound and related properties of weak Ramanujan graphs are useful in the proof of the Alon-Boppana bound for general graphs.

We will also show that the largest eigenvalue λ_{n-1} of the normalized Laplacian satisfies

$$\lambda_{n-1} \geq 1 + \sigma \left(1 - \frac{5}{k}\right). \tag{6}$$

The proof will be given in Section 7.

2 Preliminaries

For a graph $G = (V, E)$, we consider the normalized Laplacian

$$\mathcal{L} = I - D^{-1/2} A D^{-1/2}$$

where A denotes the adjacency matrix and D denotes the diagonal degree matrix with $D(v, v) = d_v$, the degree of v . We assume that there is no isolated vertex throughout this paper. For a vertex v and a positive integer l , let $B_l(v)$ denote the ball consisting of all vertices within distance l from v . For an edge $\{x, y\} \in E$ we say x is adjacent to y and write $x \sim y$.

Let $\lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_{n-1}$ denote eigenvalues of \mathcal{L} , where n denotes the number of vertices in G . It can be checked (see [2]) that $\lambda_1 > 0$ if G is connected. The Alon-Boppana bound obviously holds if $\lambda_1 = 0$. In the remainder of this paper, we assume G is connected.

Let φ_i denote the orthonormal eigenvector associated with eigenvalue λ_i . In particular, $\varphi_0 = D^{1/2} \mathbf{1} / \sqrt{\text{vol}(G)}$ where $\mathbf{1}$ is the all 1's vector and $\text{vol}(G) = \sum_{v \in V} d_v$. We can then write

$$\begin{aligned} \lambda_1 &= \inf_{g \perp \varphi_0} \frac{\langle g, \mathcal{L}g \rangle}{\langle g, g \rangle} \\ &= \inf_{f \perp D\mathbf{1}} \frac{\sum_{x \sim y} (f(x) - f(y))^2}{\sum_z f^2(z) d_z} \\ &= \inf_{f \perp D\mathbf{1}} R(f) \end{aligned}$$

where f ranges over all functions satisfying $\sum_u f(u)d_u = 0$ and the sum $\sum_{x \sim y}$ ranges over all unordered pairs $\{x, y\}$ where x is adjacent to y . Here $R(f)$ denote the *Rayleigh quotient* of f , which can be written as follows:

$$R(f) = \frac{\int |\nabla f|}{\int \|f\|^2}$$

where $\int \|f\|^2 = \sum_x f^2(x)d_x$

and $\int |\nabla f| = \sum_{x \sim y} (f(x) - f(y))^2$.

For eigenfunction φ_i , the function $f_i = D^{-1/2}\varphi_i$, called the combinatorial eigenfunction associated with λ_i , satisfies

$$\lambda_i f(u)d_u = \sum_{v \sim u} (f(u) - f(v)) \tag{7}$$

for each vertex u . In particular, for f satisfying $\sum_u f(u)d_u = 0$, we have

$$\langle f, Af \rangle \leq (1 - \lambda_1) \langle f, Df \rangle \tag{8}$$

and

$$|\langle f, Af \rangle| \leq \max_{i \neq 0} (1 - \lambda_i) \langle f, Df \rangle. \tag{9}$$

3 Vertex and edge expansions

For any subset S of vertices, there are two types of boundaries. The *edge boundary* of S , denoted by $\partial(S)$ consists of all edges with exactly one endpoint in S . The *vertex boundary* of S , denoted by $\delta(S)$ consists of all vertices not in S but adjacent to vertices in S . Namely,

$$\begin{aligned} \partial(S) &= \{\{u, v\} \in E : u \in S \text{ and } v \notin S\} = E(S, \bar{S}) \\ \delta(S) &= \{u \notin S : u \sim v \in S \text{ for some vertex } v\} \end{aligned}$$

In this section, we will examine vertex expansion and edge expansion relying only on λ_1 . These expansion properties will be needed for deriving diameter bounds for weak Ramanujan graphs which will be used in our proof of the general Alon-Boppana bound later in Section 6.

From the definition of the Cheeger constant, for all vertex subsets S , we have

$$\frac{|\partial(S)|}{\text{vol}(S)} \geq h_G \geq \frac{\lambda_1}{2}$$

Later in the proofs, we will be interested in the case that $\text{vol}(S)$ is small and therefore we will use the following version.

Lemma 1. *Let S be a subset of vertices in G . Then*

$$\frac{|\partial(S)|}{\text{vol}(S)} \geq \lambda_1 \left(1 - \frac{\text{vol}(S)}{\text{vol}(G)}\right).$$

Proof. Suppose f is defined by

$$f = \frac{\mathbf{1}_S}{\text{vol}(S)} - \frac{\mathbf{1}_{\bar{S}}}{\text{vol}(\bar{S})}$$

where $\mathbf{1}_S$ denotes the characteristic function defined by $\mathbf{1}_S(v) = 1$ if $v \in S$ and 0 otherwise.

The Rayleigh quotient $R(f)$ satisfies

$$\lambda_1 \leq R(f) = \frac{|\partial(S)|}{\text{vol}(S)} \cdot \frac{\text{vol}(G)}{\text{vol}(\bar{S})}.$$

□

For the expansion of the vertex boundary, the Tanner bound [9] for regular graphs can be generalized as follows.

Lemma 2. *Let $\bar{\lambda} = \min_{i \neq 0} |1 - \lambda_i|$. Then for any vertex subset S in a graph,*

$$\frac{\text{vol}(\delta(S))}{\text{vol}(S)} \geq \frac{1 - \bar{\lambda}^2}{\bar{\lambda}^2 + \frac{\text{vol}(S)}{\text{vol}(\bar{S})}} \quad (10)$$

The proof of the above inequality is by using the following discrepancy inequality (as seen in [2]).

Lemma 3. *In a graph G , for two subset X and Y of vertices, the number $e(X, Y) = |E(X, Y)|$ of edges between X and Y satisfies*

$$\left| e(X, Y) - \frac{\text{vol}(X)\text{vol}(Y)}{\text{vol}(G)} \right| \leq \bar{\lambda} \sqrt{\frac{\text{vol}(X)\text{vol}(Y)\text{vol}(\bar{X})\text{vol}(\bar{Y})}{\text{vol}(G)}} \quad (11)$$

where $\bar{\lambda} = \min_{i \neq 0} |1 - \lambda_i|$.

The proof of Lemma 3 follows from (9) and can be found in [2]. The proof of (12) results from (11) by setting $X = S$ and $Y = \overline{S \cup \delta(S)}$.

Here we will give a version of the vertex-expansion bounds for general graphs which only rely on λ_1 and are independent of other eigenvalues.

Lemma 4. *In a graph G with vertex set V and the first nontrivial eigenvalue λ_1 , for a subset S of V with $\text{vol}(S \cup \delta S) \leq \epsilon \text{vol}(G) \leq \text{vol}(G)/2$, the vertex boundary of S satisfies*

$$(i) \quad \frac{\text{vol}(\delta(S))}{\text{vol}(S)} \geq \frac{2\lambda_1}{1 - \lambda_1 + 2\epsilon} \quad (12)$$

(ii) *If $1/2 \leq \lambda_1 \leq 1 - 2\epsilon$, then*

$$\frac{\text{vol}(\delta(S))}{\text{vol}(S)} \geq \frac{1}{(1 - \lambda_1 + 2\epsilon)^2}. \quad (13)$$

Proof. The proof of (i) follows from Lemma 1 since

$$\begin{aligned} \frac{\text{vol}(\delta(S))}{\text{vol}(S)} &\geq \frac{|\partial(S \cup \delta(S))| + |\partial(S)|}{\text{vol}(S)} \\ &\geq \frac{\lambda_1(1 - \epsilon)(\text{vol}(S) + \text{vol}(\delta(S))) + \lambda_1(1 - \epsilon)\text{vol}(S)}{\text{vol}(S)} \end{aligned}$$

Therefore

$$\frac{\text{vol}(\delta(S))}{\text{vol}(S)} \geq \frac{2\lambda_1(1 - \epsilon)}{1 - \lambda_1(1 - \epsilon)} \geq \frac{2\lambda_1}{1 - \lambda_1 + 2\epsilon}$$

To prove (ii), we set $f = \mathbf{1}_S + \gamma\mathbf{1}_{\delta(S)}$ where $\gamma = 1 - \lambda_1$. Consider $g = f - c\mathbf{1}_V$ where $c = \sum_u f(u)d_u / \text{vol}(G)$. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} c^2 &= \frac{1}{(\text{vol}(G))^2} \left(\sum_{u \in S \cup \delta(S)} f(u)d_u \right)^2 \leq \frac{\text{vol}(S \cup \delta(S))}{(\text{vol}(G))^2} \sum_u f^2(u)d_u \\ &\leq \frac{\epsilon}{\text{vol}(G)} \sum_u f^2(u)d_u. \end{aligned}$$

Using the inequality in (8), we have

$$\begin{aligned} \langle f, Af \rangle &\leq \langle g, Ag \rangle + c^2 \text{vol}(G) \\ &\leq \gamma \langle g, Dg \rangle + c^2 \text{vol}(G) \\ &= \gamma \langle f, Df \rangle + (1 - \gamma)c^2 \text{vol}(G) \\ &\leq (\gamma + \epsilon) \langle f, Df \rangle \\ &= (\gamma + \epsilon)(\text{vol}(S) + \gamma^2 \text{vol}(\delta(S))). \end{aligned}$$

Let $e(S, T)$ denote the number of ordered pairs (u, v) where $u \in S, v \in T$ and $\{u, v\} \in E$. Since $\gamma = 1 - \lambda \leq 1/2$, we have

$$\begin{aligned} \langle f, Af \rangle &\geq e(S, S) + 2\gamma e(S, \delta(S)) \\ &\geq (1 - 2\gamma)e(S, S) + 2\gamma \text{vol}(S) \\ &\geq 2\gamma \text{vol}(S) \end{aligned}$$

Together we have

$$\begin{aligned} \frac{\text{vol}(\delta(S))}{\text{vol}(S)} &\geq \frac{\gamma - \epsilon}{\gamma^2(\gamma + \epsilon)} \\ &\geq \frac{1}{(\gamma + 2\epsilon)^2} \end{aligned}$$

since $\gamma \geq 2\epsilon$. □

Recall that weak Ramanujan graphs have eigenvalue λ_1 satisfying

$$\lambda_1 \geq 1 - \sigma \tag{14}$$

where $\sigma = 2 \sum_v d_v \sqrt{d_v - 1} / \sum_v d_v^2$. Lemma 1 implies that for S with $\text{vol}(S \cup \delta(S)) \leq \epsilon \text{vol}(G)$,

$$\frac{\text{vol}(\delta(S))}{\text{vol}(S)} \geq \frac{1}{(\sigma + 2\epsilon)^2}.$$

For k -regular Ramanujan graphs with eigenvalue $\lambda_1 = 1 - 2\sqrt{k-1}/k$, the above inequality is consistent with the bound

$$\frac{\text{vol}(\delta(S))}{\text{vol}(S)} = \frac{|\delta(S)|}{|S|} \geq \frac{1}{\left(\frac{2\sqrt{k-1}}{k} + 2\epsilon\right)^2}$$

which is about $k/4$ when $\text{vol}(S)$ is small. The factor $k/4$ in the above inequality was improved by Kahale [4] to $k/2$. There are many applications (see [1]) that require graphs having expansion factor to be $(1 - \epsilon)k$. Such graphs are called *lossless* expanders. In [1], lossless graphs were constructed explicitly by using the zig-zag construction but the method for deriving the expansion bounds does not use eigenvalues. In this paper, the expansion factor as in Lemma 4 is enough for our proof later.

4 Weak Ramanujan graphs

We recall that a graph is said to be a weak Ramanujan graph as in (14) if

$$\lambda_1 \geq 1 - \sigma \geq \frac{1}{2}$$

where

$$\sigma = 2 \frac{\sum_v d_v \sqrt{d_v - 1}}{\sum_v d_v^2}. \tag{15}$$

To prove the Alon-Boppana bound, it is enough to consider only weak Ramanujan graphs.

Lemma 5. *As defined in (15), σ satisfies*

$$\frac{2\sqrt{\bar{d}-1}}{\check{d}} \leq \sigma \leq \frac{2\sqrt{\bar{d}-1}}{\bar{d}}$$

where \bar{d} denotes the average degree in G and \check{d} denote the second order degree, i.e.,

$$\bar{d} = \frac{\sum_v d_v}{n} \quad \text{and} \quad \check{d} = \frac{\sum_v d_v^2}{\sum_v d_v}.$$

Proof. The proof is mainly by using the Cauchy-Schwarz inequality. For the upper bound, we note that

$$\begin{aligned} \sigma &= 2 \frac{\sum_v d_v \sqrt{d_v - 1}}{\sum_v d_v^2} \leq 2 \frac{\sqrt{\sum_v d_v^2 \sum_v (d_v - 1)}}{\sum_v d_v^2} \\ &= 2 \frac{\sqrt{\sum_v (d_v - 1)}}{\sqrt{\sum_v d_v^2}} \\ &\leq 2 \frac{\sqrt{\sum_v (d_v - 1)}}{\sum_v d_v / \sqrt{n}} \\ &\leq 2 \frac{\sqrt{\sum_v (d_v - 1)}}{\bar{d} \sqrt{n}} \leq \frac{2\sqrt{\bar{d} - 1}}{\bar{d}}. \end{aligned}$$

For the upper bound, we will use the fact that for $a, b > 1$ and $a + b = c$,

$$a\sqrt{a-1} + b\sqrt{b-1} \geq c\sqrt{\frac{c}{2}-1}$$

and therefore

$$\sum_v d_v \sqrt{d_v - 1} \geq \sum_v d_v \sqrt{\frac{\sum_v d_v}{n} - 1}.$$

Consequently, we have

$$\sigma = 2 \frac{\sum_v d_v \sqrt{d_v - 1}}{\sum_v d_v^2} \geq 2 \frac{\sum_v d_v \sqrt{\frac{\sum_v d_v}{n} - 1}}{\frac{\sum_v d_v^2}{\sum_v d_v}} \geq 2 \frac{\sqrt{\bar{d} - 1}}{\bar{d}}$$

as desired. □

We remark that for graphs with average degree at least 20, we have $\sigma < 1/2 < \lambda_1$.

Theorem 6. *Suppose a weak Ramanujan graph G has diameter k . Then for any $\epsilon > 0$, we have*

$$k \leq (1 + \epsilon) \frac{2 \log \text{vol}(G)}{\log \sigma^{-1}}$$

provided that the volume of G is large, i.e., $\text{vol}(G) \geq c\sigma^{\log(\sigma)}/\epsilon$ for some small constant c .

Proof. We set

$$t = \left\lceil (1 + \epsilon) \frac{\log(\text{vol}(G))}{\log \sigma^{-1}} \right\rceil.$$

It suffices to show that for every vertex v , the ball $B_t(v)$ has volume more than $\text{vol}(G)/2$.

Suppose $\text{vol}(B_t(v)) \leq \text{vol}(G)/2$. Let

$$s_j = \frac{\text{vol}(B_j(u))}{\text{vol}(G)}.$$

By part (i) of Lemma 4, we have $\text{vol}(\delta(B_u(j))) \geq 0.5\text{vol}(B_u(j))$ for $j \leq t-1$ and therefore $s_{j+1} \geq 1.5s_j$. Thus, if $j \leq t - c_1 \log(\sigma^{-1})$, then $s_j \leq \sigma^4$ where c_1 is some small constant satisfying $c_1 \leq 4(\log 1.5)^{-1}$.

Now we apply part (ii) of Lemma 4 and we have, for $j \leq t - c_1 \log(\sigma^{-1})$,

$$\frac{s_{j+1}}{s_j} = \frac{\text{vol}(B_{j+1}(u))}{\text{vol}(B_j(u))} \geq \frac{\text{vol}(\delta(B_j(u)))}{\text{vol}(B_j(u))} \geq \frac{1}{(\sigma + 2s_j)^2} \geq \frac{1}{(\sigma + 2\sigma^4)^2}.$$

This implies, for $l \leq t - c_1 \log(\sigma^{-1})$,

$$\begin{aligned} \frac{s_l}{s_0} &\geq \prod_{0 < j < l} \frac{1}{(\sigma + 2s_j)^2} \geq \prod_{0 < j < l} \frac{1}{(\sigma + 2\sigma^4)^2} \\ &\geq \frac{1}{\sigma^{2l}(1 + 2\sigma^4)^{2l}}. \end{aligned}$$

Since $s_0 \geq 1/\text{vol}(G)$ and $s_l \leq s_t \leq 1/2$, we have

$$\text{vol}(G) \geq \frac{1}{\sigma^{2l}(1 + 2\sigma^4)^{2l}}.$$

Hence

$$l \leq \frac{\log(\text{vol}(G))}{\log(\sigma^{-1}) + 2\sigma^4}.$$

However,

$$(1 + \epsilon) \frac{\log(\text{vol}(G))}{\log(\sigma^{-1})} \leq t \leq c_1 \log(\sigma^{-1}) + \frac{\log(\text{vol}(G))}{\log(\sigma^{-1}) + 2\sigma^4}$$

which is a contradiction for G with $\text{vol}(G)$ large, say, $\text{vol}(G) \geq \sigma^{2c_1 \log \sigma} / \epsilon$. Thus we conclude that $s_t \geq 1/2$ and Theorem 6 is proved. \square

Theorem 7. *For a weak Ramanujan graph with diameter k , for any vertex v and any $l \leq k/4$, the ball $B_u(l)$ has volume at most $\epsilon \text{vol}(G)$ if $k \geq c \log \epsilon^{-1}$, for $c = 1/(\log 1.5)$.*

Proof. We will prove by contradiction. Suppose that for $j_0 = \lceil k/4 \rceil$, there is a vertex u with $\text{vol}(B_u(j_0)) > \epsilon \text{vol}(G)$. Let r denote the least integer such that

$$s_r = \frac{\text{vol}(B_u(r))}{\text{vol}(G)} > \frac{1}{2}.$$

By the assumption, we have $r > k/4$ and $s_{j_0} > \epsilon$. There are two possibilities:

Case 1: $r \geq k/2$.

By part (i) of Lemma 4, we have $\text{vol}(\delta(B_u(j))) \geq 0.5\text{vol}(B_u(j))$ for $j \leq k/2$ and therefore $s_{j+1} \geq 1.5s_j$. Thus, for $j \leq k/2 - c_1 \log \epsilon^{-1}$, we have $s_j \leq \epsilon$ where $c_1 = 1/\log 1.5$. Since $k/4 \leq k/2 - c_1 \log \epsilon^{-1}$, we have a contradiction.

Case 2: $r < k/2$.

We define

$$\bar{s}_{j+1} = \frac{\text{vol}(V \setminus B_u(j))}{\text{vol}(G)}.$$

Thus $\bar{s}_j < 1/2$ for all $j \geq k/2$. We consider two subcases.

Subcase 2a: Suppose $\bar{s}_j \geq \epsilon$ for some $j \geq k/2$.

Using Lemma 4, for j where $r \leq j \leq k/2$, we have $\bar{s}_j \geq 1.5\bar{s}_{j+1}$. Thus, for some $j_1 \geq k/2 - c_1 \log \epsilon^{-1}$, we have $\bar{s}_{j_1} \geq 1/2$ or equivalently, $s_{j_1} \leq 1/2$. By using Lemma 4 again, for $j \leq j_1$, we have $s_{j+1} \geq 1.5s_j$ and therefore for any $j \leq j_1 - c_1 \log \epsilon^{-1}$ we have $s_j \leq \epsilon$. Since $j_1 - c_1 \log \epsilon^{-1} \geq k/2 - 2c_1 \log \epsilon^{-1} \geq k/4$, we again have a contradiction to the assumption $s_{j_0} \geq \epsilon$.

Subcase 2b: Suppose $\bar{s}_j < \epsilon$ for all $j \geq k/2$

We apply part (ii) of Lemma 4 and we have, for $j \geq k/2$,

$$\frac{\bar{s}_j}{\bar{s}_{j+1}} \geq \frac{1}{(\sigma + 2\epsilon)^2}.$$

This implies, for $j_2 = \lceil k/2 \rceil$,

$$\frac{\bar{s}_{j_2}}{\bar{s}_k} \geq \prod_{k/2 < j < k} \frac{1}{(\sigma + 2s_j)^2} \geq \frac{1}{(\sigma + 2\epsilon)^k}.$$

Since $\bar{s}_k \geq 1/\text{vol}(G)$, we have

$$\bar{s}_{j_1} \geq \frac{1}{\text{vol}(G)(\sigma + 2\epsilon)^k}.$$

Since the assumption of this subcase is $\bar{s}_{j_1} < \epsilon$, we have

$$k \geq \frac{\log n + \log \epsilon^{-1}}{\log \sigma^{-1}}.$$

We now use Lemma 4 and we have, for $j = k/2 - j' \geq r$

$$\bar{s}_j \geq \frac{1}{\text{vol}(G)(\sigma + 2\epsilon)^{k+2j'}}.$$

Therefore, for some $j \leq k/2 - \log \epsilon^{-1} / \log \sigma^{-1}$, we have $\bar{s}_j > 1/2$ which implies $r \geq k/2 - \log \epsilon^{-1} / \log \sigma^{-1}$.

Now we use the same argument as in Case 1 except shifting r by $\log \epsilon^{-1} / \log \sigma^{-1}$. For some $j \leq r - c_1 \log \epsilon^{-1} \leq k/2 - \log \epsilon^{-1} / \log \sigma^{-1} - c_1 \log \epsilon^{-1}$, we have $s_j < \epsilon$. Since $\log \epsilon^{-1} / \log \sigma^{-1} + c_1 \log \epsilon^{-1} < k/4$, this leads to a contradiction and Theorem 7 is proved. \square

5 Non-backtracking random walks

Before we proceed to the proof of the Alon-Boppana bound, we will need some basic facts on non-backtracking random walks.

A non-backtracking walk is a sequence of vertices $\mathbf{p} = (v_0, v_1, \dots, v_t)$ for some t such that $v_{i-1} \sim v_i$ and $v_{i+1} \neq v_{i-1}$ for $i = 1, \dots, t-2$. The non-backtracking random walk can be described as follows: For $i \geq 1$, at the i th step on v_i , choose with equal probability a neighbor u of v_i where $u \neq v_{i-1}$, move to u and set $v_{i+1} = u$. To simplify notation, we call a non-backtracking walk an NB-walk. The modified transition probability matrix \tilde{P}_k , for $k = 0, 1, \dots, t-1$, is defined by

$$\tilde{P}_k(u, v) = \begin{cases} P^k(u, v) & \text{if } k = 0 \\ \sum_{\mathbf{p} \in \mathcal{P}_{u,v}^{(k)}} w(\mathbf{p}) & \text{if } k \geq 1 \end{cases} \quad (16)$$

where the weight $w(\mathbf{p})$ for an NB-walk $\mathbf{p} = (v_0, v_1, \dots, v_t)$ with $t \geq 1$ is defined to be

$$w(\mathbf{p}) = \frac{1}{d_{v_0} \prod_{i=1}^{t-1} (d_{v_i} - 1)} \quad (17)$$

and $\mathcal{P}_{u,v}^{(k)}$ denotes the set of non-backtracking walks from u to v . For a walk $\mathbf{p} = (v_0)$ of length 0, we define $w(\mathbf{p}) = 1$.

Although a non-backtracking random walk is not a Markov chain, it is closely related to an associated Markov chain as we will describe below (also see [6]).

For each edge $\{u, v\}$ in E , we consider two directed edges (u, v) and (v, u) . Let \hat{E} denote the set consisting of all such directed edges, i.e. $\hat{E} = \{(u, v) : \{u, v\} \in E\}$. We consider a random walk on \hat{E} with transition probability matrix \mathbf{P} defined as follows:

$$\mathbf{P}((u, v), (u', v')) = \begin{cases} \frac{1}{d_v - 1} & \text{if } v = u' \text{ and } u \neq v' \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{1}_E$ denote the all 1's function defined on the edge set E as a row vector. From the above definition, we have

$$\mathbf{1}_E \mathbf{P} = \mathbf{1}_E. \quad (18)$$

In addition, we define the vertex-edge incidence matrix B and B^* for $a \in V$ and $(b, c) \in \hat{E}$ by

$$B(a, (b, c)) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise} \end{cases}$$

$$B^*((b, c), a) = \begin{cases} 1 & \text{if } c = a, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{1}_V$ denote all 1's vector defined on the vertex set V . Then

$$\mathbf{1}_V B = \mathbf{1}_E. \quad (19)$$

Although \tilde{P}_k is not a Markov chain, it is related to the Markov chain determined by \mathbf{P} on \hat{E} as follows:

Fact 1: For $l \geq 1$.

$$\tilde{P}_l = D^{-1} B \mathbf{P}^l B^* \quad (20)$$

and for the case of $l = 0$, we have $\tilde{P}_0 = I$.

By combining (19) and (20), we have

Fact 2:

$$\mathbf{1}_V D \tilde{P}_l = \mathbf{1}_E B^* = \mathbf{1}_V D. \quad (21)$$

Note that $\mathbf{1}_V D$ is just the degree vector for the graph G . Therefore (21) states that the degree vector is an eigenvector of \tilde{P}_l . Using Fact 1 and 2, we have the following:

Lemma 8.

(i) For a fixed vertex x and any integer $j \geq 0$, we have

$$\sum_u d_u \sum_{\mathbf{p} \in \mathcal{P}_{u,x}^{(j)}} w(\mathbf{p}) = d_x \quad (22)$$

(ii) For a fixed vertex u , we have

$$\sum_x \sum_{\mathbf{p} \in \mathcal{P}_{u,x}^{(j)}} w(\mathbf{p}) = \mathbf{1}_u (I + \tilde{P}_1 + \dots + \tilde{P}_l) \mathbf{1}^* = l + 1 \quad (23)$$

where $\mathbf{1}_u$ denotes the characteristic function which assumes value 1 at u and 0 elsewhere.

Proof. The proof of (22) and (23) follows from the fact that

$$\mathbf{1}_V D \tilde{P}_j(x) = \mathbf{1}_V D (D^{-1} B \mathbf{P}^j B^*) = \mathbf{1}_E \mathbf{P}^j B^* = \mathbf{1}_E B^* = \mathbf{1}_V D(x)$$

and $\mathbf{1}_u \tilde{P}_j(x) = w(\mathbf{p})$ for $\mathbf{p} \in \mathcal{P}_{u,x}^{(j)}$. □

6 An Alon-Boppana bound for λ_1

Theorem 9. In a graph $G = (V, E)$ with diameter k , the first nontrivial eigenvalue λ_1 satisfies

$$\lambda_1 \leq 1 - \sigma \left(1 - \frac{5}{k}\right)$$

provided $\sigma = 2 \sum_v d_v \sqrt{d_v - 1} / \sum_v d_v^2 \leq 1/2$ and $k(1.5)^k \geq \sigma^{-1}$ where d_v denotes the degree of the vertex v with minimum degree at least 2.

Proof. If G is not a weak Ramanujan graph, we have $\lambda_1 \leq 1 - \sigma$ and we are done. We may assume that G is weak Ramanujan.

From the definition of λ_1 , we have

$$\lambda_1 \leq \frac{\sum_{x \sim y} (f(x) - f(y))^2}{\sum_x f^2(x) d_x} = R(f) \tag{24}$$

where f satisfies $\sum_x f(x) d_x = 0$.

We will construct an appropriate f satisfying $R(f) \leq 1 - \sigma(1 - c/k)$ and therefore serve as an upper bound for λ_1 .

We consider a family of functions defined as follows. For a specified vertex u and an integer $l = \lfloor k/4 \rfloor$, we consider a function $g_u : V \rightarrow \mathbb{R}^+$, defined by

$$\begin{aligned} g_u(x) &= \left(\mathbf{1}_u (I + \tilde{P}_1 + \dots + \tilde{P}_l)(x) \right)^{1/2} \\ &= \left(\sum_{j=0}^l \sum_{\mathbf{p} \in \mathcal{P}_{u,x}^{(j)}} w(\mathbf{p}) \right)^{1/2} \end{aligned}$$

where \tilde{P}_j is as defined in (20) and $\mathbf{1}_u$ is treated as a row vector. In other words, g_u denotes the square root of the sum of non-backtracking random walks starting from u taking i steps for i ranging from 0 to l .

Claim A:

$$\sum_u d_u \sum_x g_u^2(x) d_x = \sum_{j=0}^l \sum_x \sum_{\mathbf{p} \in \mathcal{P}_{u,x}^{(j)}} d_u w(\mathbf{p}) d_x = (l+1) \sum_x d_x^2$$

where the weight $w(\mathbf{p})$ of a walk \mathbf{p} is as defined in (17).

Proof of Claim A: From the definition of g_u and (16), we have

$$\begin{aligned}
\sum_u d_u \sum_x g_u^2(x) d_x &= \sum_{j=0}^l \sum_x \sum_{\mathbf{p} \in \mathcal{P}_{u,x}^{(l)}} d_u w(\mathbf{p}) \\
&= \sum_u d_u \mathbf{1}_u B(I + \tilde{P}_1 + \dots + \tilde{P}_l)(x) \\
&= \sum_u d_u \sum_{i=1}^l \mathbf{1}_u D^{-1} B \mathbf{P}^i B^*(x) d_x + \sum_x d_x^2 \\
&= \sum_{i=1}^l \sum_u \mathbf{1}_u B \mathbf{P}^i B^*(x) d_x + \sum_x d_x^2 \\
&= \sum_{i=1}^l \mathbf{1}_E \mathbf{P}^i B^*(x) d_x + \sum_x d_x^2 \\
&= l \mathbf{1}_E B^*(x) d_x + \sum_x d_x^2 \\
&= (l+1) \sum_x d_x^2.
\end{aligned}$$

Claim A is proved.

Claim B:

$$\sum_u d_u \sum_{x \sim y} (g_u(x) - g_u(y))^2 \leq (l+1 - l\sigma) \sum_x d_x^2.$$

where $\sum_{x \sim y}$ denotes the sum ranging over unordered pairs $\{x, y\}$ where x is adjacent to y .

Proof of Claim B:

We will use the following fact for $a_i, b_i > 0$.

$$\left(\sqrt{\sum_i a_i} - \sqrt{\sum_i b_i} \right)^2 \leq \sum_i \left(\sqrt{a_i} - \sqrt{b_i} \right)^2 \tag{25}$$

which can be easily checked.

For a fixed vertex u , we apply Claim B:

$$\begin{aligned}
& \sum_{x \sim y} (g_u(x) - g_u(y))^2 \\
&= \sum_{x \sim y} \left(\sqrt{\sum_{\substack{\mathbf{p} \in \mathcal{P}_{u,x}^{(t)} \\ t \leq l}} w(\mathbf{p})} - \sqrt{\sum_{\substack{\mathbf{p}' \in \mathcal{P}_{u,y}^{(t)} \\ t \leq l}} w(\mathbf{p}')} \right)^2 \\
&\leq \sum_{t \leq l-1} \sum_{r \in V} \sum_{\substack{\mathbf{p} \in \mathcal{P}_{u,r}^{(t)} \\ \mathbf{p}' = \mathbf{p} \cup s \in \mathcal{P}_{u,s}^{(t+1)}}} \left(\sqrt{w(\mathbf{p})} - \sqrt{w(\mathbf{p}')} \right)^2 + \sum_{\mathbf{p} \in \mathcal{P}_{u,x}^{(l)}} w(\mathbf{p})(d_x - 1) \\
&\leq \sum_{t \leq l-1} \sum_x \sum_{\mathbf{p} \in \mathcal{P}_{u,x}^{(t)}} \left(\sqrt{w(\mathbf{p})} - \sqrt{\frac{w(\mathbf{p})}{d_x - 1}} \right)^2 (d_x - 1) + \sum_{\mathbf{p} \in \mathcal{P}_{u,x}^{(l)}} \sqrt{w(\mathbf{p})}(d_x - 1) \\
&\leq \sum_{t \leq l-1} \sum_x \sum_{\mathbf{p} \in \mathcal{P}_{u,x}^{(t)}} w(\mathbf{p}) \left(1 + \frac{1}{d_x - 1} - \frac{2}{\sqrt{d_x - 1}} \right) (d_x - 1) + \sum_{\mathbf{p} \in \mathcal{P}_{u,x}^{(l)}} w(\mathbf{p})(d_x - 1) \\
&\leq \sum_{t \leq l-1} \sum_x \sum_{\mathbf{p} \in \mathcal{P}_{u,x}^{(t)}} w(\mathbf{p}) (d_x - 2\sqrt{d_x - 1}) + \sum_{\mathbf{p} \in \mathcal{P}_{u,x}^{(l)}} w(\mathbf{p})(d_x - 1).
\end{aligned}$$

Using Fact 3, we have

$$\begin{aligned}
& \sum_u d_u \sum_{x \sim y} (g_u(x) - g_u(y))^2 \\
&\leq \sum_{t \leq l-1} \sum_u d_u \sum_{\mathbf{p} \in \mathcal{P}_{u,x}^{(t)}} w(\mathbf{p}) (d_x - 2\sqrt{d_x - 1}) + \sum_u d_u \sum_{\mathbf{p} \in \mathcal{P}_{u,x}^{(l)}} w(\mathbf{p})(d_x - 1) \\
&= l \sum_x d_x (d_x - 2\sqrt{d_x - 1}) + \sum_x d_x^2 \\
&= l(1 - \sigma) \sum_x d_x^2 + \sum_x d_x^2 \\
&= (l + 1 - l\sigma) \sum_x d_x^2
\end{aligned}$$

This proves Claim B.

Claim C: There is a vertex u satisfying

$$R(g_u) \leq 1 - \sigma \left(1 - \frac{1}{l+1} \right)$$

Proof of Claim C:

Combining Claim A and B, we have

$$\begin{aligned}
 & \sum_u d_u \sum_{x \sim y} (g_u(x) - g_u(y))^2 \\
 & \leq (l + 1 - l\sigma) \sum_x d_x^2 \\
 & \leq (l + 1 - l\sigma) \left(\frac{1}{l + 1} \right) \sum_u d_u \sum_x g_u^2(x) d_x \\
 & = \left(1 - \frac{l\sigma}{l + 1} \right) \sum_u d_u \sum_x g_u^2(x) d_x
 \end{aligned} \tag{26}$$

Thus we deduce that there is a vertex u such that

$$\begin{aligned}
 R(g_u) &= \frac{\sum_{x \sim y} (g_u(x) - g_u(y))^2}{\sum_x g_u^2(x) d_x} \\
 &\leq 1 - \frac{l\sigma}{l + 1}.
 \end{aligned} \tag{27}$$

We define

$$\alpha_v = \frac{\sum_x g_v(x) d_x}{\sum_x d_x} = \frac{\sum_x g_v(x) d_x}{\text{vol}(G)}$$

We consider the function g'_u defined by

$$g'_u(x) = g_u(x) - \alpha_u$$

Clearly, g'_u satisfies the condition that

$$\sum_x g'_u(x) d_x = 0$$

Hence, we have

$$\begin{aligned}
 \lambda_1 \leq R(g'_u) &= \frac{\sum_{x \sim y} (g'_u(x) - g'_u(y))^2}{\sum_x g'^2_u(x) d_x} \\
 &= \frac{\sum_{x \sim y} (g_u(x) - g_u(y))^2}{\sum_x g_u^2(x) d_x - \alpha_u^2 \text{vol}(G)}.
 \end{aligned} \tag{28}$$

Note that by the Cauchy-Schwarz inequality, we have

$$\left(\sum_{x \in B_u(l)} g_u(x) d_x \right)^2 \leq \text{vol}(B_u(l)) \sum_{x \in B_u(l)} g_u^2(x) d_x.$$

and therefore

$$\alpha_u^2 \leq \frac{\text{vol}(B_u(l))}{\text{vol}(G)^2} \sum_x g_u^2(x) d_x.$$

By substitution into (28) and using (27), we have

$$\lambda_1 \leq R(g'_u) \leq \frac{R(g)}{1 - \frac{\text{vol}(B_u(l))}{\text{vol}(G)}} \leq \frac{1 - \sigma(1 - \frac{1}{l+1})}{1 - \frac{\text{vol}(B_u(l))}{\text{vol}(G)}} \quad (29)$$

$$\leq 1 - \sigma(1 - \frac{1}{l+1}) + \frac{\text{vol}(B_u(l))}{\text{vol}(G)} \quad (30)$$

$$\leq 1 - \sigma(1 - \frac{4}{k}) + \epsilon \quad (31)$$

$$\leq 1 - \sigma(1 - \frac{5}{k}) \quad (32)$$

The last inequality follows from Theorem 7 and the choice of $\epsilon = \sigma/k$ which satisfies $k \geq (\log \epsilon^{-1})/\log 1.5$. This completes the proof of Theorem 9. \square

7 A lower bound for λ_{n-1}

If a graph is bipartite, it is known (see [2]) that $\lambda_i = 2 - \lambda_{n-i-1}$ for all $0 \leq i \leq n-1$ and, in particular, $\lambda_{n-1} = 2 - \lambda_0 = 2$. If G is not bipartite, it is easy to derive the following lower bound:

$$\lambda_{n-1} \geq 1 + 1/(n-1)$$

by using the fact that the trace of \mathcal{L} is n . This lower bound is sharp for the complete graph. However if G is not the complete graph, is it possible to derive a better lower bound? The answer is affirmative. Here we give an improved lower bound for λ_{n-1} .

Theorem 10. *In a connected graph $G = (V, E)$ with diameter k , the largest eigenvalue λ_{n-1} of the normalized Laplacian \mathcal{L} of G satisfies*

$$\lambda_{n-1} \geq 1 + \sigma(1 - \frac{5}{k})$$

provided $\sigma = 2 \sum_v d_v \sqrt{d_v - 1} / \sum_v d_v^2 \leq 1/2$ and $k(1.5)^k \geq \sigma^{-1}$ where d_v denotes the degree of the vertex v with minimum degree at least 2.

Proof. By definition, λ_{n-1} satisfies

$$\lambda_{n-1} \geq \frac{\sum_{x \sim y} (f(x) - f(y))^2}{\sum_x f^2(x) d_x} = R(f) \quad (33)$$

for any $f : V \rightarrow \mathbb{R}$.

We will construct an appropriate f such that $R(f) \geq 1 + \sigma(1 - c/\gamma)$ by considering the following function $f_u : V \rightarrow \mathbb{R}^+$, for a fixed vertex u , defined by

$$\eta_u(x) = \begin{cases} (-1)^t \chi_u(\tilde{P}_t(x))^{-1/2} & \text{if } \text{dist}(u, x) = t \leq l \\ 0 & \text{otherwise} \end{cases}$$

where $l \leq \gamma/2$. Note that $|\eta_u(x)| = g_u(x)$ since we assume that $l \leq \gamma/2$. Using the same proof in Claim A, we have

Claim A':

$$\sum_u d_u \sum_x \eta_u^2(x) d_x = \sum_{j=0}^l \sum_x \sum_{\mathbf{p} \in \mathcal{P}_{u,x}^{(j)}} d_u w(\mathbf{p}) d_x = (l+1) \sum_x d_x^2.$$

Claim B':

$$\sum_u d_u \sum_{x \sim y} (\eta_u(x) - \eta_u(y))^2 \geq (l+1 + l\sigma) \sum_x d_x^2.$$

Proof of Claim B':

The proof is quite similar to that of Claim B. For a fixed vertex u , the sum over unordered pair $\{x, y\}$ where $x \sim y$,

$$\begin{aligned} & \sum_{x \sim y} (\eta_u(x) - \eta_u(y))^2 \\ & \leq \sum_{t \leq l-1} \sum_{r \in V} \sum_{\substack{\mathbf{p} \in \mathcal{P}_{u,r}^{(t)} \\ \mathbf{p}' = \mathbf{p} \cup s \in \mathcal{P}_{u,s}^{(t+1)}}} \left(\sqrt{w(\mathbf{p})} + \sqrt{w(\mathbf{p}')} \right)^2 - \sum_{\mathbf{p} \in \mathcal{P}_{u,x}^{(l)}} w(\mathbf{p})(d_x - 1) \\ & \leq \sum_{t \leq l-1} \sum_x \sum_{\mathbf{p} \in \mathcal{P}_{u,x}^{(t)}} \left(\sqrt{w(\mathbf{p})} + \sqrt{\frac{w(\mathbf{p})}{d_x - 1}} \right)^2 (d_x - 1) - \sum_{\mathbf{p} \in \mathcal{P}_{u,x}^{(l)}} \sqrt{w(\mathbf{p})}(d_x - 1) \\ & \leq \sum_{t \leq l-1} \sum_x \sum_{\mathbf{p} \in \mathcal{P}_{u,x}^{(t)}} w(\mathbf{p}) \left(1 + \frac{1}{d_x - 1} + \frac{2}{\sqrt{d_x - 1}} \right) (d_x - 1) - \sum_{\mathbf{p} \in \mathcal{P}_{u,x}^{(l)}} w(\mathbf{p})(d_x - 1) \\ & \leq \sum_{t \leq l-1} \sum_x \sum_{\mathbf{p} \in \mathcal{P}_{u,x}^{(t)}} w(\mathbf{p}) (d_x + 2\sqrt{d_x - 1}) - \sum_{\mathbf{p} \in \mathcal{P}_{u,x}^{(l)}} w(\mathbf{p})(d_x - 1). \end{aligned}$$

Using Fact 3, we have

$$\begin{aligned}
& \sum_u d_u \sum_{x \sim y} (\eta_u(x) - \eta_u(y))^2 \\
& \geq \sum_{t \leq l-1} \sum_u d_u \sum_{\mathbf{p} \in \mathcal{P}_{u,x}^{(t)}} w(\mathbf{p}) (d_x + 2\sqrt{d_x - 1}) - \sum_u d_u \sum_{\mathbf{p} \in \mathcal{P}_{u,x}^{(l)}} w(\mathbf{p}) (d_x - 1) \\
& = l \sum_x d_x (d_x + 2\sqrt{d_x - 1}) - \sum_x d_x^2 \\
& = l(1 + \sigma) \sum_x d_x^2 - \sum_x d_x^2 \\
& = (l - 1 + l\sigma) \sum_x d_x^2
\end{aligned}$$

This proves Claim B'.

Combining Claims A' and B', we have

$$\begin{aligned}
& \sum_u d_u \sum_{x \sim y} (\eta_u(x) - \eta_u(y))^2 \\
& \geq (l - 1 + l\sigma) \sum_x d_x^2 \\
& \geq (l - 1 + l\sigma) \left(\frac{1}{l+1} \right) \sum_u d_u \sum_x \eta_u^2(x) d_x \\
& = \left(1 + \frac{l\sigma}{l-1} \right) \sum_u d_u \sum_x \eta_u^2(x) d_x
\end{aligned} \tag{34}$$

Thus we deduce that there is a vertex u such that

$$\begin{aligned}
R(\eta_u) &= \frac{\sum_{x \sim y} (\eta_u(x) - \eta_u(y))^2}{\sum_x \eta_u^2(x) d_x} \\
&\leq 1 + \frac{l\sigma}{l-1}.
\end{aligned} \tag{35}$$

We consider the function η'_u defined by

$$\eta'_u(x) = \eta_u(x) - \alpha_u$$

where

$$\alpha_v = \frac{\sum_x \eta_v(x) d_x}{\sum_x d_x} = \frac{\sum_x \eta_v(x) d_x}{\text{vol}(G)}$$

so that η'_u satisfies the condition that

$$\sum_x \eta'_u(x) d_x = 0$$

Hence, we have

$$\begin{aligned}\lambda_{n-1} &\geq R(\eta'_u) = \frac{\sum_{x \sim y} (\eta'_u(x) - \eta'_u(y))^2}{\sum_x \eta'^2_u(x) d_x} \\ &= \frac{\sum_{x \sim y} (\eta_u(x) - \eta_u(y))^2}{\sum_x \eta'^2_u(x) d_x - \alpha_u^2 \text{vol}(G)} \\ &\geq 1 + \sigma \left(1 + \frac{c}{l}\right) - \frac{\text{vol}(B_u(l))}{\text{vol}(G)}.\end{aligned}$$

This completes the proof of Theorem 10. □

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