The drop polynomial of a weighted digraph

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Abstract

Given a directed graph $D = (V, E)$ with n vertices and a permutation $\pi : V \to V$ on its vertices, we say that π has a *drop* at a vertex $u \in V$ is $(u, \pi(u))$ is an edge of D. Letting $\left\langle {D \atop k} \right\rangle$ denote the number of permutations on V with k drops, we can define the binomial drop polynomial $B_D(x) = \sum_k \left\langle {D \atop k} \right\rangle \binom{x+k}{n}$. In this paper we study various properties of $B_D(x)$ and its generalization to the case when the edges of D are assigned weights. In particular, $B_D(x)$ satisfies a natural deletion/contraction recursion, quite similar to this type of recursion for the celebrated Tutte polynomial (for graphs) and the path/cycle cover polynomial for digraphs.

1 Introduction

Suppose $P = (V, \prec)$ is a partially-ordered set (= poset) on a vertex set V of size n. For a permutation (= bijection) $\pi : V \to V$, we say that π has a *drop* at v if $\pi(v) \prec v$. We denote by $\left\langle {P \atop k} \right\rangle$ the number of permutations π on V which have exactly k drops. Also we denote by $\overline{P}$, the *incomparability graph* of P. That is, $\overline{P} = (V, E)$ where the edge set E consists of all those pairs of vertices $\{u, v\}$ where neither $u \prec v$ nor $v \prec u$ hold.

To the best of the authors' knowledge, the following surprising result first appeared in [14] (see also [4, 5]).

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Theorem. For a poset P on n vertices,

$$\sum_{0\le k<n} \left\langle {P \atop k} \right\rangle \binom{\lambda+k}{n} = \chi_{\overline{P}}(\lambda) \tag{1}$$

where $\chi_{\overline{P}}(\lambda)$ denotes the *chromatic polynomial* of $\overline{P}$, i.e., $\chi_{\overline{P}}(\lambda)$ is the number of ways to color the vertices of $\overline{P}$ using colors from a set of λ colors so that adjacent vertices have different colors. In particular, when P is a *chain* (= linearly ordered set) on n points, then the $\left\langle {P \atop k} \right\rangle$ are just $\left\langle {n \atop k} \right\rangle$, the usual Eulerian numbers (see [16]). In this case, $\overline{P}$ consists of n independent points so that $\chi_{\overline{P}}(\lambda) = \lambda^n$. Thus, (1) becomes

$$\sum_{0\le k<n} \left\langle {n \atop k} \right\rangle \binom{\lambda+k}{n} = \lambda^n, \tag{2}$$

a relation that is usually called Worpitsky's identity (see [16]). In fact, this identity was used in the analysis of certain juggling patterns [3, 4] and it was in this context that the current paper was motivated.

Note that if we set $\lambda = -1$ in (1) then the LHS becomes

$$\sum_{0\le k<n} \left\langle {P \atop k} \right\rangle \binom{k-1}{n} = (-1)^n \left\langle {P \atop 0} \right\rangle \tag{3}$$

since

$$\binom{t}{n} = \frac{\prod_{i=0}^{n-1}(t-i)}{n!} = \begin{cases} (-1)^n & \text{if } t=-1, \\ 0 & \text{if } 0 \le t < n, \end{cases} \tag{4}$$

On the other hand, the RHS of (1) becomes $\chi_{\bar{P}}(-1)$ which by a classic result of Stanley [19] is just equal to $(-1)^n$ times the number of *acyclic orientations* of $\bar{P}$. Hence, we have the

Theorem. **[4]** *The number of drop-free permutations on P is equal to the number of acyclic orientations on the complementary graph $\bar{P}$.*

(A nice bijective proof of this appears in [21]).

Our main purpose in this paper is to generalize these ideas to arbitrary weighted directed graphs (=digraphs) D. Previously, in the effort of finding an analog of the Tutte polynomial for digraphs, the deletion/contraction rules were defined for digraphs and the (path/cycle) cover polynomial of a digraph was introduced in [9]. Since then, a number of papers further investigated the cover polynomial relating to various digraph properties (see [8, 10]). In particular, the cover polynomial of a digraph is intimately related to enumerating various path/cycle vertex covers of a digraph with natural connections to coloring enumerations and rook polynomials (see [6, 7, 11, 12, 13, 15, 17, 20]). In this paper, we study digraph polynomials with a perspective somewhat different from that of the cover polynomial. Our emphasis here is on polynomials that reveal invariants for enumerating various types of permutations on the vertex set of a given digraph.

In this paper, we will define a binomial drop polynomial $B_D(x)$ for weighted digraphs and study its properties. A key fact we establish for $B_D(x)$ is a general deletion/contraction recursion from which these polynomials can be derived. When D is restricted to the class of *simple* digraphs (i.e., with all edge weights equal to 0 or 1), the deletion/contraction rules are the same as in [9]. However they are different for digraphs with general edge weights.

2 The binomial drop polynomial

For a digraph $D = (V, E)$, each edge $e = (u, v) \in E$ will be indicated by a directed arc from u to v, i.e., $u \to v$. We will usually assume that $|V| = n$. If $u = v$, the edge $e = (u, u)$ is called a *loop* at u. For now, we will assume that D is *simple*, i.e., all the edge weights are 0 or 1. (Later, this restriction will be removed). Thus, the *adjacency matrix* $M = M_D$ associated to D is a matrix indexed by the vertices of D with $M(u, v) = 1$ if (u, v) is an edge of D and is 0 otherwise. We say that a permutation π on D (really, on V) has a *drop* at u if $(u, \pi(u)) \in E$. In particular, if (u, u) is a loop in D and $\pi(u) = u$, then this is also drop at u. For $0 \le k \le n = |V|$, we define $\left\langle {D \atop k} \right\rangle$ to be the number of permutations on D which have exactly k drops. In particular, $\left\langle {D \atop 0} \right\rangle$

is the number of drop-free permutations on D, something that we will return to later in the paper. As noted earlier, when D is the transitive closure of a chain then $\left\langle {D \atop k} \right\rangle = \left\langle {n \atop k} \right\rangle$, the usual Eulerian number. Thus, $\left\langle {D \atop k} \right\rangle$ can be viewed as a generalization of Eulerian numbers for simple digraphs. For example, the

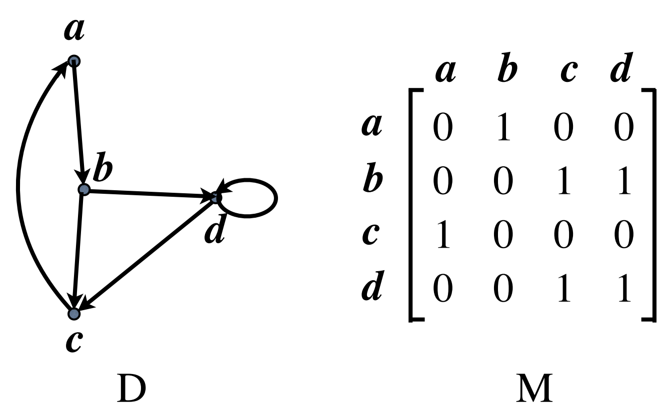

Figure 1: A simple digraph D and its associated matrix M.

digraph shown in Fig. 1 has 4 permutations with 4 drops, no permutations with 3 drops, 10 permutations with 2 drops, 8 permutations with 1 drop, and 4 permutations with no drops, as listed in the following table.

k	$\left\langle {D \atop k} \right\rangle$	permutations with k drops
4	2	$(abc), (abdc)$
3	0	$\emptyset$
2	10	$(bc), (bdc), (ab), (ab)(cd), (abcd), (abd), (ac), (ac)(bd), (adc), (adbc)$
1	8	*identity*, $(cd), (bcd), (bd), (acb), (acbd), (adcb), (ad)(bc)$
0	4	$(acdb), (acd), (adb), (ad)$

Table 1: Table of permutations on D.

We can summarize this information in the form of a polynomial which we call the *binomial drop* polynomial for D, denoted by $B_D(x)$, and defined as follows:

Definition.

$$B_D(x) = \sum_k \left\langle {D \atop k} \right\rangle \binom{x+k}{n} \tag{5}$$

where, as usual, n denotes the number of vertices of D.
In the case of the digraph shown in Fig 1, we have

$$\begin{aligned} B_D(x) &= 2\binom{x+4}{4} + 10\binom{x+2}{4} + 8\binom{x+1}{4} + 4\binom{x}{4} \\ &= x^4 + 4x^2 + 3x + 2 \end{aligned}$$

when expanded out.

The above definition for the binomial drop polynomial can be applied for digraphs with multiple edges. In this case, suppose $e = (u, v)$ and there are s edges from u to v. If $\pi(u) = v$ then this counts as contributing s drops to the number of drops of π. In Figure 2 we have an example of a digraph with multiple edges together with its binomial drop polynomial.

We will consider five families of (increasingly general) digraphs:

$\mathscr{D}_0$ — the family of *transitive* digraphs, i.e., such that $(u, v) \in E$ and $(v, w) \in E$ implies that $(u, w) \in E$. This is the case when D is derived from a poset.

$\mathscr{D}_1$ — the family of *acyclic* digraphs.

$\mathscr{D}_2$ — the family of *simple* digraphs with loops allowed.

$\mathscr{D}_3$ — the family of digraphs with *multiple edges and loops* allowed. This family of digraphs can be viewed as having positive integral edge weights.

$\mathscr{D}_4$ — the family of digraphs for which we allow *real-valued weights* on the edges and loops.

Clearly, $\mathscr{D}_0 \subset \mathscr{D}_1 \subset \mathscr{D}_2 \subset \mathscr{D}_3 \subset \mathscr{D}_4$.

We will see that all of these five families are closed under the operations of deletion and contraction, something which will be defined in the next section.

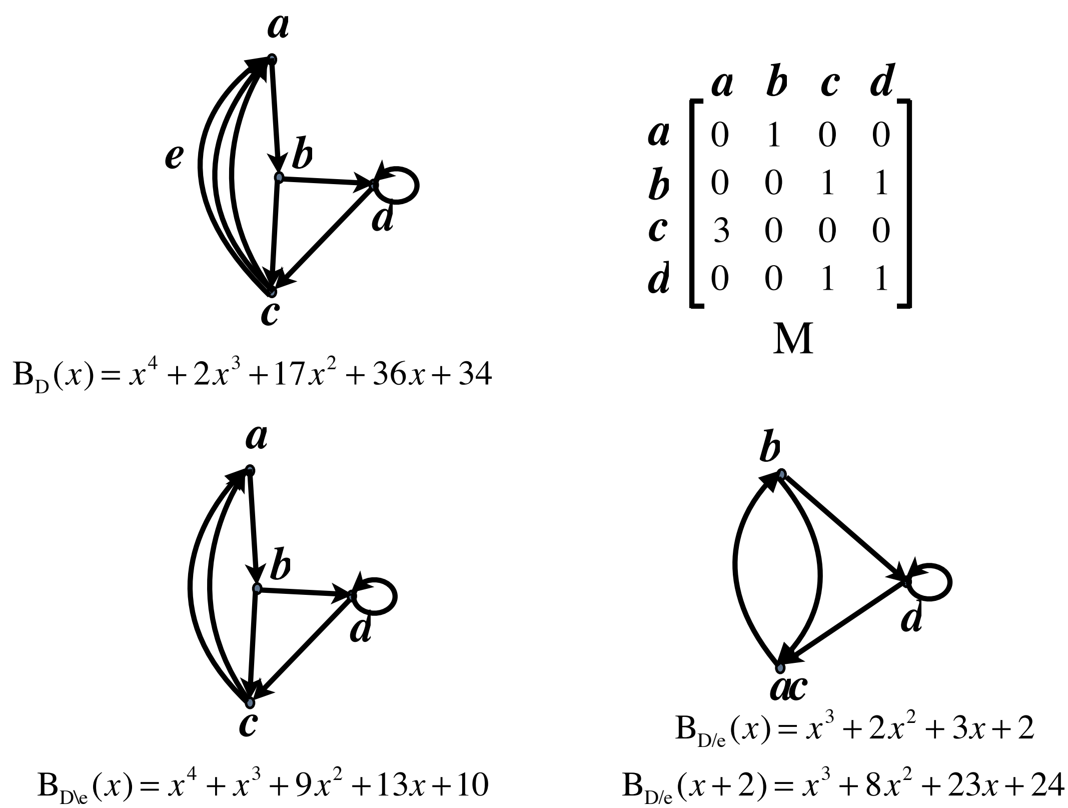

Figure 2: A digraph with multiple edges.

3 Deletion and contraction in digraphs

In order to manipulate $B_D(x)$, we will first need to define the operations of *deletion* and *contraction* on a digraph D. First, suppose $e = (u, v)$ is a non-loop edge of D (so $u \neq v$). The vertex set for the *deleted* digraph $D \setminus e$ is still V and the edge set is just $E \setminus e$, i.e., we just remove the edge e from E.

The *contracted* digraph D/e is slightly more complicated. For the vertex set of D/e we replace the two vertices u and v by a single vertex uv. Any edge in D of the form (x, u) becomes an edge (x, uv) in D/e. Also, any edge in D of the form (v, y) becomes an edge (uv, y) in D/e. All other edges incident to either u or v (including loops) are deleted. If (v, u) happens to be an edge in D, it becomes a loop at uv in D/e. All other edges (not incident to u or v) in D are retained in D/e (see Fig 3).

If $e = (u, u)$ is a loop at u in D, then we do the following. For the edge set of the *deleted* digraph $D \setminus e$, we just remove e. To form the *contracted* digraph D/e, we simply delete the vertex u and *all* edges incident to u (see Fig 4).

We point out that these definitions of deletion and contraction in a digraph are the same as those used in our previous papers on path/cycle cover polynomials [9, 10].

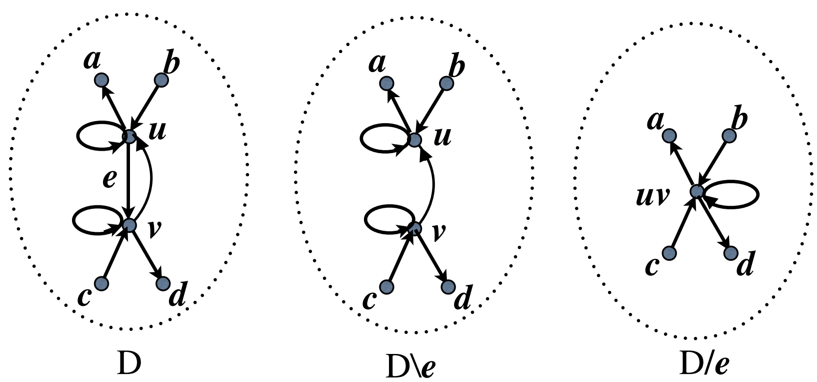

Figure 3: Deletion and contraction of a non-loop edge e in a digraph D.

4 A recursion for $B_D(x)$.

In this section we will derive a recursion for $B_D(x)$ which allows one to express it in terms of $B_{D\setminus e}$ and $B_{D/e}$ for an edge $e \in E$.. This is very similar to what happens for the Tutte polynomial [23] and the path/cycle cover polynomial $C_D(x, y)$ (see [9]). We point out that in [9] $C_D(x, y)$ was only defined for simple digraphs $D \in \mathscr{D}_2$. This restriction was removed in [10],

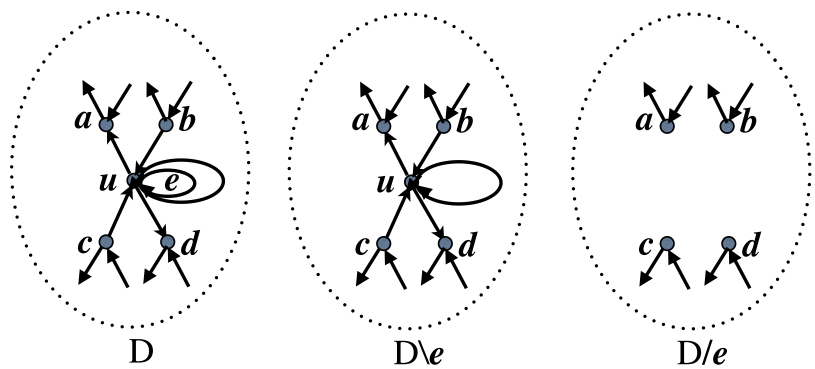

Figure 4: Deletion and contraction of a loop $e = (u, u)$ in a digraph D.

where $C_D(x, y)$ was defined for all $D \in \mathscr{D}_4$. In this section, we will derive recurrences for digraphs in $\mathscr{D}_3$ having multiple edges and loops.

We first will prove several useful facts.

Lemma 1. *Suppose there are s edges from u to v and one of these edges is denoted by e (where u is not required to be different from v). For an integer $k \geq 0$, the number of permutations π with $k + s$ drops on D satisfying the condition $\pi(u) = v$ is exactly $\left\langle {D/e \atop k} \right\rangle$.*

Proof. Let σ be a permutation with k drops in D/e. We consider two possibilities.

Case 1: $e = (u, v)$ is a non-loop edge.
Let uv denote the contracted vertex in D/e formed when e was contracted. Thus, $V = V(D) = V(D/e) \cup \{uv\}$. We can extend σ to a permutation π on V as follows (see Fig. 6). Define

$$\pi(x) = \begin{cases} \sigma(uv) & \text{if } x = v, \\ v & \text{if } x = u, \\ u & \text{if } \sigma(x) = uv, \\ \sigma(x) & \text{otherwise.} \end{cases}$$

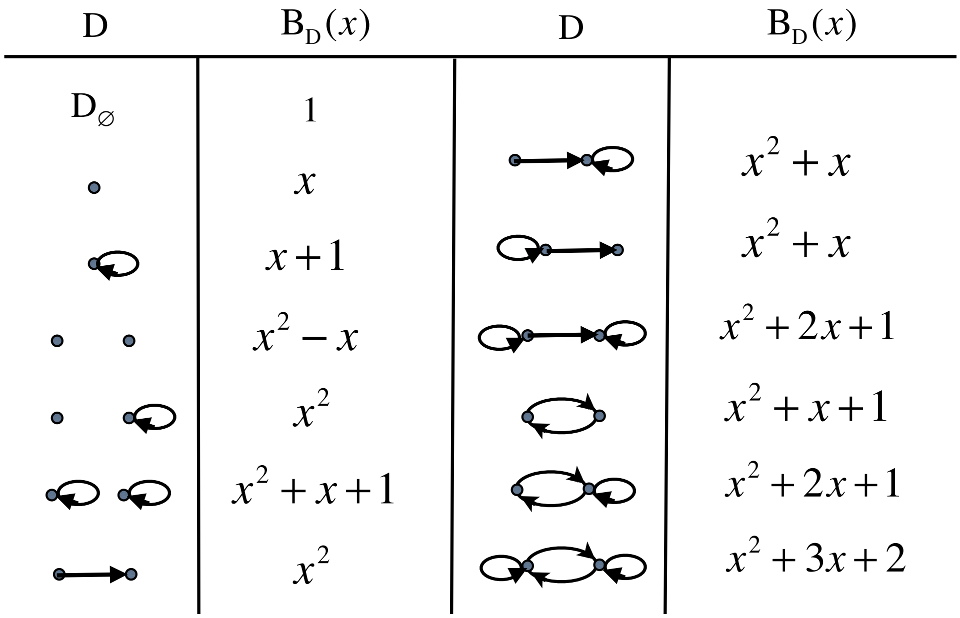

Figure 5: Binomial drop polynomials for small digraphs.

Any drop in D/e is still a drop in D. Conversely, any permutation π on V can be mapped to a permutation σ on $V(D/e)$ by defining

$$\sigma(y) = \begin{cases} \pi(v) & \text{if } y = uv, \\ uv & \text{if } \pi(y) = u, \\ \pi(y) & \text{otherwise.} \end{cases}$$

Case 2: $e = (u, u)$ is a loop.
In this case, $V(D) = V(D/e) \cup \{u\}$ and

$$\pi(x) = \begin{cases} \sigma(x)) & \text{if } x \neq u, \\ u & \text{if } x = u. \end{cases}$$

This is an easy bijection and the proof of the lemma is complete. □

Now for our first main result.

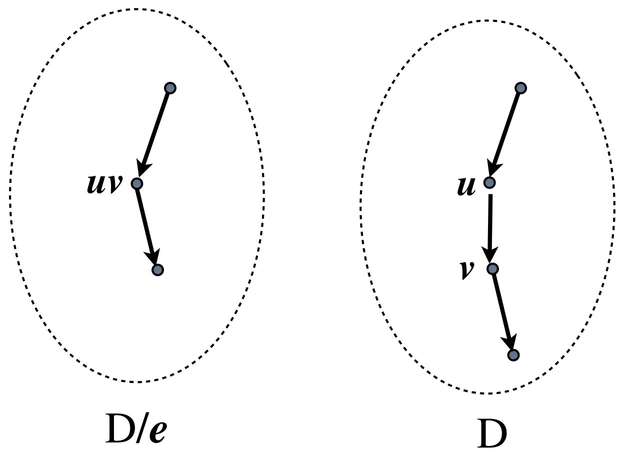

Figure 6: Extending the permutation σ.

Theorem 1. *Suppose there are s edges from u to v and one of these edges is denoted by e (where u is not required to be different from v). For any $k \geq 0$, we have*

$$\left\langle {D \atop k} \right\rangle = \left\langle {D \setminus e^{(s)} \atop k} \right\rangle - \left\langle {D/e \atop k} \right\rangle + \left\langle {D/e \atop k-s} \right\rangle \tag{6}$$

where $D \setminus e^{(s)}$ denotes the digraph resulting by removing all s loops from v in D and we use the convention that $\left\langle {D \atop j} \right\rangle = 0$ if $j < 0$.

Proof.

$$\begin{aligned}
\left\langle {D \setminus e^{(s)} \atop k} \right\rangle &= \#\{\pi \text{ has } k \text{ drops in } D \setminus e^{(s)}\} \\
&= \#\{\pi \text{ has } k \text{ drops in } D \setminus e^{(s)} \text{ and } \pi(u) = v\} \\
&+ \#\{\pi \text{ has } k \text{ drops in } D \setminus e^{(s)} \text{ and } \pi(u) \neq v\} \\
&= \#\{\pi \text{ has } k + s \text{ drops in } D \text{ and } \pi(u) = v\} \\
&+ \#\{\pi \text{ has } k \text{ drops in } D \text{ and } \pi(u) \neq v\} \\
&= \#\{\pi \text{ has } k \text{ drops in } D/e\} \text{ (by Lemma 1)} \\
&+ \#\{\pi \text{ has } k \text{ drops in } D\} - \#\{\pi \text{ has } k \text{ drops in } D \text{ and } \pi(u) = v\} \\
&= \left\langle {D/e \atop k} \right\rangle + \left\langle {D \atop k} \right\rangle - \left\langle {D/e \atop k-s} \right\rangle
\end{aligned}$$

using the fact that $\left\langle {D/e \atop i} \right\rangle = 0$ if $i < 0$. This completes the proof of Theorem 1. □

Theorem 2. *For a digraph D on n vertices, suppose there are s edges from u to v and one of the edges is denoted by e. Then*

$$\left\langle {D \atop k} \right\rangle = \left\langle {D \setminus e \atop k} \right\rangle - \left\langle {D/e \atop k-s+1} \right\rangle + \left\langle {D/e \atop k-s} \right\rangle.$$

Proof. We apply Theorem 1 to D and $D \setminus e$, respectively, to obtain

$$\begin{aligned}
\left\langle {D \atop k} \right\rangle &= \left\langle {D \setminus e^{(s)} \atop k} \right\rangle - \left\langle {D/e \atop k} \right\rangle + \left\langle {D/e \atop k-s} \right\rangle \\
&= \left\langle {D \setminus e \atop k} \right\rangle + \Big(- \left\langle {D \setminus e \atop k} \right\rangle + \left\langle {D \setminus e^{(s)} \atop k} \right\rangle \\
&- \left\langle {D/e \atop k} \right\rangle + \left\langle {D/e \atop k-s+1} \right\rangle \Big) - \left\langle {D/e \atop k-s+1} \right\rangle + \left\langle {D/e \atop k-s} \right\rangle \\
&= \left\langle {D \setminus e \atop k} \right\rangle - \left\langle {D/e \atop k-s+1} \right\rangle + \left\langle {D/e \atop k-s} \right\rangle
\end{aligned}$$

as claimed. □

We point out that for the case $k = 0$, the recurrence for $\left\langle {D \atop 0} \right\rangle$ (= the number of drop-free permutations on D), is particularly interesting. From Theorem 1 we find

$$\left\langle {D \setminus e \atop 0} \right\rangle = \begin{cases} \left\langle {D \atop 0} \right\rangle + \left\langle {D/e \atop 0} \right\rangle & \text{if } e \text{ is not a loop,} \\ \\ \left\langle {D \atop 0} \right\rangle + \left\langle {D \setminus u \atop 0} \right\rangle & \text{if } e \text{ is a loop at } u. \end{cases} \tag{7}$$

Recall the definition of the binomial drop polynomial $B_D(x)$ from (5):

$$B_D(x) = \sum_k \left\langle {D \atop k} \right\rangle \binom{x+k}{n}$$

where we assume that D has n vertices. We next show how $B_D(x)$ can be computed recursively using deletion and contraction.

Theorem 3. *The binomial drop polynomial $B_D(x)$ satisfies the following properties:*

(i) *Suppose there are s edges from u to v and one of the edges is denoted by e (where u and v are not required to be different). Then*

$$B_D(x) = B_{D \setminus e}(x) + B_{D/e}(x + s - 1).$$

In particular, for $s = 1$, we have

$$B_D(x) = B_{D \setminus e}(x) + B_{D/e}(x).$$

(ii) *If $D = I_n$ consists of n isolated points, then*

$$B_{I_n}(x) = n! \binom{x}{n} = x^{\underline{n}}$$

where $x^{\underline{n}}$ denotes the falling factorial $x(x-1)(x-2)\cdots(x-n+1)$.

(iii) *If $D = D_\emptyset$ is the digraph with no vertices, then by convention we set $B_{D_\emptyset}(x) = 1$.*

Furthermore, $B_D(x)$ is the only digraph function which satisfies all the Properties **(i), (ii) and (iii)**.

Proof. "$\Longrightarrow$"

From the definition of $B_D(x)$ we have

$$\begin{aligned}
B_D(x) &= \sum_k \left\langle {D \atop k} \right\rangle \binom{x+k}{n} \\
&= \sum_k \left(\left\langle {D \setminus e \atop k} \right\rangle + \left\langle {D/e \atop k-s} \right\rangle - \left\langle {D/e \atop k-s+1} \right\rangle \right) \binom{x+k}{n} \qquad \text{by Theorem 2} \\
&= B_{D\setminus e}(x) + \sum_k \left\langle {D/e \atop k} \right\rangle \left(\binom{x+k+s}{n} - \binom{x+k+s-1}{n} \right) \\
&= B_{D\setminus e}(x) + \sum_k \left\langle {D/e \atop k} \right\rangle \binom{x+k+s-1}{n-1} \\
&= B_{D\setminus e}(x) + B_{D/e}(x+s-1).
\end{aligned}$$

This completes the proof in the "$\Longrightarrow$" direction.

To prove the theorem in the "$\Longleftarrow$" direction, suppose that Properties **(i–iii)** hold for some unknown function $F_D(x)$ on all digraphs. We will assume by induction that $F_D(x) = B_D(x)$ for all digraphs having fewer than r edges. The induction hypothesis certainly holds for digraphs with no edges (by **(i)** and **(iii)**). We consider a digraph D which has r edges. If D has s edges from u to v and e is one of these edges where $s \geq 1$. Then by **(i)** we have by induction:

$$\begin{aligned}
F_D(x) &= F_{D\setminus e}(x) + F_{D/e}(x+s-1) \\
&= \sum_k \left\langle {D\setminus e \atop k} \right\rangle \binom{x+k}{n} + \sum_k \left\langle {D/e \atop k} \right\rangle \binom{x+k+s-1}{n-1} \\
&= \sum_k \left\langle {D\setminus e \atop k} \right\rangle \binom{x+k}{n} + \sum_k \left\langle {D/e \atop k} \right\rangle \left(\binom{x+k+s}{n} - \binom{x+k+s-1}{n} \right) \\
&= \sum_k \left(\left\langle {D\setminus e \atop k} \right\rangle + \left\langle {D/e \atop k-s} \right\rangle - \left\langle {D/e \atop k-s+1} \right\rangle \right) \binom{x+k}{n} \\
&= \sum_k \left\langle {D \atop k} \right\rangle \binom{x+k}{n} \qquad \text{by Theorem 1} \\
&= B_D(x).
\end{aligned}$$

This completes the induction step and we can conclude that $F_D(x) = B_D(x)$ for all D. This completes the proof of Theorem 3. □

We point out that because the polynomial $B_D(x)$ can be defined in terms of counting drops of permutations on D, then the *order* in which the deletions and contractions are applied to reduce D to collections of independent sets I_k does not matter. Ordinarily, if one were to define some arbitrary deletion and contraction rules for graphs or digraphs, the order in which they were applied would matter, and so the resulting polynomials wouldn't be well defined. However, there are a number of cases where you do get uniqueness, such as the Tutte polynomial, the path/cycle cover polynomial and (this) binomial drop polynomial. It would be interesting to develop a more general theory for this phenomenon (cf. [2]). In Figures 7 and 8, we show the digraph from Fig. 1 and several of its deletions and contractions, both with an edge and with a loop. We also list the corresponding polynomials (which fortunately add up correctly!).

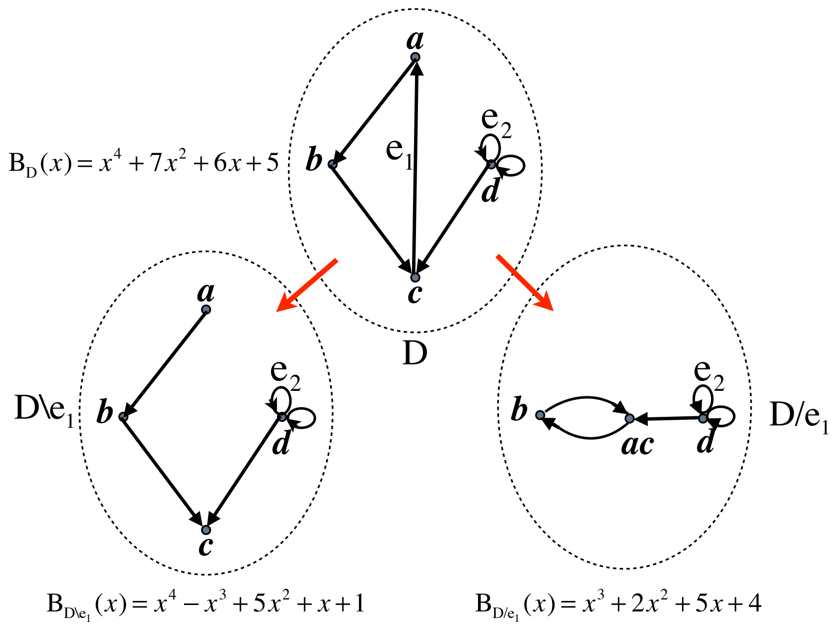

Figure 7: Deletion and contraction using the edge e_1.

5 Acyclic orientations for acyclic graphs

We have noted earlier that when D is transitive then:

$$\left\langle {D \atop k} \right\rangle = \#\{\text{acyclic orientations of the (undirected) complementary graph } \bar{D}\}.$$

We will show that this equality holds only if D is transitive. This will follow from considerations for more general digraphs D. In particular, in this section we will focus on the family $\mathscr{D}_2$, the set of digraphs which are simple. Clearly, $\mathscr{D}_2$ is closed under deletion and contraction. If $D \in \mathscr{D}_2$ has n vertices then we define a (vertex) *path/cycle cover* C of D to be a sub-digraph consisting of vertex disjoint (directed) paths and (directed) cycles which cover all n vertices of D (see [9, 10]). We note that in a path/cycle cover C, each vertex has indegree and outdegree at most 1.

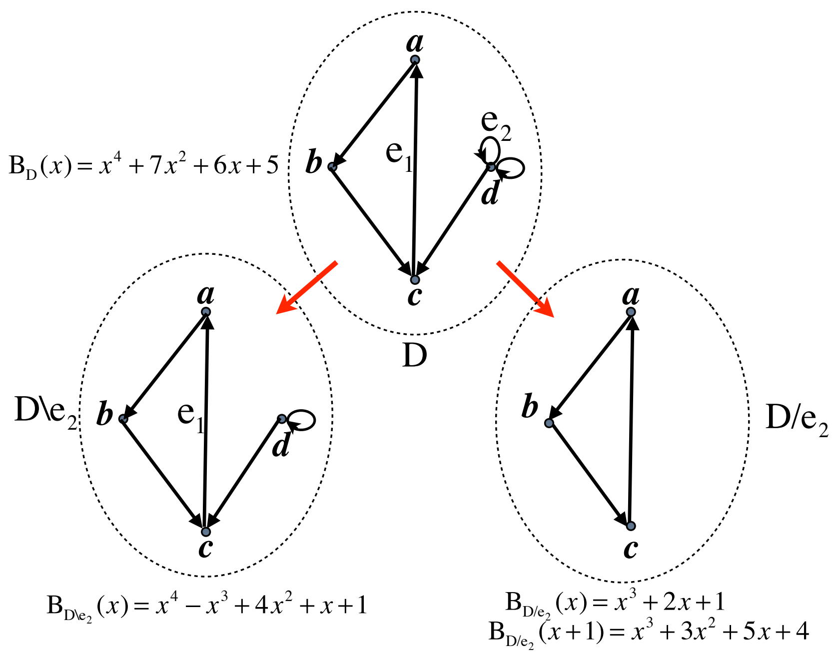

Figure 8: Deletion and contraction using the loop e_2.

Theorem 4. *Suppose that $e_1, e_2, \ldots, e_k$ are distinct edges in $D \in \mathscr{D}_2$. Then*

$$\left\langle \begin{matrix} D \setminus e_1 \setminus e_2 \ldots \setminus e_k \\ 0 \end{matrix} \right\rangle = \sum_{S \subseteq \{e_1, e_2, \ldots, e_k\}} \left\langle \begin{matrix} D/S \\ 0 \end{matrix} \right\rangle \tag{8}$$

where S ranges over all subsets of path/cycle covers of D and for $S = \{e_{i_1}, e_{i_2} \ldots, e_{i_r}\}$, we define $D/S = D/e_{i_1}/e_{i_2} \ldots /e_{i_r}$.

It is easy to check that the definition of D/S makes sense when S consists of edges in a path/cycle cover C of D. It suffices to observe that in this case if e_i and e_j are two edges in C, then $D/e_i/e_j = D/e_j/e_i$. We will use the convention that $\left\langle \begin{smallmatrix} D/S \\ k \end{smallmatrix} \right\rangle = 0$ if S is not a subset of path/cycle cover of D.

Proof. We will prove (8) by induction on k. For the base case $k = 1$, we have by Lemma 1,

$$\left\langle {D \setminus e_1 \atop 0} \right\rangle = \left\langle {D \atop 0} \right\rangle + \left\langle {D/e_1 \atop 0} \right\rangle = \sum_{\substack{S \subseteq \{e_1\} \\ S \subseteq \text{path/cycle cover}}} \left\langle {D/e_1 \atop 0} \right\rangle$$

as required.

Now assume that (8) holds for all $k' < k$ for some fixed $k \geq 1$. Then

$$\begin{aligned} \left\langle {D \setminus e_1 \setminus \ldots \setminus e_k \atop 0} \right\rangle &= \left\langle {D \setminus e_1 \setminus \ldots \setminus e_{k-1} \atop 0} \right\rangle + \left\langle {(D \setminus e_1 \setminus \ldots \setminus e_{k-1})/e_k \atop 0} \right\rangle \\ &= \sum_{\substack{S \subseteq \{e_1, \ldots e_{k-1}\} \\ S \subseteq \text{path/cycle cover}}} \left\langle {D/S \atop 0} \right\rangle + \left\langle {(D \setminus e_1 \setminus \ldots \setminus e_{k-1})/e_k \atop 0} \right\rangle \end{aligned}$$

by the induction hypothesis. We now focus on the term

$$\left\langle {(D \setminus e_1 \setminus \ldots \setminus e_{k-1})/e_k \atop 0} \right\rangle. \tag{9}$$

Write $e_k = (u, v)$ and consider the set S^* consisting of those e_i which do not start at u and which do not end at v. That is,

$$S^* = \{e_i = (u_i, v_i), 1 \leq i \leq k-1, \text{where } u_i \neq u \text{ and } v_i \neq v\}.$$

Without loss of generality (by relabeling if necessary) we can write $S^* = \{e_1, e_2, \ldots e_t\}$ for some $t \leq k-1$.

Claim A.

$$(D \setminus e_1 \setminus \ldots \setminus e_{k-1})/e_k = (D/e_k) \setminus e_1^* \setminus \ldots \setminus e_t^*$$

where for an edge (u, v) we define $e^* = (u^*, v^*)$ by

$$u^* = \begin{cases} u & \text{if } u \neq v, \\ uv & \text{if } u = v, \end{cases} \qquad \text{and} \qquad v^* = \begin{cases} v & \text{if } v \neq u, \\ uv & \text{if } v = u. \end{cases}$$

and uv is the (contracted) vertex arising from the contracted edge $e_k = (u, v)$.

Proof of Claim A. It is straightforward to check that the map of $e = (u, v)$

to $e^* = (u^*, v^*)$ is a bijection between the edge sets of $(D \setminus e_1 \setminus \ldots \setminus e_{k-1})/e_k$ and $(D/e_k) \setminus e_1^* \setminus \ldots \setminus e_t^*$.

Proof of Claim B. "$\Longrightarrow$" follows from the definition of S^*. For the "$\Longleftarrow$" direction, note that if $S \subseteq \{e_1, \ldots, e_{k-1}, e_k\}$ is a path/cycle cover of D and $e_k \in S$ then for any $e_j \in S$, we have $e_j = (u_j, v_j)$ with $u_j \neq u$ and $v_j \neq v$. This proves Claim B.

We now go back to the expression (9). We have

$$\begin{aligned}
&\left\langle {(D \setminus e_1 \setminus \ldots \setminus e_{k-1})/e_k \atop 0} \right\rangle \\
&= \left\langle {(D/e_k) \setminus e_1^* \setminus \ldots \setminus e_t^* \atop 0} \right\rangle && \text{(by the def. of } S\text{)} \\
&= \sum_{\substack{T \subseteq \{e_1^*, \ldots e_t^*\} \\ T \subseteq \text{path/cycle cover of} D/e_k}} \left\langle {(D/e_k)/T \atop 0} \right\rangle && \text{(by induction)} \\
&= \sum_{\substack{S \subseteq \{e_1, \ldots e_{k-1}\} \\ S \cup \{e_k\} \subseteq \text{path/cycle cover of} D}} \left\langle {D/e_k/S \atop 0} \right\rangle + \left\langle {(D \setminus e_1 \setminus \ldots \setminus e_{k-1})/e_k \atop 0} \right\rangle
\end{aligned}$$

by Claim B.

Returning to the main proof, we can now write

$$\begin{aligned}
\left\langle {D \setminus e_1 \setminus \ldots \setminus e_k \atop 0} \right\rangle &= \sum_{\substack{S \subseteq \{e_1, \ldots e_{k-1}\} \\ S \subseteq \text{path/cycle cover}}} \left\langle {D/S \atop 0} \right\rangle + \sum_{\substack{S \subseteq \{e_1, \ldots e_{k-1}\} \\ S \cup \{e_k\} \subseteq \text{path/cycle cover}}} \left\langle {D/e_k/S \atop 0} \right\rangle \\
&= \sum_{\substack{S \subseteq \{e_1, \ldots e_k\} \\ S \subseteq \text{path/cycle cover of D}}} \left\langle {D/S \atop 0} \right\rangle
\end{aligned}$$

□

as desired. This proves Theorem 4.

Finally, we turn to acyclic orientations. As mentioned earlier, it is known that if D is transitive then the number of drop-free permutations on D is equal to the number of acyclic orientations of the complementary (undirected) graph $\bar{D}$. In other words, if we let $\mathbf{AO}(G)$ denote the number of *acyclic orientations* of a graph G, then for any transitive digraph D, we have

$$\left\langle {D \atop 0} \right\rangle = \mathbf{AO}(\bar{D}). \tag{10}$$

The next result is an immediate consequence of Theorem 4 and generalizes (10). What it says that for any acyclic digraph D, the number of drop-free permutations on D is equal to a sum of the number of acyclic orientations of a (large) collection of subgraphs of $\bar{D}$.

Theorem 5. *For any acyclic digraph D,*

$$\left\langle {D \atop 0} \right\rangle = \sum_{\substack{S \subseteq T^* \\ S \subseteq \textit{path/cycle cover of } D^*}} \mathbf{AO}(\overline{D^*/S}) \tag{11}$$

where D^* denotes the transitive closure of D and $T^* = E(D^*) \setminus E(D)$.

Proof. We note that D^*/e is also transitive for any e and therefore $\left\langle {D^*/e \atop 0} \right\rangle$ is equal to the number of acyclic orientations of D^*/e. From Theorem 4 we have

$$\begin{aligned} \left\langle {D \atop 0} \right\rangle &= \left\langle {D^* \setminus T^* \atop 0} \right\rangle \\ &= \sum_{\substack{S \subseteq T^* \\ S \subseteq \text{path/cycle cover of } D^*}} \left\langle {D^*/S \atop 0} \right\rangle \\ &= \sum_{\substack{S \subseteq T^* \\ S \subseteq \text{path/cycle cover of } D^*}} \mathbf{AO}(\overline{D^*/S}). \end{aligned}$$

and Theorem 5 follows. □

6 Drops and descents in acyclic graphs.

Suppose $D = ([n], E)$ is an acyclic digraph on the vertex set $[n] = \{1, 2, \ldots, n\}$. By an *arrangement* $A = (a_1, a_2, \ldots, a_n)$ of D, we mean a rearrangement of the vertices of D. We will say that A has a *descent* at i if (a_i, a_{i+1}) is an edge of D. Denote by $Des_D(k)$ the number of arrangements of D have exactly k descents. Let us assume that the vertex set of D is $[n]$ and that the labeling of the vertices of D forms a linear extension of D, i.e., if (u, v) is an edge of D then $u > v$. The following lemma is the natural extension of the well-known fact that the number of permutations of $[n]$ with k drops is equal to the number of arrangements of $[n]$ with k descents.

Lemma 2. *For any acyclic digraph* $D = ([n], E)$*,*

$$Des_D(k) = \left\langle {D \atop k} \right\rangle, \quad 0 \le k < n.$$

Proof. We follow the standard proof for permutations of $[n]$. Given a permutation π on D, we write $\pi = (a_1, a_2, \ldots, a_r)(b_1, b_2, \ldots, b_s)\ldots(c_1, c_2, \ldots, c_t)$ in *standard form*, namely so that the first element in each cycle is the largest element in that cycle, and $a_1 < b_1 < \ldots < c_1$. We now form the corresponding arrangement $A = A(\pi)$ by simply removing the parentheses, i.e., $A = (a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_s, \ldots, c_1, c_2, \ldots, c_t)$. It is now enough to observe that since $\pi(a_r) = a_i, \pi(b_s) = b_1, \ldots$ are not drops of π, and $(a_r, b_1), \ldots$ are not descents of A, then the bijection follows. □

Theorem 6. *For any acyclic digraph* $D = ([n], E)$*,*

$$\left\langle {D \atop 0} \right\rangle \ge \mathbf{AO}(\overline{D}).$$

Furthermore, equality holds if and only if D is transitive.

Proof. We will construct an injection from the set of acyclic orientations of $\overline{D} = G$ to the set of permutations on D. Suppose $\overrightarrow{G}$ is an acyclic orientation of G. We will form a unique descent-free arrangement A from $\overrightarrow{G}$. Consider the set S_1 of *sinks* in $\overrightarrow{G}$, i.e., vertices with outdegree 0. The vertices of S_1 form an independent set in $\overrightarrow{G}$. Hence, these vertices form a linearly-ordered set in D. Let s_1 be the smallest element of S_1. This will be the first element of A. Now, delete s_1 and all incident edges in $\overrightarrow{G}$. In this new acyclic graph, let S_2 denote the set of sinks and, as before, since S_2 is linearly ordered in D, take the smallest element in S_2 and use it as the second element in A. Continue this process until we have formed an arrangement A of all the vertices of D. Consider two consecutive elements in A, say a_i and a_{i+1}. When a_i was selected (as the smallest element in the chain S_i in D), there were two possibilities. If $\{a_i, a_{i+1}\}$ was an edge in G then there was no edge between them in D, and so there is no descent in A in going from a_i to a_{i+1}. On the other hand, suppose that $\{a_i, a_{i+1}\}$ is not an edge in G. Then a_i was smaller then all the other sinks at that time, and so, must be smaller than a_{i+1}. Thus, there is no descent from a_i to a_{i+1} in this case as well. In other words, A is descent-free. Therefore, by Lemma 2, the corresponding permutation on D is drop-free. To recover the acyclic orientation $\overrightarrow{G}$ of G from A, we simply do the following. For each edge $\{x, y\}$ in G, we orient it from x to y if x occurs before y in A (that is, if $x = a_i$ and $y = a_j$ and $i < j$). It is clear from the construction that this will generate the original acyclic orientation $\overrightarrow{G}$ of G from A.

Suppose now that D is *not* transitive. Then there exist three vertices u, v and w such that $u > v > w$ and $(u, v) \in E$, $(v, w) \in E$ but $(u, w) \notin E$. Consider the two arrangements

$$A_1 = (1, 2, \ldots, u-1, \mathbf{u}, \mathbf{w}, \mathbf{v}, v+1 \ldots, u-1, u+1, \ldots, n)$$

and

$$A_2 = (1, 2, \ldots, u-1, \mathbf{v}, \mathbf{u}, \mathbf{w}, v+1 \ldots, u-1, u+1, \ldots, n).$$

It is easy to see that they are both descent-free and they both generate the same acyclic orientation of $G = \overline{D}$ (since $\{u, v\}$ and $\{v, w\}$ are not edges in G). Hence, in this case there are strictly more drop-free permutations on D than there are acyclic orientations of $\overline{D}$. This proves Theorem 6. □

7 The binomial drop polynomial for weighted digraphs

In this section, we will deal with the most general weighted digraphs $D = (V, E) \in \mathscr{D}_4$, namely those digraphs in which each edge $e = (u, v)$ is assigned a real-valued weight $w(e)$, where $u = v$ is allowed. For a permutation π on D, we can define

$$drop(\pi) = \sum_{\substack{u \in V \\ e=(u,\pi(u))}} w(e).$$

We then define

$$B_D(x) = \sum_{\pi} \binom{x + drop(\pi)}{n} \tag{12}$$

where, as usual, we assume that D has n vertices, and for an arbitrary α,

$$\binom{x+\alpha}{n} = \frac{1}{n!}(x+\alpha)^{\underline{n}} = \frac{1}{n!}(x+\alpha)(x+\alpha-1)\ldots(x+\alpha-n+1).$$

First, we show that the binomial drop polynomial obeys a reduction-contraction rule.

Theorem 7. *For a weighted digraph D in $\mathscr{D}_4$, the binomial drop polynomial $B_D(x)$ satisfies the following reduction-contraction rule: Suppose $e = (u, v)$ is an edge with $w(e) = \alpha$ (where $u = v$ is allowed). Let D' denote the weighted digraph with all edge weights the same as those in D, except that the weight of e is now $\alpha - 1$. Let D/e denote the digraph formed by contracting the edge e. Then we have*

$$B_D(x) = B_{D'}(x) + B_{D/e}(x + \alpha - 1) \tag{13}$$

Proof. By definition, we have

$$\begin{aligned}
B_D(x) - B_{D'}(x) &= \sum_{\pi} \binom{x + drop(\pi)}{n} - \sum_{\pi'} \binom{x + drop(\pi')}{n} \\
&= \sum_{\pi(u)=v} \binom{x + drop(\pi)}{n} - \sum_{\pi'(u)=v} \binom{x + drop(\pi')}{n} \\
&= \sum_{\pi(u)=v} \binom{x + \dots \alpha \dots}{n} - \sum_{\pi'(u)=v} \binom{x + \dots \alpha - 1 \dots}{n} \\
&= \sum_{\pi(u)=v} \left(\binom{x + \dots \alpha \dots}{n} - \binom{x + \dots \alpha - 1 \dots}{n} \right) \\
&= \sum_{\pi(u)=v} \binom{x + \dots \alpha - 1 \dots}{n-1}
\end{aligned}$$

by a standard binomial coefficient identity. However, there is a natural bijection from the set of permutations π on the vertex set of D with $\pi(u) = v$ to the set of permutations π'' on D/e such that for the (contracted) vertex uv, we have $\pi''(uv) = \pi(v)$ and $(\pi'')^{-1}(uv) = \pi^{-1}(u)$. Hence, we can rewrite the above sum as:

$$\begin{aligned}
B_D(x) - B_{D'}(x) &= \sum_{\pi''} \binom{x + \dots \alpha - 1 \dots}{n-1} \\
&= \sum_{\pi''} \binom{x + \alpha - 1 + drop(\pi'')}{n-1} \\
&= B_{D/e}(x + \alpha - 1).
\end{aligned}$$

This completes the proof of Theorem 7. □

We remark that for digraphs in $\mathscr{D}_4$ with arbitrary real-valued weights, the preceding reduction-contraction rule does not constitute a definition for $B_D(x)$. However, for digraphs D in $\mathscr{D}_3$ with *positive integral* weights, the definition $B_{I_n}(x) = x^{\underline{n}}$ for the base case I_n, allows one to define the polynomial $B_D(x)$ for all $D \in \mathscr{D}_3$. In this reduction/contraction recursion, the multiplicity of a chosen edge is reduced by one, rather than having the edge removed altogether. However, this can be remedied as follows. We first need to define a sequence of polynomials $B_D^{(k)}(x)$ for D.

Definition. For $k \geq 0$, define the sequence of polynomials:

$$B_D^{(k)}(x) = \sum_{\pi} \binom{x + drop(\pi)}{n+k} \tag{14}$$

In particular, $B_D^{(0)}(x) = B_D(x)$.

Theorem 8. *Suppose that e is an edge or loop in a digraph $D \in \mathscr{D}_4$ with weight $w(e)$. Then for $k \geq 0$, we have*

$$B_D^{(k)}(x) = B_{D\backslash e}^{(k)}(x) + B_{D/e}^{(k+1)}(x + w(e)) - B_{D/e}^{(k+1)}(x) \tag{15}$$

Proof. Observe by (14) that if D has no edges, i.e., $D = I_n$ consists of n independent vertices, then

$$B_{I_n}^{(k)}(x) = n!\binom{x}{n+k}. \tag{16}$$

Now suppose D is a weighted digraph with an edge $e = (u, v)$ of weight $w(e)$. Then

$$\begin{aligned}
B_D^{(k)}(x) - B_{D\backslash e}^{(k)}(x) &= \sum_{\pi} \binom{x + drop(\pi)}{n+k} - \sum_{\pi'} \binom{x + drop(\pi')}{n+k} \\
&= \sum_{\pi(u)\neq v} \binom{x + drop(\pi)}{n+k} + \sum_{\pi(u)=v} \binom{x + drop(\pi)}{n+k} \\
&- \sum_{\pi'(u)\neq v} \binom{x + drop(\pi')}{n+k} - \sum_{\pi'(u)=v} \binom{x + drop(\pi')}{n+k} \\
&= \sum_{\pi(u)=v} \binom{x + drop(\pi)}{n+k} - \sum_{\pi'(u)=v} \binom{x + drop(\pi')}{n+k} \\
&= \sum_{\pi''} \binom{x + w(e) + drop(\pi'')}{n-1+k+1} - \sum_{\pi''} \binom{x + drop(\pi'')}{n-1+k+1} \\
&= B_{D/e}^{(k+1)}(x + w(e)) - B_{D/e}^{(k+1)}(x).
\end{aligned}$$

Here, π'' ranges over all permutations of the vertex set V'' of D/e in which the two vertices u and v have been contracted to the single vertex uv, and we use the usual bijection between permutations π on V with $\pi(x) = u$, $\pi(u) = v$, and $\pi(v) = y$, and permutations π'' on V'' with $\pi''(x) = uv$ and $\pi''(uv) = y$. $\square$

Setting $k = 0$ in Theorem 8, we obtain:

Corollary. *Suppose that e is an edge or loop in a digraph $D \in \mathscr{D}_4$ with weight $w(e)$. Then*

$$B_D(x) = B_{D\setminus e}(x) + B^{(1)}_{D/e}(x + w(e)) - B^{(1)}_{D/e}(x). \tag{17}$$

As an example of using this expansion to compute $B_D(x)$ for a small weighted digraph, let D consist of the three vertices $\{1, 2, 3\}$ with edges $e_1 = (1, 2)$ with weight $w(e_1) = a$ and $e_2 = (2, 3)$ with weight $w(e_2) = b$. Then

$$\begin{aligned} B_D(x) &= B^{(0)}_D(x) = B^{(0)}_{D\setminus e_1}(x) + B^{(1)}_{D/e_1}(x + a) - B^{(1)}_{D/e_1}(x) \\ &= B^{(0)}_{D\setminus e_1\setminus e_2}(x) + B^{(1)}_{D\setminus e_1/e_2}(x + b) - B^{(1)}_{D\setminus e_1/e_2}(x) \\ &+ B^{(1)}_{D/e_1\setminus e_2}(x + a) + B^{(2)}_{D/e_1/e_2}(x + a + b) - B^{(2)}_{D/e_1/e_2}(x + a) \\ &- B^{(1)}_{D/e_1\setminus e_2}(x) - B^{(2)}_{D/e_1/e_2}(x + b) + B^{(2)}_{D/e_1/e_2}(x) \\ &= 3!\binom{x}{3} + 2!\binom{x+b}{3} - 2!\binom{x}{3} \\ &+ 2!\binom{x+a}{3} + 1!\binom{x+a+b}{3} - 1!\binom{x+a}{3} \\ &- 2!\binom{x}{3} - 1!\binom{x+b}{3} + 1!\binom{x}{3} \\ &= 3\binom{x}{3} + \binom{x+a}{3} + \binom{x+b}{3} + \binom{x+a+b}{3} \end{aligned} \tag{18}$$

since $D \setminus e_1 \setminus e_2$ is an independent set of 3 vertices, $D \setminus e_1/e_2$ and $D/e_1 \setminus e_2$ are independent sets of 2 vertices and $D/e_1/e_2$ is an independent set of one vertex, and we apply (16).

To compute $B_D(x)$ from its definition in (12), we observe that the permutations $(1)(2)(3), (13)(2)$ and (132) each have drop value equal to 0, the permutation $(12)(3)$ has drop value a, the permutation $(1)(23)$ has drop value b, and the permutation (123) has drop value $a+b$. Thus, we can immediately see that the value of $B_D(x)$ is as is given in (18).

A reciprocity formula. We note here there is a type of reciprocity that holds for $B_D(x)$. Namely, for a weighted digraph D, let us define its *dual* $\overline{D}$ to be the digraph having the same set of vertices and with each edge weight $\overline{w}(e) = 1 - w(e)$.

Theorem 9. *If a weighted digraph D has n vertices then*

$$B_{\overline{D}}(x) = (-1)^n B_D(-x-1). \tag{19}$$

Proof. From (12) we have

$$\begin{aligned}
B_{\overline{D}}(x) &= \sum_{\pi} \binom{x+n-drop(\pi)}{n} \\
&= \frac{1}{n!}(x+n-drop(\pi))^{\underline{n}} \\
&= \frac{(-1)^n}{n!}(-x-n+drop(\pi))^{\overline{n}} \\
&= \frac{(-1)^n}{n!}(-x-1+drop(\pi))^{\underline{n}} \\
&= (-1)^n B_D(-x-1)
\end{aligned}$$

where $z^{\overline{n}}$ denotes the *rising* factorial $z(z+1)(z+2)\dots(z+n-1)$. □

8 Concluding remarks

We should remark that Theorem 8 can be thought of as result dealing with square matrices M with entries taken from some commutative ring $\mathscr{R}$ with identity. This is similar to what was done in [10]. In this case, for an entry $e = (u,v)$, we can form the *deleted* matrix $M \setminus e$ by replacing $M(u,v)$ by

0. To form the *contracted* matrix M/e, we first replace row u of M by row v of M, and then delete row v and column v of M. We then have the corresponding polynomial $B_M(x)$ satisfying the previously mentioned definitions and recurrences.

We conclude by mentioning several further directions which could be interesting to explore.
For example, one could ask if there is a natural analog for Theorem 5 for $\left\langle {D \atop k} \right\rangle$ for $k > 0$?

In another direction, it would be interesting to know what the complexity of evaluating $B_D(x)$ is for general digraphs D? For example, $B_D(0)$ is the number of cycle covers of D. For the case of the Tutte polynomial, we mention the following result of Jaeger, Vertigan and Welsh [18]:

Theorem 10. *The problem of evaluating the Tutte polynomial $T_G(x, y)$ of a general graph G at a point (a, b) is #P-hard except when (a, b) is on the special hyperbola $(x - 1)(y - 1) = 1$ or when (a, b) is one of the eight special points $(1, 1), (-1, -1), (0, -1), (-1, 0), (i, -i), (-i, i), (j, j^2)$ and (j^2, j), where $j = e^{\frac{2\pi i}{3}}$. In each of these exceptional cases, the evaluation can be done in polynomial time.*

In the case of the cover polynomial $C_D(x, y)$ (see [9]), the situation is even worse! In this case, Bläser and H. Dell [1] have shown that it is #P-hard to evaluate $C_D(x, y)$ for general digraphs $D \in \mathscr{D}_3$ (i.e., with multiple edges and loops) at a point (a, b) unless (a, b) is one of the *three* special points $(0, 0), (0, -1)$ and $(1, -1)$ (and for these three points, the computation can be done in polynomial time). Presumably, the same result also holds when restricted to simple digraphs $D \in \mathscr{D}_2$. It would be interesting to know when the computation of $B_D(x)$ for $x \in [0, n]$ can be done in polynomial time.

It is possible to define a *2-variable* drop polynomial $B_D(x, y)$ for weighted digraphs D which generalizes the polynomial $B_D(x)$ we have defined here in such a way that $B_D(x, 1) = B_D(x)$. The polynomial $B_D(x, y)$ depends on more subtle properties of D (such as the cycle structure in D), and is related to the path-cycle cover polynomial $C_D(x, y)$ of D (see [9, 10]). In fact, for simple digraphs D, $B_D(x, y) = C_D(x, y)$. $B_D(x, y)$ also satisfies analogous reduction/ contraction and deletion/contraction rules as well as some intriguing reciprocity relations and various symmetric function generalizations (cf. Stanley [20], Chow [8], and the authors [10]). We plan to address some of these extensions in a future paper.

References

[1] M. Bläser and H. Dell, Complexity of the Cover Polynomial, *Automata, Languages and Programming*, Lecture Notes in Computer Science **4596**, (2007), 801–812.

[2] B. Bollobás, L Pebody and O. Riordan, Contraction–Deletion Invariants for Graphs, *Jour. Combinatorial Th.* B **80** (2000), 320–345.

[3] J. Buhler, D. Eisenbud, R. Graham and C. Wright, Juggling drops and descents, *Amer. Math. Monthly*, **101** (1994), 729–756.

[4] J. P. Buhler and R. L. Graham, A note on the drop polynomial of a poset, *Jour. Combinatorial Th.* (A) **66** (1994), 321–326.

[5] J. P. Buhler and R. L. Graham, Juggling patterns, passing and posets, *Mathematical Adventures for Students and Amateurs*, MAA (2004), 99–116.

[6] F. Butler, Rook theory and cycle-counting permutation statistics, *Adv. Applied Math.* **33** (2004), 655–675.

[7] T. Y. Chow, The path/cycle symmetric function of a digraph, *Advances in Math.* **118** (1996), 71–98.

[8] T. Y. Chow, Descents, Quasi-symmetric Functions, Robinson-Shensted for Posets, and the Chromatic Symmetric Function, *J. Algebraic Combinatorics* **10** (1999), 227–240.

[9] F. R. K. Chung and R. L. Graham, On the Cover Polynomial of a Digraph, *Jour. Combinatorial Th.* (B) **65** (1995), 273–290.

[10] F. R. K. Chung and R. L. Graham, The matrix cover polynomial, (to appear in Journal of Combinatorics (2016)).

[11] O. M. D'Antona and E. Munarini, The cycle-path indicator polynomial of a digraph, *Advances in Applied Math.* **25** (2000), 41–56.

[12] F. M. Dong, M. D. Hendy, K. L. Teo and C. H. C. Little, the vertex-cover polynomial of a graph, *Discrete Math.* **250** (2002), 71–78.

[13] M. Dworkin, factorization of the cover polynomial, *J. Comb. Th.* **71** (1997), 17–53.

[14] J. R. Goldman, J. T. Joichi and D. E. White, Rook Theory III, Rook polynomials and the chromatic structure of graphs, *Jour. Combinatorial Th.* (B) **25** (1978), 135–142.

[15] J. Goldman and J. Haglund, Generalized rook polynomials, *J. of Comb. Th.* **91** (2000), 509–530.

[16] R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics, A Foundation for Computer Science*, Addison-Wesley, Boston, 1994, xiii+ 657 pp.

[17] J. Haglund, Further investigations involving rook polynomials with only real zeros, *European J. of Comb.* **21** (2000), 1017–1037.

[18] F. Jaeger, D. L. Vertigan and D. J. A. Welsh, On the computational complexity of the Jones and Tutte polynomials, *Math. Proc. Cambridge Philos. Soc.* **j108**, (1990), 35–53.

[19] R. P. Stanley, Acyclic orientations of graphs, Discrete Math. **5** (1973), 171–178.

[20] R. P. Stanley, A Symmetric Function Generalization of the Chromatic Polynomial of a Graph, *Adv. in Math.*, **111** (1995), 166–194.

[21] R. P. Stanley, Enumerative Combinatorics. Vol 1, second edition, *Cambridge Studies in Advanced Mathematics*, **62** Cambridge Univ. Press, Cambridge, 2012, xiv+626 pp.

[22] D. J. A. Welsh, The Tutte polynomial, *Random Structures and Algorithms*, **15**, (1999), 210–228.

[23] W. T. Tutte, A ring in graph theory, *Proc. Cambridge Philos. Soc.* **43** (1947), 26–40.