Well dispersed sequences in $[0, 1]^d$

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Abstract

Given a sequence $\bar{z} = (z_1, z_2, \ldots)$ where $z_k \in [0, 1]^d$, the *dispersion* of $\bar{z}$ is defined by

$$\mu_d(\bar{z}) = \inf_{n \geq 1} \inf_{m \geq 1} n^{1/d} ||z_m - z_{m+n}||.$$

where $\|\cdot\|$ is taken to be the L_1 metric $\|\cdot\|_1$on $[0, 1]^d$. This is a natural measure for how "spread out" the sequence $\bar{z}$ is. In this paper, we investigate $\alpha_d := \sup_{\bar{z}} \mu_d(\bar{z})$. Note that the requirement $\mu_d(\bar{z}) = \alpha_d > 0$ is a very stringent condition on the distribution of the z_i. For example, it implies that $||z_m - z_{m+1}|| \geq \alpha_d$ for all $m \geq 1$. We show by construction that $\alpha_d \geq \left((2^{d-1}(2^d - 1))^{1/d}\left(1 + \sum_{k \geq 1} \frac{1}{F_{2k}}\right)\right)^{-1} > 0.098$ where F_n denotes the n^{th} Fibonacci number. We also introduce a combinatorial problem for d-tuples of permutations which yields bounds on α_d. This work extends previous results of the authors where the value of α_1 is determined.

1 Introduction.

A fundamental question in the study of the distribution of sequences is the quantitative estimation of the extent by which an arbitrary sequence must deviate from some appropriately defined measure of regularity. As an example, de Bruijn and Erdős [9] considered the following measure. For a sequence $\bar{x} = (x_1, x_2, x_3, \ldots)$ with x_k on the unit circle C, define

$$\omega(\bar{x}) = \liminf_{n \to \infty} \inf_{1 \leq i < j \leq n} n \, |x_i - x_j|.$$

where the distance between two points on C is the length of the shorter arc joining them. Then they proved that for all $\bar{x}$ in C,

$$\omega(\bar{x}) \leq \frac{1}{\ln 4} = 0.72135 \ldots.$$

Furthermore, this bound is best possible, as shown by taking $\bar{x}$ to be the sequence defined by setting $x_n = \{\frac{\log(2n-1)}{\log 2}\}$ where $\{\cdot\}$ denotes the fractional part. (The reader can consult [14, 15, 17, 18] for further references on this result.) Notice that the consecutive terms of $\bar{x}$ can be arbitrarily close together, provided they occur sufficiently far out. From this

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perspective, the x_k are not well dispersed. In previous work [4, 5, 6], the authors considered the following much stricter measure of dispersion. For $\bar{x} \in [0,1]$, define

$$\mu(\bar{x}) = \inf_{n\geq 1} \inf_{m\geq 1} n\ |x_m - x_{m+n}|. \tag{1}$$

Then it was shown in [4] that for all $\bar{x} \in [0,1]$, we have

$$\mu(\bar{x}) \leq \alpha_1 := \left(1 + \sum_{k\geq 1} \frac{1}{F_{2k}}\right)^{-1} = 0.3944\ldots. \tag{2}$$

where F_n denotes the usual Fibonacci numbers defined recursively by $F_0 = 0, F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Furthermore, this bound is best possible. We will describe the construction of sequences satisfying (2) since this will be useful for our results later in the paper.

It is well known that any positive integer n can be expressed uniquely as a sum $n = \sum_k a_k F_k$ where $a_k \in \{0,1\}$ and if $a_i = a_j = 1$ for some $i < j$ then there is a k with $i < k < j$ such that $a_k = 0$. In other words, you cannot use two consecutive Fibonacci numbers in the representation. This is the so-called Zeckendorf representation of n which was actually first published by Ostrowski [16] (also, see [1], sec. 3.9).

Less well known is the following analogous representation theorem. Every positive integer n can be expressed uniquely as a sum $n = \sum_k a_k F_{2k}$ where $a_k \in \{0,1,2\}$ and if $a_i = a_j = 2$ for some $i < j$ then there is a k with $i < k < j$ such that $a_k = 0$. We call this the *even Fibonacci* representation of n. (For a simple proof, see [4].) Now define a sequence $\bar{f} = (f(1), f(2), \ldots)$ by setting

$$f(n) = \alpha_1 \sum_{k\geq 1} \frac{a_k}{F_{2k}} \tag{3}$$

where $n = \sum_k a_k F_{2k}$ is the even Fibonacci representation of n. The factor of α_1 guarantees that $f(n) \in [0,1]$. It is shown in [4] that

$$\mu(\bar{f}) = \alpha_1. \tag{4}$$

Note that this construction has a similar flavor to the construction of the van der Corput sequence $\bar{C} = (c_1, c_2, \ldots,)$. For this sequence, we express $n = \sum_{k\geq 0} a_k 2^k$ in binary where $a_k \in \{0,1\}$ and then define $c_n = \sum_{k\geq 0} a_k 2^{-k-1}$ (see [13]). However, the van der Corput sequence won't work for the measure (1) since, for example, for $m = 2^r - 2$ and $n = 3$, we have $c_n = 1/2 - 1/2^r$ and $c_{n+3} = 1/2 + 1/2^{r+1}$. Thus, terms with indices differing by 3 can be arbitrarily close together, violating (1).

Our goal in this paper is to derive dispersion bounds for sequences in higher dimensions. It is well known from the extensive literature in simultaneous Diophantine approximation that problems of this type can present difficult challenges (e.g., see [3, 7, 8, 12]). We will first deal with the case of two dimensions.

2 Dispersion in two dimensions

For a sequence $\bar{z} = (z_1, z_2, \ldots)$ with $z_m = (x_m, y_m) \in [0,1]^2$, define the dispersion measure $\mu_2(\bar{z})$ by

$$\mu_2(\bar{z}) = \inf_{n\geq 1} \inf_{m\geq 1} n^{1/2} \|z_m - z_{m+n}\| \tag{5}$$

where $\|\cdot\|$ denotes the L_1 norm, i.e., $\|z_m - z_{m+n}\| = |x_m - x_{m+n}| + |y_m - y_{m+n}|$. Our main goal is to bound α_2 defined by

$$\alpha_2 := \sup_{\bar{z}} \mu_2(\bar{z}).$$

where $\bar{z}$ ranges over all sequences $(z_1, z_2, \ldots)$ with $z_m = (x_m, y_m) \in [0,1]^2$. It is not obvious a priori that $\alpha_2 > 0$! We remedy this gap with

Theorem 1. $\alpha_2 \geq \alpha_1/\sqrt{6} = 0.16102\ldots$.

To prove Theorem 1, we need to construct a suitable sequence $\bar{z}$. To do this, we will first need to prove a result concerning integers. Denote the *binary expansion* of the positive integer $m = \sum_{k=0}^{r} m_k 2^k$ by $m = m_r m_{r-1} \ldots m_1 m_0$.

We next split the binary expansion of m into two binary expansions

$$\begin{aligned} m^{(0)} &= \ldots m_k^{(0)} \ldots m_1^{(0)} m_0^{(0)} = \ldots m_{2k} \ldots m_2 m_0 \\ \text{and} \quad m^{(1)} &= \ldots m_k^{(1)} \ldots m_1^{(1)} m_0^{(1)} = \ldots m_{2k+1} \ldots m_3 m_1. \end{aligned} \tag{6}$$

That is, we form the binary expansions of the $m^{(i)}$ by taking alternating digits from the binary expansion of m. For a positive integer n, we set $s = m + n$ and we form in the same way $s^{(0)}$ and $s^{(1)}$ from the binary expansion of $s = s_r s_{r-1} \ldots s_1 s_0$. Note that $m_k^{(0)} = m_{2k}$ and $m_k^{(1)} = m_{2k+1}$, with similar expressions for the $s_k^{(i)}$.

Theorem 2. *For all $m, n \geq 1$, we have for some $i \in \{0, 1\}$,*

$$0 < s^{(i)} - m^{(i)} \leq \sqrt{6n}.$$

Furthermore, the constant $\sqrt{6}$ is best possible.

We first make some preliminary remarks. Assume without loss of generality that $2^t \leq n < 2^{t+1}$ for some $t \geq 0$. Writing out the binary expansions of m, n and $s = m + n$, we can represent the addition of m and n in the following way:

$$D(m,n) = \begin{array}{cccccccc} m_r & \ldots & m_{t+1} & m_t & m_{t-1} & \ldots & m_1 & m_0 \\ 0 & \ldots & 0 & 1 & n_{t-1} & \ldots & n_1 & n_0 \\ s_r & \ldots & s_{t+1} & s_t & s_{t-1} & \ldots & s_1 & s_0 \end{array}$$

We will refer to this figure as the *diagram* $D(m,n)$ for the (binary) addition of m and n.

For example, for $m = 364 = 101101100_{(2)}, n = 22 = 10110_{(2)}$, we have $s = 386 =$

$110000010_{(2)}$. The addition of $364 + 22 = 386$ is represented by the diagram $D(364, 22)$ below.

$$D(364,22) = \begin{array}{ccccccccc} \mathbf{8} & \mathbf{7} & \mathbf{6} & \mathbf{5} & \mathbf{4} & \mathbf{3} & \mathbf{2} & \mathbf{1} & \mathbf{0} \\ \hline 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}$$

For ease of reference, we have placed the column indices (in bold) above each column. In this diagram, for example, column **8**, denoted by C_8, is $\begin{smallmatrix}1\\0\\1\end{smallmatrix}$. Similarly, columns **2, 3** and **4** are $C_2 = \begin{smallmatrix}1\\1\\0\end{smallmatrix}$, $C_3 = \begin{smallmatrix}1\\0\\0\end{smallmatrix}$ and $C_4 = \begin{smallmatrix}0\\1\\0\end{smallmatrix}$, etc.
Notice for C_2 there was a *carry out* to C_3 and for C_5 there was a *carry in* from C_4 and a *carry out* to C_6. On the other hand C_1 had no carries, in or out.

In all, there are 8 possible columns which could occur in a diagram. They are:

$$c_0 = \begin{matrix}0\\0\\0\end{matrix},\ c_1 = \begin{matrix}0\\0\\1\end{matrix},\ c_2 = \begin{matrix}0\\1\\0\end{matrix},\ c_3 = \begin{matrix}0\\1\\1\end{matrix},\ c_4 = \begin{matrix}1\\0\\0\end{matrix},\ c_5 = \begin{matrix}1\\0\\1\end{matrix},\ c_6 = \begin{matrix}1\\1\\0\end{matrix},\ c_7 = \begin{matrix}1\\1\\1\end{matrix}.$$

We can classify the column types according to the carries they generate and accept. Of course, carries *out* propagate to the left, and carries *in* arrive from the right.

Column	*Carry Out*	*Carry In*
c_0	*no*	*no*
c_1	*no*	*yes*
c_2	*yes*	*yes*
c_3	*no*	*no*
c_4	*yes*	*yes*
c_5	*no*	*no*
c_6	*yes*	*no*
c_7	*yes*	*yes*

.

A diagram of length r consists of a *compatible* concatenation of r of these columns. By compatible, we simply mean that if a column generates a carry out (to the left), then the adjacent column to the left must accept a carry in (from the right). An example of this in the diagram $D(364, 22)$ is column **3**. In other words, compatible means that all the columns of D must arise from the binary addition of m and n.

Since $n > 0$, we have $s = m + n > m$. Thus, there is a largest index r in the diagram $D(m, n)$ with $m_r = 0, s_r = 1$. We are going to sample every other digit of m and s to form $m^{(i)}$ and $s^{(i)}$ as above. In particular, we are always going to choose $i \equiv r \pmod 2$. In this case we will denote $m^{(i)}$ and $s^{(i)}$ by m' and s', respectively. Note that this choice guarantees that $s' > m'$. We define $\Delta = s' - m'$. It follows from our earlier remarks that if m_k occurs in the m' sequence, then it is the term $m'_{\lfloor k/2 \rfloor}$ (with the same remark applying to $s_k = s'_{\lfloor k/2 \rfloor}$). Given a diagram $D = D(m, n)$, we define the $Score(D)$ of D by

$$Score(D) = \frac{(s' - m')^2}{n} = \frac{\Delta^2}{n}.$$

With these definitions we can restate Theorem 2 as:

Theorem 3. *For any diagram $D = D(m,n)$, $Score(D) \le 6$. Furthermore, the constant 6 is best possible.*

Proof: We will first give examples which show that the constant 6 is best possible. We consider the following diagram:

$$D_r = \begin{array}{cccccccccc} \mathbf{2r+4} & \mathbf{2r+3} & \mathbf{2r+2} & \mathbf{2r+1} & \mathbf{2r} & \mathbf{2r-1} & \cdots & \mathbf{2} & \mathbf{1} & \mathbf{0} \\ \hline 0 & 1 & 1 & 1 & \mathbf{0} & \mathbf{1} & \cdots & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & 0 & 1 & \mathbf{0} & \mathbf{1} & \cdots & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 1 & 0 & 0 & 0 & \mathbf{1} & \mathbf{0} & \cdots & \mathbf{1} & \mathbf{0} & \mathbf{0} \end{array}$$

The diagram D_r consists of the 4 columns $\begin{smallmatrix} 0&1&1&1 \\ 0&0&0&1 \\ 1&0&0&0 \end{smallmatrix}$ followed by r copies of the pairs of columns $\begin{smallmatrix} 0&1 \\ 0&1 \\ 1&0 \end{smallmatrix}$ with a final column $\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}$. We will compute $\Delta = \Delta^{(i)} = s^{(i)} - m^{(i)}$ for both choices of $i = 0$ and 1. This will show neither of the choices for sampling the digits of m and s can asymptotically beat the constant 6.

For $i = 0$ we have:

$s^{(0)} = 2^{r+2} + 2^r + 2^{r-1} + \ldots + 2 + 1 = 3 \cdot 2^{r+1} - 1$, $m^{(0)} = 2^{r+1}$, $n = 2^{2r+1} + 2^{2r-1} + \ldots + 2 = 2 \cdot (4^{r+1} - 1)/3$.

Thus,

$$\frac{\Delta^2}{n} = \frac{3 \cdot (2^{r+2} - 1)^2}{2 \cdot (4^{r+1} - 1)} \sim 6 \text{ as } r \to \infty.$$

On the other hand, for $i = 1$ we have:

$$s^{(1)} = 0, \ m^{(1)} = 2^{r+1} + 2^r + \ldots + 2 + 1 = 2^{r+2} - 1, \quad n = 2 \cdot (4^{r+1} - 1)/3.$$

Thus,

$$\frac{\Delta^2}{n} = \frac{3 \cdot (2^{r+2} - 1)^2}{2 \cdot (4^{r+1} - 1)} \sim 6 \text{ as } r \to \infty.$$

Therefore the score $S(D_r) = 6 - o(1)$ as $r \to \infty$. This shows that the constant 6 cannot be improved.

To complete the proof of the theorem, we are going to make a detailed analysis of the columns in D. We first will simplify the diagram D without changing its score. To begin, any columns C_j preceding C_r have no effect on the score of D since they have $s_j = m_j$ and $n_j = 0$. Thus, we can delete these from D. Also, we can replace any column $\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}$ by $\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}$ without affecting the score of D. The same is true if we replace any column $\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}$ by $\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}$. Hence we can assume that the columns $\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}$ and $\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}$ do not occur in D.

Assume that $2^t \le n < 2^{t+1}$ for some $t \ge 0$. Thus, in our "normalized" diagram D, the first (i.e., left-most) column is either $c_3 = \substack{0\\1\\1}$ or $c_1 = \substack{0\\0\\1}$.

Case 1. The first column C_r in D is $c_3 = \substack{0\\1\\1}$. In this case, $r = t$. Therefore

$$\Delta \le 2^{\lfloor t/2 \rfloor} + 2^{\lfloor (t-2)/2 \rfloor} + 2^{\lfloor (t-4)/2 \rfloor} + \ldots < 2^{\lfloor (t+2)/2 \rfloor}$$

and $n \ge 2^t$. Thus,

$$S(D) = \frac{\Delta^2}{n} < \frac{2^{t+2}}{2^t} = 4 < 6.$$

Case 2. The first column C_r in D is $c_1 = \substack{0\\0\\1}$. Note that in this case, we must have $b = r - t \ge 1$. The only columns that can occur after C_r and before C_t are $c_4 = \substack{1\\0\\0}$. Thus, for some $b \ge 1$, the diagram D looks like:

$$D = \begin{array}{cccccccccc} \mathbf{r=t+b} & \mathbf{t+b-1} & \mathbf{t+b-2} & \ldots & \mathbf{t+1} & \mathbf{t} & \mathbf{t-1} & \ldots & \mathbf{1} & \mathbf{0} \\ \hline 0 & 1 & 1 & \ldots & 1 & m_t & m_{t-1} & \ldots & m_1 & m_0 \\ 0 & 0 & 0 & \ldots & 0 & 1 & n_{t-1} & \ldots & n_1 & n_0 \\ 1 & 0 & 0 & \ldots & 0 & s_t & s_{t-1} & \ldots & s_1 & s_0. \end{array}$$

Side remark. At this point it is already easy to show that the score for any diagram $D = D(m, n)$ is at most 9. We note that we must have $m'_t \ge s'_t$ and compute

$$\Delta \le 2^{\lfloor (t+2)/2 \rfloor} + 2^{\lfloor t/2 \rfloor} = 3 \cdot 2^{\lfloor t/2 \rfloor}, \quad n \ge 2^t$$

so that

$$Score(D) = \frac{\Delta^2}{n} < \frac{9 \cdot 2^t}{2^t} = 9.$$

To replace the 9 by 6 requires a more careful analysis which we now proceed to carry out. Keep in mind that by our parity choice for i, the only columns C_j that contribute to Δ are those with $j \equiv r \pmod 2$. Thus, the contribution to Δ of the columns $C_j, j > t$, is either $2^{\lfloor (t+1)/2 \rfloor}$ if $r \equiv t+1 \pmod 2$ or $2^{\lfloor (t+2)/2 \rfloor}$ if $r \equiv t \pmod 2$. Also, these columns do not contribute to n. Hence, we can remove any *even* number of these columns without changing the score of the diagram. Therefore, there are two remaining possibilities for D. The first is:

Subcase 2.1.

$$D = \begin{array}{cccccc} \mathbf{t+2} & \mathbf{t+1} & \mathbf{t} & \mathbf{t-1} & \mathbf{t-2} & \cdots \\ \hline 0 & \mathbf{1} & m_t & m_{t-1} & m_{t-2} & \cdots \\ 0 & \mathbf{0} & 1 & n_{t-1} & n_{t-2} & \cdots \\ 1 & \mathbf{0} & s_t & s_{t-1} & s_{t-2} & \cdots . \end{array}$$

There are two choices for column C_t, namely either $C_t = c_6 = \substack{1\\1\\0}$ or $C_t = c_2 = \substack{0\\1\\0}$. This implies $s_t = 0$.

If $m_t = 1$, we have $\Delta \le 2^{\lfloor (t+2)/2 \rfloor}$ and $n \ge 2^t$. Hence, $\Delta^2/n \le 2^{t+2}/2^t = 4 < 6$. We may assume $m_t = 0$ which implies $s_t = 0$. Thus

$$D = \begin{array}{cccccc} \mathbf{t+2} & \mathbf{t+1} & \mathbf{t} & \mathbf{t-1} & \mathbf{t-2} & \cdots \\ \hline 0 & 1 & \mathbf{0} & m_{t-1} & m_{t-2} & \cdots \\ 0 & 0 & \mathbf{1} & n_{t-1} & n_{t-2} & \cdots \\ 1 & 0 & \mathbf{0} & s_{t-1} & s_{t-2} & \cdots . \end{array}$$

There are now 3 possibilities for column C_{t-1}. They are $\begin{smallmatrix}0\\1\\0\end{smallmatrix}$, $\begin{smallmatrix}1\\1\\0\end{smallmatrix}$ and $\begin{smallmatrix}1\\0\\0\end{smallmatrix}$. This implies $s_{t-1} = 0$.

If $n_{t-1} = 1$, we have

$$\Delta \leq 2^{\lfloor (t+2)/2 \rfloor} + 2^{\lfloor t/2 \rfloor} = 3 \cdot 2^{\lfloor t/2 \rfloor} \text{ and } n \geq 2^t + 2^{t-1} = 3 \cdot 2^{t-1}.$$

Consequently, $\Delta^2/n \leq \frac{9 \cdot 2^t}{3 \cdot 2^{t-1}} = 6$. We may assume $n_{t-1} = 0$ which implies $m_{t-1} = 1$. Thus,

$$D = \begin{array}{cccccc} \mathbf{t+2} & \mathbf{t+1} & \mathbf{t} & \mathbf{t-1} & \mathbf{t-2} & \cdots \\ \hline 0 & 1 & 0 & \mathbf{1} & m_{t-2} & \cdots \\ 0 & 0 & 1 & \mathbf{0} & n_{t-2} & \cdots \\ 1 & 0 & 0 & \mathbf{0} & s_{t-2} & \cdots . \end{array}$$

There are 3 possibilities for column C_{t-2}. They are $\begin{smallmatrix}0\\1\\0\end{smallmatrix}$, $\begin{smallmatrix}1\\1\\0\end{smallmatrix}$ and $\begin{smallmatrix}1\\0\\0\end{smallmatrix}$. We may assume $s_{t-2} = 0$.

If $n_{t-1} = 1$, then,

$$\Delta \leq 2^{\lfloor (t+2)/2 \rfloor} + 2^{\lfloor (t-2)/2 \rfloor} = 5 \cdot 2^{\lfloor (t-2)/2 \rfloor} \text{ and } n \geq 2^t + 2^{t-2} = 5 \cdot 2^{t-2}.$$

Consequently, $\Delta^2/n \leq \frac{25 \cdot 2^{t-2}}{5 \cdot 2^{t-2}} = 5 < 6$. We only need to consider $n_{t-1} = 0$ which implies $m_{t-2} = 1$.

Thus,

$$D = \begin{array}{cccccc} \mathbf{t+2} & \mathbf{t+1} & \mathbf{t} & \mathbf{t-1} & \mathbf{t-2} & \cdots \\ \hline 0 & 1 & 0 & 1 & \mathbf{1} & \cdots \\ 0 & 0 & 1 & 0 & \mathbf{0} & \cdots \\ 1 & 0 & 0 & 0 & \mathbf{0} & \cdots . \end{array}$$

Therefore, $\Delta \leq 2^{\lfloor (t+2)/2 \rfloor}$ and $n \geq 2^t$ so that $\Delta^2/n \leq 2^{t+2}/2^t = 4 < 6$.

This completes the analysis of Subcase 2.1. We are left with

Subcase 2.2.

$$D = \begin{array}{ccccc} \mathbf{t+1} & \mathbf{t} & \mathbf{t-1} & \mathbf{t-2} & \cdots \\ \hline 0 & m_t & m_{t-1} & m_{t-2} & \cdots \\ 0 & 1 & n_{t-1} & n_{t-2} & \cdots \\ 1 & s_t & s_{t-1} & s_{t-2} & \cdots . \end{array}$$

As before, there are two choices for column C_t, namely either $C_t = \begin{smallmatrix}0\\1\\0\end{smallmatrix}$ or $C_t = \begin{smallmatrix}1\\1\\0\end{smallmatrix}$. Therefore we have $s_t = 0$.

If $n_{t-1} = 1$, then $\Delta \leq 2^{\lfloor (t+3)/2 \rfloor}$ and $n \geq 2^t + 2^{t-1} = 3 \cdot 2^{t-1}$ so that $\Delta^2/n \leq \frac{16 \cdot 2^{t-1}}{3 \cdot 2^{t-1}} < 6$. We may assume $n_{t-1} = 0$.

If $m_{t-1} = 1$ or $s_{t-1} = 0$, $\Delta \leq 2^{\lfloor (t+1)/2 \rfloor} + 2^{\lfloor (t-1)/2 \rfloor}$ and $n \geq 2^t$ so that $\Delta^2/n \leq \frac{9 \cdot 2^{t-1}}{2^t} < 6$. We may assume $m_{t-1} = 0$ which implies $m_t = 1$ and $s_{t-1} = 1$. We have

$$D = \begin{array}{ccccc} \mathbf{t+1} & \mathbf{t} & \mathbf{t-1} & \mathbf{t-2} & \cdots \\ \hline 0 & \mathbf{1} & 0 & m_{t-2} & \cdots \\ 0 & \mathbf{1} & 0 & n_{t-2} & \cdots \\ 1 & \mathbf{0} & 1 & s_{t-2} & \cdots . \end{array}$$

We now consider $m^* = m - 2^t, n^* = n - 2^t$ and $s^* = s - 2^{t+1}$. Clearly, $s^* = m^* + n^*$ and $s^* < s$. By induction we have

$$\frac{(\Delta^*)^2}{n^*} \leq 6.$$

We now consider

$$\begin{aligned} \Delta^2 - 6n &\leq (\Delta^* + 2^{(t+1)/2})^2 - 6(n^* + 2^t) \\ &\leq 2^{(t+3)/2}\Delta^* + 2^{t+1} - 6 \cdot 2^t \\ &\leq 2^{(t+3)/2}(\Delta^* - 2^{(t+1)/2}) \\ &\leq 0. \end{aligned}$$

This shows that $Score(D) \leq 6$ and the proof of Theorem 3 is complete. □

Proof of Theorem 1:

To complete the proof of Theorem 1, we will construct a well dispersed sequence $\bar{z} = (z_1, z_2, \ldots)$ with $z_k \in [0,1]^2$ as follows.

For the positive integer k, we form the integers $k^{(0)}$ and $k^{(1)}$ by splitting the digits in the binary expansion of k, as described in (6). We consider the sequence $\bar{f}$ which achieves the best possible dispersion bound α in the one-dimensional case, as defined in (3).

Recall that f maps the integer m into $f(m) \in [0,1]$ by expanding m into its even Fibonacci representation and then summing the inverses of the Fibonacci numbers occurring in the representation (and then multiplying by α_1 to make sure the result is in $[0,1]$). The point z_k is now defined by

$$z_k = (f(k^{(0)}), f(k^{(1)})).$$

It follows from (1) and (4) that for any pair of integers $a, b \geq 1$, we have

$$b|f(a) - f(a+b)| \geq \alpha_1$$

or

$$|f(a) - f(a+b)| \geq \alpha_1/b. \tag{7}$$

Hence, with $s = m + n$, we have

$$\begin{aligned} \|z_m - z_s\| &= |f(m^{(0)}) - f(s^{(0)})| + |f(m^{(1)}) - f(s^{(1)})| \\ &\geq \max_i |f(m^{(i)}) - f(s^{(i)})| \\ &= |f(m^{(i_0)}) - f(s^{(i_0)})| \qquad \text{for some } i_0 \\ &\geq \frac{\alpha_1}{|m^{(i_0)} - s^{(i_0)}|} \\ &\geq \frac{\alpha_1}{\sqrt{6n}} \end{aligned}$$

by Theorem 3 and the properties of f. In other words,

$$\sqrt{n}\ \|z_m - z_s\| \geq \alpha_1/\sqrt{6}$$

which proves Theorem 1. □

We show pictures of the $f(k), 0 \leq k < 15$, and $1000 \leq k \leq 1099$ in Figures 1 and 2.

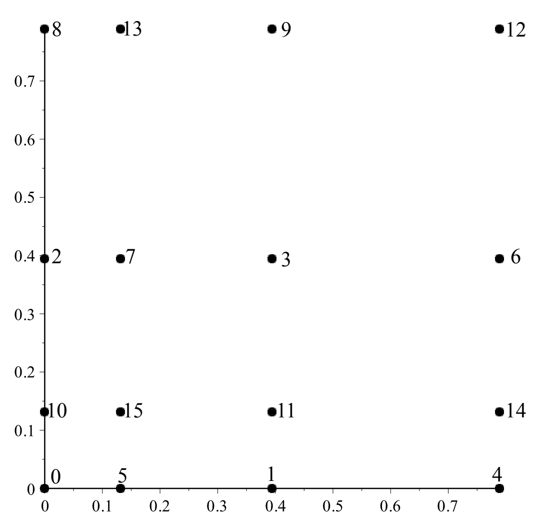

Figure 1: *The first 16 points of f.*

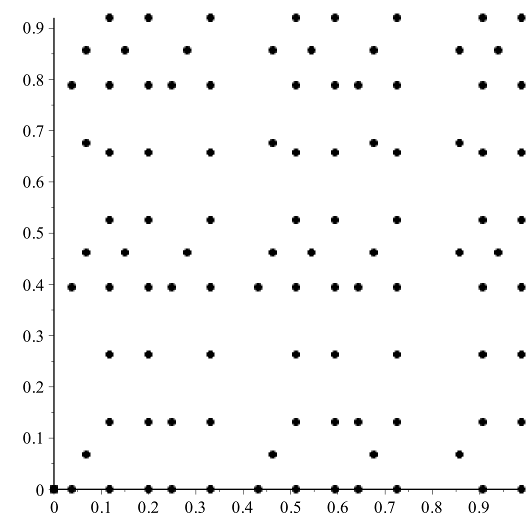

Figure 2: *Points 1000 to 1099 of f.*

3 Dispersion in higher dimensions.

In this section we consider the problem of estimating bounds on the dispersion measure for sequences in $[0,1]^d$ with $d > 2$. For a sequence $\bar{z} = (z_1, z_2, \ldots)$ of points in $[0,1]^d$, we define

$$\mu_d(\bar{z}) = \inf_{n\geq 1} \inf_{m\geq 1} n^{1/d} \|z_m - z_{m+n}\| \tag{8}$$

and

$$\alpha_d = \sup_{\bar{z}} \mu_d(\bar{z}).$$

where $\bar{z}$ ranges over all sequences $(z_1, z_2, \ldots)$ with $z_k \in [0,1]^d$.

Theorem 4. *For all $d \geq 1$,*

$$\alpha_d \geq \frac{\alpha_1}{\left(2^{d-1}(2^d - 1)\right)^{1/d}}.$$

In particular, $\alpha_d > \frac{\alpha_1}{4} = 0.098\ldots$.

We will use the techniques from Section 2 to apply to this more general case. Namely, by first splitting the binary expansion of m into d binary expansions $m^{(i)}, 1 \leq i \leq d$, and then sampling every d^{th} digit (and doing the same for $s = m + n$), we will show that for some i,

$$0 < |s^{(i)} - m^{(i)}| \leq (\beta_d n)^{1/d}. \tag{9}$$

where $\beta_d = 2^{d-1}(2^d - 1)$. This will be used to prove Theorem 4.

To prove (9), we will use the usual setup from the preceding section in analyzing the binary addition of $m + n = s$. Namely we will construct the diagram $D = D(m, n)$, where we assume that $2^t \leq n < 2^{t+1}$ for some $t \geq 0$. Since $s > m$, there is a largest index $r \geq t$ such that $s_r = 1$ and $m_r = 0$. We normalize D by removing all columns C_j for $r > j > t + d$ since they have no effect on the score. Also, we replace any column $\begin{smallmatrix}1\\0\\1\end{smallmatrix}$ by $\begin{smallmatrix}0\\0\\0\end{smallmatrix}$ and any column $\begin{smallmatrix}1\\1\\1\end{smallmatrix}$ by $\begin{smallmatrix}0\\1\\0\end{smallmatrix}$ which also does not affect the score. Thus, the diagram will look like:

$D =$

$\mathbf{r = t + b}$	$\mathbf{t + b - 1}$	$\mathbf{t + b - 2}$	$\dots$	$\mathbf{t + 1}$	$\mathbf{t}$	$\mathbf{t - 1}$	$\dots$	$\mathbf{1}$	$\mathbf{0}$
0	m_{t+b-1}	m_{t+b-2}	$\dots$	m_{t+1}	m_t	m_{t-1}	$\dots$	m_1	m_0
0	0	0	$\dots$	0	1	n_{t-1}	$\dots$	n_1	n_0
1	s_{t+b-1}	s_{t+b-2}	$\dots$	s_{t+1}	s_t	s_{t-1}	$\dots$	s_1	s_0

where $b \geq 0$. We will always choose the value of i for selecting the digits of m and s to form $m^{(i)}$ and $s^{(i)}$ to satisfy $i \equiv r \pmod d$. In this case, we will denote $m^{(i)}$ and $s^{(i)}$ by m' and s', respectively. Keep in mind that if the term m_k is selected to be in m' then $m_k = m'_{\lfloor k/d \rfloor}$ (with the same remark applying to $s_k = s'_{\lfloor k/d \rfloor}$). Note that $s' > m'$. As before, we set $\Delta = s' - m'$ and define the score of D by

$$Score(D) = \frac{\Delta^d}{n}.$$

Theorem 5. *For every diagram $D = D(m, n)$, we have*

$$Score(D) \leq 2^{d-1}(2^d - 1).$$

Furthermore, the value of the constant $\beta_d = 2^{d-1}(2^d - 1)$ is best possible.[1]

Proof: First we will present the examples which show that the value of $\beta_d = 2^{d-1}(2^d - 1)$ is best possible. Consider the following diagram:

$D_r =$

$\mathbf{(r+2)d}$	$\dots$	$\mathbf{rd+1}$	$\dots$	$\mathbf{(k+1)d}$	$\dots$	$\mathbf{kd+1}$	$\dots$	$\mathbf{d}$	$\dots$	$\mathbf{1}$	$\mathbf{0}$
0	$\dots$	1	$\dots$	0	$\dots$	1	$\dots$	0	$\dots$	1	0
0	$\dots$	1	$\dots$	0	$\dots$	1	$\dots$	0	$\dots$	1	0
1	$\dots$	0	$\dots$	1	$\dots$	0	$\dots$	1	$\dots$	0	0

That is, D_r begins with a block of $2d$ columns with the form $\begin{smallmatrix}0&1&\cdots&1&1\\0&0&\cdots&0&1\\1&0&\cdots&0&0\end{smallmatrix}$ where there are $2d - 2$ columns $\begin{smallmatrix}1\\0\\0\end{smallmatrix}$ between the two ends of the block. D_r then continues with r blocks of length d which have the form $\begin{smallmatrix}0&1&\cdots&1&1\\0&0&\cdots&0&1\\1&0&\cdots&0&0\end{smallmatrix}$ where now there are only $d - 2$ columns of $\begin{smallmatrix}1\\0\\0\end{smallmatrix}$ between the

[1] We were pleased that this bound was so perfect!

ends. Finally, D_r has a final column $\begin{smallmatrix}0\\0\\0\end{smallmatrix}$.

We note that $n = \frac{2(2^{(r+1)d}-1)}{2^d-1}$. We must now compute $\Delta^{(i)} = |s^{(i)} - m^{(i)}|$ for $0 \le i < d$.

Case (i). $i = 0$. In this case,

$$\Delta^{(0)} = 2^{\lfloor (r+2)d/d \rfloor} = 2^{r+2}$$

so that

$$\frac{(\Delta^{(0)})^d}{n} = \frac{2^{(r+2)d}(2^d-1)}{2 \cdot (2^{(r+1)d}-1)} \to 2^{d-1}(2^d-1) \text{ as } r \to \infty.$$

Case (ii). $1 \le i \le d-1$. In this case,

$$s^{(i)} = 0, \; m^{(i)} = 2^{\lfloor ((r+1)d+i)/d \rfloor + \mathbf{1}} - 1 = 2^{r+2} - 1$$

so that

$$\frac{(\Delta^{(i)})^d}{n} = \frac{(2^{r+2}-1)^d(2^d-1)}{2 \cdot (2^{(r+1)d}-1)} \to 2^{d-1}(2^d-1) \text{ as } r \to \infty.$$

Therefore

$$Score(D_r) \to 2^{d-1}(2^d-1) \text{ as } r \to \infty.$$

To complete the proof of the theorem, we must show that any diagram $D = D(m,n)$ has $Score(D) \le \beta_d = 2^{d-1}(2^d-1)$. We will consider two cases:

Case 1. $b = 0$. Thus, $r = t$ and

$$D = \begin{array}{ccccc} \mathbf{t} & \mathbf{t-1} & \dots & \mathbf{1} & \mathbf{0} \\ \hline 0 & m_{t-1} & \dots & m_1 & m_0 \\ 1 & n_{t-1} & \dots & n_1 & n_0 \\ 1 & s_{t-1} & \dots & s_1 & s_0 \end{array}$$

Then

$$\Delta \le 2^{\lfloor t/d \rfloor + 1}, \quad n \ge 2^t,$$

and so

$$\frac{\Delta^d}{n} \le \frac{2^{t+d}}{2^t} = 2^d < \beta_d = 2^{d-1}(2^d-1)$$

for $d \ge 2$.

Case 2. $b \ge 1$. Therefore, $r \ge t+1$. By the carry restrictions on the columns of D, all columns C_j for $t < j < r$ must be $\begin{smallmatrix}1\\0\\0\end{smallmatrix}$. Thus,

$$D = \begin{array}{ccccccccc} \mathbf{r=t+b} & \mathbf{t+b-1} & \dots & \mathbf{t+1} & \mathbf{t} & \mathbf{t-1} & \dots & \mathbf{1} & \mathbf{0} \\ \hline 0 & 1 & \dots & 1 & m_t & m_{t-1} & \dots & m_1 & m_0 \\ 0 & 0 & \dots & 0 & 1 & n_{t-1} & \dots & n_1 & n_0 \\ 1 & 0 & \dots & 0 & s_t & s_{t-1} & \dots & s_1 & s_0 \end{array}$$

Now observe that we can delete any d consecutive columns $\begin{smallmatrix}1\\0\\0\end{smallmatrix}$ without affecting the score of D (as before, you might have to think about this for a minute). Hence, we can assume that $1 \le b \le d-1$.

Subcase 2.1 $b = d-1$. Thus,

$$D = \begin{array}{cccccccccc} \mathbf{t+d} & \mathbf{t+d-1} & \ldots & \mathbf{t+1} & \mathbf{t} & \mathbf{t-1} & \ldots & \mathbf{1} & \mathbf{0} \\ \hline 0 & 1 & \ldots & 1 & m_t & m_{t-1} & \ldots & m_1 & m_0 \\ 0 & 0 & \ldots & 0 & 1 & n_{t-1} & \ldots & n_1 & n_0 \\ 1 & 0 & \ldots & 0 & s_t & s_{t-1} & \ldots & s_1 & s_0 \end{array}$$

However, we cannot have $s_t = 1$ since the column $\begin{smallmatrix}0\\1\\1\end{smallmatrix}$ doesn't generate a carry, and we are not using the column $\begin{smallmatrix}1\\1\\1\end{smallmatrix}$. Thus we must have $s_t = 0$. But then

$$\Delta \le 2^{\lfloor (t+d)/d \rfloor} + 2^{\lfloor t/d \rfloor} = 3 \cdot 2^{\lfloor t/d \rfloor}, \quad n \ge 2^t$$

so that

$$\frac{\Delta^d}{n} \le \frac{3^d 2^t}{2^t} = 3^d < 2^{d-1}(2^d - 1)$$

for $d \ge 3$.

Subcase 2.2. $1 \le b \le d-3$. In this case, we have

$$\Delta \le 2^{\lfloor (t+b)/d+1 \rfloor}, \quad n \ge 2^t$$

so that

$$\frac{\Delta^d}{n} \le \frac{2^{b+d+t}}{2^t} \le 2^{2d-3} < 2^{d-1}(2^d - 1)$$

for $d \ge 1$.

This leaves us with the following (crucial) subcase.

Subcase 2.3. $b = d-2$. Thus,

$$D = \begin{array}{cccccccccc} \mathbf{t+d-1} & \mathbf{t+d-2} & \ldots & \mathbf{t+1} & \mathbf{t} & \mathbf{t-1} & \ldots & \mathbf{1} & \mathbf{0} \\ \hline 0 & 1 & \ldots & 1 & m_t & m_{t-1} & \ldots & m_1 & m_0 \\ 0 & 0 & \ldots & 0 & 1 & n_{t-1} & \ldots & n_1 & n_0 \\ 1 & 0 & \ldots & 0 & s_t & s_{t-1} & \ldots & s_1 & s_0 \end{array}$$

Since there are two choices for column C_t, namely either $C_t = \begin{smallmatrix}0\\1\\0\end{smallmatrix}$ or $C_t = \begin{smallmatrix}1\\1\\0\end{smallmatrix}$, we have $s_t = 0$.

If $n_{t-1} = 1$, then $\Delta \le 2^{\lfloor (t+d-1)/d \rfloor + 1}$ and $n \ge 2^t + 2^{t-1} = 3 \cdot 2^{t-1}$ so that

$$\frac{\Delta^d}{n} \le \frac{2^{t+2d-1}}{3 \cdot 2^{t-1}} = \frac{2^{2d}}{3} \le 2^{d-1}(2^d - 1).$$

We may assume $n_{t-1} = 0$.

If $m_{t-1} = 1$ or $s_{t-1} = 0$, $\Delta \le 2^{\lfloor (t-1)/d \rfloor + 1} + 2^{\lfloor (t-1)/d \rfloor}$ and $n \ge 2^t$ so that

$$\frac{\Delta^2}{n} \le \frac{3^d \cdot 2^{t-1}}{2^t} < 2^{d-1}(2^d - 1).$$

We may assume $m_{t-1} = 0$ and $s_{t-1} = 1$ which implies $m_t = 1$. We have

$$D = \begin{array}{cccccccc} \mathbf{t+d-1} & \mathbf{t+d-2} & \dots & \mathbf{t+1} & \mathbf{t} & \mathbf{t-1} & \dots & \mathbf{1} & \mathbf{0} \\ \hline 0 & 1 & \dots & 1 & 1 & 0 & \dots & m_1 & m_0 \\ 0 & 0 & \dots & 0 & 1 & 0 & \dots & n_1 & n_0 \\ 1 & 0 & \dots & 0 & 0 & 1 & \dots & s_1 & s_0 \end{array}$$

We now consider $m^* = m - (2^{t+d-2} + \ldots + 2^t), n^* = n - 2^t$ and $s^* = s - 2^{t+d-1}$. Clearly, $s^* = m^* + n^*$ and $s^* < s$. By induction we have

$$\frac{(\Delta^*)^d}{n^*} \le 2^{d-1}(2^d - 1).$$

Note that $\Delta = \Delta^* + 2^{\lfloor (t+d-1)/d \rfloor}$. We now use the above inequality and consider

$$\begin{aligned} \Delta^d - 2^{d-1}(2^d-1)n &\le (\Delta^* + 2^{(t+d-1)/d})^d - 2^{d-1}(2^d-1)(n^* + 2^t) \\ &\le \sum_{1 \le i \le d} \binom{d}{i} 2^{i(t+d-1)/d} (\Delta^*)^i - 2^{t+d-1}(2^d - 1) \\ &\le \sum_{1 \le i \le d-1} \binom{d}{i} 2^{i(t+d-1)/d} \Big((\Delta^*)^i - 2^{(d-i)(t+d-1)/d} \Big) \\ &\le 0. \end{aligned}$$

The last inequality follows from the fact that $\Delta^* \le 2^{(t+d-1)/d}$ and $2^d = \sum_i \binom{d}{i}$.

This completes the proof of Theorem 5. □

We can now use this result to prove Theorem 4, much in the same way that Theorem 3 was used to prove Theorem 1 for the case of $d = 2$.

Namely, we will construct a well dispersed sequence $\bar{z} = (z_1, z_2, \ldots)$ with $z_k \in [0,1]^d$ as follows. For the positive integer k, we form the integers $k^{(i)}$, $i = 0, 1, \ldots, d-1$, by sampling every d^{th} digit in the binary expansion of k. That is $k_j^{(i)} = k_{jd+i}$. We consider the sequence $\bar{f}$ which achieves the best possible dispersion bound α_1 in the one-dimensional case, as defined in (3).

Recall that f maps the integer m into $f(m) \in [0,1]$ by expanding m into its even Fibonacci representation and then summing the inverses of the Fibonacci numbers occurring in the representation (and then multiplying by α_1 to make sure the result is in $[0,1]$). The point z_k is now defined by

$$z_k = (f(k^{(0)}), f(k^{(1)}), \ldots, f(k^{(d-1)})).$$

It follows from (1) and (4) that for any pair of integers $a, b \geq 1$, we have

$$b|f(a) - f(a+b)| \geq \alpha_1$$

or

$$|f(a) - f(a+b)| \geq \alpha_1/b. \tag{10}$$

Hence, with $s = m + n$, we have

$$\begin{aligned}
\|z_m - z_s\| &= |f(m^{(0)}) - f(s^{(0)})| + |f(m^{(1)}) - f(s^{(1)})| + \ldots + |f(m^{(d-1)}) - f(s^{(d-1)})| \\
&\geq \max_i |f(m^{(i)}) - f(s^{(i)})| \\
&= |f(m^{(i_0)}) - f(s^{(i_0)})| \qquad \text{for some } i_0 \\
&\geq \frac{\alpha_1}{|m^{(i_0)} - s^{(i_0)}|} \\
&\geq \frac{\alpha_1}{\left(2^{d-1}(2^d - 1)n\right)^{1/d}}
\end{aligned}$$

by Theorem 5 and the properties of f. In other words,

$$n^{1/d} \, \|z_m - z_s\| \geq \frac{\alpha_1}{\left(2^{d-1}(2^d - 1)\right)^{1/d}}$$

which proves Theorem 4. □

4 Bounding dispersion via permutations.

In this section we will describe another approach for upper bounding α_2. Namely, we will consider a related problem for permutations which generalizes an argument used for the one-dimensional problem considered in [4].

First we need a few definitions. For $\pi \in S_n$, we denote π by the sequence

$$\pi = \big(\pi(1), \pi(2), \ldots, \pi(n)\big).$$

For example, 6423157 is a permutation in S_7 with $\pi(2) = 4$, etc.

A sequence $(a_1, \ldots, a_t)$ is said to be an *admissible* subsequence of π if there exist $i_1, i_2, \ldots, i_t$ such that
(1) $i_1 < i_2 < \ldots < i_t$ or $i_1 > i_2 > \ldots > i_t$, and
(2) $\pi(i_j) = a_j$ for $j = 1, \ldots, t$.

For example, 621 and 534 both are admissible subsequences of $\pi = 6423157$.

For two permutations π and σ in S_n, we say a sequence $(a_1, \ldots, a_t)$ is (π, σ)-admissible if $(a_1, \ldots, a_t)$ is an admissible subsequence of both π and σ. Let $I(\pi, \sigma)$ denote the set consisting of all (π, σ)-admissible subsequences.

For $\pi, \sigma \in S_n$, we define

$$\phi(\pi, \sigma) = \max_{(a_1, \ldots, a_t) \in I(\pi, \sigma)} \sum_{i=1}^{t-1} |a_i - a_{i+1}|^{-1/2}. \tag{11}$$

For example, $\phi(6423157, 2713564) = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{6}}$ which is achieved by the admissible subsequence 4317.

For any positive integer n, we define

$$\phi^{(n)} = \min_{\pi,\sigma\in S_n} \phi(\pi,\sigma). \tag{12}$$

Computation ([2]) shows that:

n	2	3	4	5	6	7	8	9	10
$\phi^{(n)}$	1	1	1	$\frac{1}{2}+\frac{1}{\sqrt{3}}$	$\frac{1}{2}+\frac{1}{\sqrt{3}}$	$\frac{1}{2}+\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}$

An example of a pair of permutations π and σ in S_{10} which achieve the value $\phi^{(10)} = \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}} = 1.2844\ldots$ is $(10, 3, 6, 8, 1, 4, 5, 9, 2, 7)$ and $(1, 8, 5, 3, 10, 7, 6, 2, 9, 4)$ with the best admissible subsequence $(3, 6, 4)$.

Lemma 1.

$$\phi^{(n)} \le \phi^{(n+1)}.$$

Proof. For $\pi, \sigma \in S_n$ and $\pi', \sigma' \in S_{n+1}$, suppose π and σ are subsequences of π' and σ', respectively. If a sequence $(a_1, \ldots, a_t)$ is (π, σ)-admissible, then $(a_1, \ldots, a_t)$ is (π', σ')-admissible. Thus, $I(\pi, \sigma) \subset I(\pi', \sigma')$ which implies $\phi(\pi, \sigma) \le \phi(\pi', \sigma')$ and $\phi^{(n)} \le \phi^{(n+1)}$. □

From Lemma 1, we know that the limit of $\phi(n)$ exists as n approaches infinity. We define

$$\phi = \lim_{n\to\infty} \phi^{(n)}.$$

Then we can use ϕ to bound the dispersion. First we consider a finite version of the dispersion μ_2. For a positive integer N, we define

$$\mu_2^{(N)}(\bar{z}) = \inf_{n\ge 1} \inf_{1\le m\le N-n} \sqrt{n}\, \|z_m - z_{m+n}\|$$

and

$$\alpha_2^{(N)} = \sup_{\bar{z}} \mu_2^{(N)}(\bar{z}).$$

where as usual, $\|\cdot\|$ denotes the L_1 norm. Clearly,

$$\alpha_2^{(N+1)} \le \alpha_2^{(N)}.$$

Lemma 2.

$$\alpha_2 = \lim_{N\to\infty} \alpha_2^{(N)}.$$

We will show the following:

Theorem 6. *For any N, we have*

$$\alpha_2^{(N)} \le \frac{2}{\phi^{(N)}}. \tag{13}$$

Proof. Suppose $\alpha_2^{(N)} > \frac{2}{\phi^{(N)}}$. Then there exists a sequence $\bar{z}$ such that for all $n \leq N, m \leq N - n$, we have

$$\sqrt{n}\, \|z_{m+n} - z_m\| > \frac{2}{\phi}.$$

In the other direction, from the definition of $\phi^{(N)}$, we know that for any π, σ in S_N, there is a sequence $\bar{a} = (a_1, \ldots, a_t)$ in $I(\pi, \sigma)$ such that

$$\phi(\pi, \sigma) = \sum_k |a_k - a_{k+1}|^{-1/2} \geq \phi^{(N)}.$$

Now, we choose π, σ in S_N by using $z_i = (x_i, y_i), 1 \leq i \leq N$, where $\bar{z} = (z_1, z_2, \ldots)$ and π and σ are defined by

$$x_{\pi(1)} \leq x_{\pi(2)} \leq \ldots \leq x_{\pi(N)},$$
$$y_{\sigma(1)} \leq y_{\sigma(2)} \leq \ldots \leq y_{\sigma(N)}$$

where ties are broken arbitrarily. Namely, π is determined by the order of the projection of z's on the x-axis and σ is determined by the order of the projection on the y-axis.

Claim 1: Suppose a sequence $(a_1, \ldots, a_t)$ is (π, σ)-admissible. Then we have

$$\sum_{k=1}^{t-1} \|z_{a_k} - z_{a_{k+1}}\| \leq 2.$$

The proof of the claim follows from monotonicity. Namely,

$$\begin{aligned}\sum_{k=1}^{t-1} \|z_{a_k} - z_{a_{k+1}}\| &= \sum_{k=1}^{t-1} \left(|x_{a_k} - x_{a_{k+1}}| + |y_{a_k} - y_{a_{k+1}}|\right) \\ &= |x_{a_1} - x_{a_k}| + |y_{a_1} - y_{a_k}| \\ &\leq 2.\end{aligned}$$

Now we proceed to prove Theorem 6. Note that for any k, from the assumption on $\bar{z}$, we have

$$\|z_{a_k} - z_{a_{k+1}}\| > \frac{2}{\phi^{(N)}|a_k - a_{k+1}|^{1/2}}.$$

Putting everything together, we have

$$\begin{aligned}2 &\geq \sum_{k=1}^{t-1} \|z_{a_k} - z_{a_{k+1}}\| \qquad \text{from Claim 1} \\ &> \sum_{k=1}^{t-1} \frac{2}{\phi^{(N)}|a_k - a_{k+1}|^{1/2}} \\ &\geq \frac{2}{\phi^{(N)}}\phi^{(N)} \\ &= 2\end{aligned}$$

which is a contradiction. Theorem 6 is proved. □

As an immediate consequence of Lemmas 1 and 2, and Theorem 6, we have the following:

Theorem 7. *For any integer N, we have*

$$\alpha_2^{(N)} \le \alpha_2 \le \frac{2}{\phi} \le \frac{2}{\phi^{(N)}}.$$

In particular, this implies that

$$\alpha_2 \le \frac{2}{\phi^{(10)}} = \frac{2}{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}} = 1.557\ldots.$$

Remark. Computation ([2]) has produced the following permutations $\pi, \sigma \in S_{86}$:

$$\begin{aligned}\pi = (&6, 62, 83, 32, 74, 39, 21, 43, 52, 68, 79, 70, 44, 28, 50, 75, 15, 60, 36, 56, 9, 25, 76, 46, 30,\\ &67, 53, 38, 49, 3, 72, 18, 59, 84, 85, 33, 66, 34, 12, 63, 23, 81, 40, 29, 8, 48, 77, 51, 69, 20,\\ &64, 14, 35, 86, 55, 26, 61, 4, 58, 41, 80, 17, 65, 37, 47, 73, 19, 78, 42, 54, 11, 22, 7, 27, 13,\\ &24, 82, 1, 5, 31, 10, 57, 2, 45, 16, 71),\\ \sigma = (&20, 64, 86, 48, 77, 35, 1, 8, 59, 84, 14, 55, 26, 18, 50, 75, 33, 66, 40, 28, 56, 43, 68, 83, 23,\\ &21, 62, 5, 61, 31, 78, 81, 10, 57, 19, 51, 42, 69, 29, 16, 71, 24, 82, 45, 54, 3, 72, 63, 36, 79,\\ &46, 52, 37, 73, 47, 12, 85, 6, 32, 74, 15, 60, 39, 25, 76, 70, 17, 65, 34, 53, 41, 80, 9, 44, 30,\\ &2, 13, 27, 58, 7, 22, 4, 67, 38, 11, 49),\end{aligned}$$

with an optimal admissible subsequence $(86, 77, 81, 72, 76, 70)$. Thus, $\phi(\pi, \sigma) = 5/3 + 1/\sqrt{6} = 2.0749\ldots$ so that $\phi^{(86)} \le 2.0749\ldots$. If in fact this were the actual value of $\phi^{(86)}$ then we would have $\alpha_2 \le 2/\phi^{(86)} = 0.9638\ldots$.

5 Common admissible subsequences of several permutations

The permutation problem in the preceding section can be naturally generalized to the case of $d \ge 2$. We need several additional definitions.

For the case of general $d \ge 2$, we consider the admissible set $I(\pi_1, \pi_2, \ldots, \pi_d)$ consisting of all sequences which are π_j-admissible for every j, $1 \le j \le d$, where the π_j are in S_N. We define

$$\phi(\pi_1, \ldots, \pi_d) = \max_{\substack{(a_1, \ldots, a_t) \in I(\pi_i) \\ i=1,\ldots,d}} \sum_{i=1}^{t-1} |a_i - a_{i+1}|^{-1/d}. \tag{14}$$

Then we define

$$\phi_d^{(N)} = \min_{\substack{\pi_i \\ i=1,\ldots,d}} \phi(\pi_1, \ldots, \pi_d)$$

$$\text{and} \quad \phi_d = \lim_{N \to \infty} \phi_d^{(N)}.$$

For the one-dimensional case, we have previously shown [4, 5]

$$\phi_1 = \frac{1}{\alpha_1} = 1 + \sum_{j\geq 1} \frac{1}{F_{2j}} = 2.535\ldots.$$

For general d, the following can be shown.

Theorem 8. *For any integer N, we have*

$$\alpha_d^{(N)} \leq \alpha_d \leq \frac{d}{\phi_d} \leq \frac{d}{\phi_d^{(N)}}. \tag{15}$$

Proof. The proof is by using similar arguments as those in the proofs of Theorems 6 and 7. Instead of considering common admissible subsequence of two permutations, we consider common admissible subsequences of d permutations, $\pi_1, \ldots, \pi_d$, determined by the projections of the first N terms of the sequence $\bar{z} = (z_1, z_2, \ldots)$ which satisfies

$$n^{1/d}\|z_{m+n} - z_m\| \geq \frac{d}{\phi_d^{(N)}}.$$

The Claim 1 in the proof of Theorem 6 is to be replaced by the following:

Claim 1': Suppose a sequence $(a_1, \ldots, a_t)$ is $(\pi_1, \ldots, \pi_d)$-admissible. Then for $z_i = (x_i)$. we have

$$\sum_{k=1}^{t-1} \|z_{a_k} - z_{a_{k+1}}\| \leq d.$$

The proof of the claim follows from monotonicity of admissible common subsequences. The rest of the proof follows in the same fashion as in Theorem 6. □

6 Connections to simultaneous Diophantine approximation.

A classic problem in simultaneous Diophantine approximation is the determination of the smallest possible value of the constant c_0 such that for all irrational numbers α and β, the inequalities

$$\left|\alpha - \frac{p}{r}\right| < \frac{c_0}{r^{3/2}}, \quad \left|\beta - \frac{q}{r}\right| < \frac{c_0}{r^{3/2}}, \tag{16}$$

hold for infinitely many values of integers p, q and r (e.g., see [8]). The best possible value of c_0 is still not known. However, these results are related to our problem as follows. In ([7]) it is shown that for any irrational α and β, the inequality

$$\left(\alpha - \frac{p}{r}\right)^2 + \left(\beta - \frac{q}{r}\right)^2 \leq \frac{2}{r^3\sqrt{23}} \tag{17}$$

holds for infinitely many p, q and r. Furthermore, the value $\frac{2}{\sqrt{23}}$ is best possible. This means for $c' < \frac{2}{\sqrt{23}}$, there exist α' and β' such that

$$\left(\alpha' - \frac{p}{r}\right)^2 + \left(\beta' - \frac{q}{r}\right)^2 > \frac{c'}{r^3} \tag{18}$$

holds for all sufficiently large r. Multiplying (18) by r^3 and taking square roots, we have

$$r^{1/2}(\{r\alpha'\} + r\beta'\}) \geq \sqrt{c'} \tag{19}$$

for $r > r_0$, where $\{\cdot\}$ denotes fractional part.

Let us define the sequence $\bar{z} = (z(0), z(1), \dots)$ by setting $z(k) = (\{k\alpha'\}, \{k\beta'\}) \in [0,1]^2$ for $k \geq 0$. Notice that $\|z(m+r) - z(m)\| = \|z(r)\|$ follows from the definition of z (where $\|\cdot\|$ denotes the L_1 distance). Thus, from (19) we find

$$r^{1/2}\|z(m) - z(m+r)\| = r^{1/2}|\{r\alpha'\} + \{r\beta'\}| \geq \sqrt{c'}$$

for $r \geq r_0$. This implies that

$$\liminf_{r\to\infty} \inf_{m\geq 0} r^{1/2}\|z(m) - z(m+r)\| = \liminf_{r\to\infty} r^{1/2}\|z(r)\| \geq \sqrt{c'}.$$

Note that this condition is weaker than what is required in our definition of μ_2.

The value of $\sqrt{\frac{2}{\sqrt{23}}}$ is approximately 0.6458. The values of α' and β' used in [8] depend on properties of the cubic field $\mathbb{Q}(\phi)$ where $\phi = 1.3247\dots$ is the real root of the polynomial $x^3 - x - 1$. However, using the values $\alpha' = 5.621\dots \times 10^{-9}, \beta' = 0.2802\dots$ from [7], you find that for $r = 25$, for example, $\sqrt{25}||z(0) - z(25)|| = 0.025\dots$ which is a lot smaller than our lower bound of $0.1610\dots$ on α_2 in Theorem 1. Of course, in [7], small values of r weren't of concern whereas for us they are. More general results of this type for simultaneous Diophantine approximations for $d \geq 2$ dimensions can be found, for example, in [7]. The corresponding applications for our measure μ_d can be made (but we won't do it).

7 Problems and remarks.

We conclude with a number of questions which remain unanswered.

(1) What is the true value of α_2? All we currently know is that

$$\frac{1}{\sqrt{6}\left(1 + \sum_{k\geq 1}\frac{1}{F_{2k}}\right)} = 0.16102\dots < \alpha_2 < \frac{2}{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}} = 1.557\dots.$$

No doubt, the truth is much closer to the lower bound. We should remark that it is possible to modify the van der Corput construction to form a sequence $\bar{z}$ with $\mu_2(\bar{z}) \geq 1/16$, which is not as good as the bound in Theorem 1.

Another possible construction is the sequence $x_n = (\{an\}, \{bn\})$ for appropriate choices of irrational numbers a and b as discussed in Section 6 on the connection with simultaneous Diophantine approximation. Can sequences like this be used to improve the lower bound for α_2? In fact, the authors have often considered this type of sequence in the search for a good lower bound for α_2. For some choices of the irrational pairs, computations are encouraging. However, it does not seem easy to prove any positive lower bound.

(2) More generally, what are values of α_d for $d \geq 2$? Similar problems and constructions to those for α_2 can be asked for α_d. Computations could be done for d-tuples of admissible permutations as was done for $d = 2$ to give some (not very sharp) upper bounds.

(3) There are numerous problems for permutations concerning the invariant ϕ_d as defined in (11) and (14). For the one-dimensional case, the value of ϕ_1 is known since we have the equality $\phi_1 = \alpha_1^{-1} = 1 + \sum_{j\geq 1} F_{2j}^{-1} = 2.535\ldots$. For $d = 2$, the invariant ϕ_2 concerns admissible common subsequences of two permutations which is an interesting question in its own right. As it turns out, ϕ_2 has a natural generalization to higher dimensions. The invariant ϕ_d concerns admissible common subsequences of d permutations. What are the true values of ϕ_d for $d \geq 2$? For $d \geq 2$, we only can prove the inequality $\phi_d \leq d\alpha_d^{-1}$ while we do not know the value of α_d. What is the true relationship between ϕ_d and α_d? Could they be equal? (We would be pleasantly surprised if they were.) For $d = 1$, we have a construction of permutations which achieve ϕ_1 (as shown in [4]). What are the permutations which achieve ϕ_d for $d \geq 2$?

(4) What are the corresponding results for different spaces and different norms? For example, in [6], the authors studied these questions for the case that $\bar{x}$ lies on the unit circle C (where the distance between two points on C is the length of the shorter arc joining them). Here, the corresponding constant was $\alpha(C) = \frac{3-\sqrt{5}}{2} = 0.3819\ldots$ which is somewhat smaller than $\alpha_1 = 0.3944\ldots$ (as would be expected). We also showed in this case that this value of $\alpha(C)$ was best possible. A natural next step would be to consider sequences in the 2-dimensional torus.

(5) In this paper, we have restricted our norm $\|\cdot\|$ to be the L_1 norm. Actually, our lower bound proof applies to the L_∞ norm as well. On the other hand, the permutation approach seems to require L_1. For sequences considered in the simultaneous Diophantine approximation approach in Section 6, the L_2-norm was used. Would the use of different norms make these problems easier (or harder)?

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