Discrepancy inequalities for directed graphs

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Abstract:

We establish several discrepancy and isoperimetric inequalities for directed graphs by considering the associated random walk. We show that various isoperimetric parameters, as measured by the stationary distribution of the random walks, including the Cheeger constant and discrepancy, are related to the singular values of the normalized probability matrix and the normalized Laplacian. Further, we consider the *skew-discrepancy* of directed graphs which measures the difference of flow among two subsets. We show that the skew-discrepancy is intimately related to $\mathcal{Z}$, the skew-symmetric part of the normalized probability transition matrix. In particular, we prove that the skew-discrepancy is within a logarithmic factor of $\|\mathcal{Z}\|$. Finally, we apply our results to construct extremal families of directed graphs with large differences between the discrepancy of the underlying graph and the skew-discrepancy.

1. Introduction

The discrepancy of a graph bounds the largest difference between the number of edges between two subsets of vertices and its expected value among all possible choice of subsets. The study of discrepancy in graph theory has been an extensively useful tool in spectral graph theory with wide applications in extremal graph theory, in the analysis of approximation algorithms, and in statistical tests (see [10]).

In the undirected case, there are several different ways to measure the discrepancy. One way is to consider the size of the two subsets, and take the "expected" number of edges to be proportional to the their product. This is a natural consideration for regular graphs, as in this case, the discrepancy of $A, B \subset V$ is bounded above by $\lambda_2\sqrt{|A||B|}$ where λ_2 is the second largest absolute eigenvalue of the adjacency matrix of G [3]. For a graph with a general degree sequence, the normal eigenvalue bound using the adjacency matrix does

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not apply. Instead, one may consider the *volume*, the sum of the degrees, of each subset; in which case, the "expectation" is taken to be proportional to the product of the volumes. Under this notion, the discrepancy can be bounded using the spectral gap of the *normalized Laplacian matrix* (see [7]).

This study focuses on directed graphs by adapting the concepts above. However, this presents additional challenges. First, directed graphs do not have a natural notion of degree or volume, as the in- and out-degree of each vertex do not necessarily coincide. Further, many of the graph theoretic matrices of directed graphs are not symmetric; therefore, many of the tools and techniques used to study undirected graphs cannot be applied to the general directed case.

To address the first challenge, in their work on the Cheeger constant, Chung [8] followed by Li and Zhang [12] consider a *random walk* and apply the stationary distribution in order to measure the *volume* of a subset of vertices instead of size of the sets. Specifically, in our case, we use the stationary distribution of a typical random walk on a directed graph G to define two types of discrepancy: $\mathrm{disc}(G)$ and $\mathrm{disc}'(G)$. Roughly speaking, $\mathrm{disc}(G)$ bounds the difference of the flow from one subset of vertices S to another subset T from the expected quantity while $\mathrm{disc}'(G)$ measures the difference between the flow from S to T and the reverse flow from T to S. Note that the expected quantity depends on the stationary distribution and can be quite different in one direction from the other. Precise definitions are given in Section 2.

To overcome the second obstacle, one can symmetrize the matrix associated with the directed graph (or the associated random walk) as in [1], [8], or [12]. As a result, one can apply the techniques used with symmetric matrices. In doing so, one loses information regarding the directed nature of the graph. In order to capture the directed nature of a directed graph, Li and Zhang in [12] considered the skew-symmetric part of the normalized probability transition matrix.

In this paper, we expand upon this method of using three graph theoretic matrices and show their eigen- and singular values bound the notions of discrepancy described above. We will show that these two types of discrepancies are intimately related to several types of eigenvalues. The first type is derived using the (normalized) Laplacian $\mathcal{L}$ as defined earlier in [8] to establish a generalized Cheeger's inequality for directed graphs. For undirected graphs, the eigenvalues of the Laplacian $\mathcal{L}$ can be used to bound the discrepancy. Here, we can still use eigenvalues of $\mathcal{L}$ to bound the discrepancy with regard to the flow from a subset S to its complement, $\overline{S}$. To deal with the discrepancy from a subset S to another subset T in a directed graph G, we will use singular values of the normalized transition probability matrix, $\mathcal{P}$, of the random walk on G. Another type of eigenvalue depends on the skew-symmetric matrix $\mathcal{Z}$ that will be defined in Section 2 and is useful for bounding $\mathrm{disc}'(G)$.

The paper is organized as follows: In Section 2, we give basic definitions. In Section 3, we derive facts regarding the graph theoretic matrices we use. In

Section 4, we deal with the discrepancy between a subset and its complement and derive the relation with eigenvalues of the Laplacian $\mathcal{L}$. We then bound the discrepancy for any two general subsets in terms of the singular values of the transition probability matrix in Section 5. In Sections 6 and 7, we give several upper bounds for the skew-discrepancy, $\mathrm{disc}'(G)$, in terms of the matrix $\mathcal{Z}$ and its maximal singular value, $\|\mathcal{Z}\|$. Finally, we give constructions and applications of these results in Section 8.

2. Preliminaries

For a directed graph $G = (V, E)$ with edge weights $w_{u,v} > 0$ we consider the associated random walk on G whose transition probability matrix, denoted $\mathbf{P}$, is given by:

$$\mathbf{P}(u, v) = \frac{w_{u,v}}{d_u}$$

where $d_u = \sum_v w_{u,v}$ denotes the total weights among out-going arcs of u.

We say a directed graph is *aperiodic* if the greatest common divisor of the lengths of all closed walks is 1. Otherwise, it is *periodic*. A random walk is said to be *ergodic* if its directed graph is strongly connected and aperiodic [1]. An ergodic random walk has a unique stationary distribution, ϕ, obeying

$$\phi\mathbf{P} = \phi. \tag{1}$$

For the purposes of this paper, in order to guarantee a unique stationary distribution, we only consider directed graphs which are aperiodic and strongly connected.

Note that ϕ can be used to define a special type of flow on the edges of G, called *circulation* as follows (see [8]). For an edge (u, v), the flow $f_\phi(u, v)$ is

$$f_\phi(u, v) = \phi(u)\mathbf{P}(u, v).$$

For two subsets S and T of vertices in G the flow from S to T is denoted by:

$$f(S, T) = \sum_{u \in S, v \in T} f(u, v).$$

We extend the notion of ϕ to subsets of vertices by defining

$$\phi(S) = \sum_{v \in S} \phi(v).$$

For two subsets of vertices S, T, we define $\mathrm{disc}(S, T)$ to be the quantity

$$\mathrm{disc}(S, T) = \mid f_\phi(S, T) - \phi(S)\phi(T) \mid .$$

Notice that for an undirected graph, $\phi(v) = \frac{d_v}{\text{vol } G}$, where vol $G = \sum_v d_v$ and therefore $f_\phi(u,v) = 1/\text{vol } G$. Hence, the above notion of discrepancy is consistent with that for undirected graphs using the normalized Laplacian as seen in [7].

Further, we define the *skew-discrepancy*, denoted $\text{disc}'(S,T)$, to be the quantity

$$\text{disc}'(S,T) = \left| \frac{f_\phi(S,T) - f_\phi(T,S)}{2} \right|.$$

The discrepancy of G, denoted $\text{disc}(G)$, is the minimal value of α for which

$$\text{disc}(S,T) \leq \alpha\sqrt{\phi(S)\phi(T)}$$

for all subsets of vertices $S, T \subset V(G)$.

Similarly, we define $\text{disc}'(G)$ to be the smallest β such that for all two subsets of vertices S and T,

$$\text{disc}'(S,T) \leq \beta\sqrt{\phi(S)\phi(T)}.$$

We remark that the concept of skew-discrepancy is similar to, yet distinct from, the *digraph gap* in [12].

We will show that these two types of discrepancies are closely related to the eigenvalues of the Laplacian and its variations. Throughout, we will use $\mathbf{A}^*$ to denote the conjugate transpose of $\mathbf{A}$, χ_S to indicate the the characteristic vector (or function) for the set S, and $\mathbf{1}$ to denote the all-one vector.

Let $\mathbf{I}$ denote the identity matrix and $\Phi = \text{diag}(\phi)$ denote the diagonal matrix with entries $\Phi_{ii} = \phi(i)$, and $\Phi_{ij} = 0$ if $i \neq j$. The normalized Laplacian $\mathcal{L}$ is defined as follows (see [8]):

$$\begin{aligned} \mathcal{L} &= \mathbf{I} - \frac{\Phi^{1/2}\mathbf{P}\Phi^{-1/2} + \Phi^{-1/2}\mathbf{P}^*\Phi^{1/2}}{2} \\ &= \mathbf{I} - \frac{\Phi^{-1/2}(\Phi\mathbf{P} + \mathbf{P}^*\Phi)\Phi^{-1/2}}{2} \\ &= \mathbf{I} - \frac{\mathcal{P} + \mathcal{P}^*}{2} \end{aligned} \tag{2}$$

where $\mathcal{P}$ is the normalized transition probability matrix given by

$$\mathcal{P} = \Phi^{1/2}\mathbf{P}\Phi^{-1/2}$$

Further, we consider $\mathcal{Z}$, the asymmetric part of $\mathcal{P}$:

$$\begin{aligned} \mathcal{Z} &= \frac{\Phi^{1/2}\mathbf{P}\Phi^{-1/2} - \Phi^{-1/2}\mathbf{P}^*\Phi^{1/2}}{2} \\ &= \frac{\mathcal{P} - \mathcal{P}^*}{2} \end{aligned} \tag{3}$$

As we are concerned with the spectra and singular values of these matrices, we will denote the singular values of the matrix $\mathbf{M}$ as $\sigma_0(\mathbf{M}) \geq \sigma_1(\mathbf{M}) \ldots \geq \sigma_{n-1}(\mathbf{M})$. When we omit the matrix from σ_i we will refer to $\mathcal{P}$; in which case we have, $1 = \sigma_0 \geq \sigma_1 \geq \ldots \geq \sigma_{n-1} \geq 0$. We denote the eigenvalues of $\mathcal{L}$ as $0 = \lambda_0 < \lambda_1 \leq \ldots \leq \lambda_{n-1} \leq 2$. We will use $\overline{\lambda} := \max\{1-\lambda_1, \lambda_{n-1}-1\}$ and we denote the maximal singular value of $\mathcal{Z}$ as $\|\mathcal{Z}\|$.

Lastly, in many of our constructions in the latter part of this paper, we will be concerned with the asymptotic behavior of various functions. In order to describe these asymptotic relations, we will use "little-o" notation. We say $f(n) = o(g(n))$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

3. Facts about the matrices $\mathcal{P}$, $\mathcal{L}$, $\mathcal{Z}$

We state some basic facts regarding $\mathcal{P}$, $\mathcal{L}$, $\mathcal{Z}$ to be used later.

Fact 1. *The following are facts regarding the matrices $\mathcal{P}$ and $\mathcal{L}$:*

(i) *The vector $\mathbf{1}^*\Phi^{1/2}$ is both a left and right eigenvector of $\mathcal{P}$ with eigenvalue 1.*

(ii) $\|\mathcal{P}\| = 1$.

(iii) *The vector $\mathbf{1}^*\Phi^{1/2}$ is both a left and right eigenvector of $\mathcal{L}$ with eigenvalue 0.*

(iv) *For a strongly connected aperiodic directed graph, the eigenvalue 0 has multiplicity 1, and all nonzero eigenvalues of $\mathcal{L}$, denoted λ_i for $i \neq 0$ are real and obey $0 < \lambda_i < 2$.*

(v) *For any vector $\mathbf{x} \in \mathbb{C}^n$, the inner product $\langle \mathbf{x}, \mathcal{L}\mathbf{x}\rangle$ is real.*

(ii) is not obvious, but is a consequence of Perron-Frobenius theory and (i).

In comparing (iv) to the undirected case of the normalized Laplacian, the multiplicity of the eigenvalue 0 indicates the number of components and the multiplicity of the eigenvalue 2 indicates the number of connected bipartite components [7]. Since we require strongly connected aperiodic graphs, there must be only one component.

Lemma 1. *For a directed graph G, the matrix $\mathcal{Z}$, as defined in (3), satisfies the following conditions:*

(i) *The transpose $\mathcal{Z}^*$ of $\mathcal{Z}$ is equal to $-\mathcal{Z}$.*

(ii) *The vector $\Phi^{1/2}\mathbf{1}$ is an eigenvector of $\mathcal{Z}$ with eigenvalue 0.*

(iii) *The eigenvalues of the matrix $\mathcal{Z}$ are purely imaginary. Hence, since $\mathcal{Z}$ is normal, for any vector $\mathbf{x} \in \mathbb{C}^n$, the inner product $\langle \mathbf{x}, \mathcal{Z}\mathbf{x} \rangle$ is purely imaginary.*

(iv) *For a strongly connected aperiodic directed graph, $\|\mathcal{Z}\| < 1$.*

(v) *For $\mathbf{f} : V \to \mathbb{R}$, we have $\langle \mathbf{f}\mathcal{Z}, \mathbf{f} \rangle = 0$.*

Proof. (i) and (ii) can be verified by substitutions. (iii) follows from (i). (iv) follows from the triangle inequality. Strict inequality follows from Perron-Frobenius theory. To see (v), it can be checked that $\langle \mathbf{f}\mathcal{Z}, \mathbf{f} \rangle = \mathbf{f}\Phi^{-1/2}\mathbf{P}\Phi^{1/2}\mathbf{f}^* - \mathbf{f}\phi^{1/2}\mathbf{P}^*\Phi^{-1/2}\mathbf{f}^* = 0$. □

Lemma 2. *In a directed graph G, the spectral radius $\|\mathcal{Z}\|$ of $\mathcal{Z}$ (or similarly for any skew-symmetric matrix) satisfies the following Rayleigh quotients:*

(a) *The spectral radius $\|\mathcal{Z}\|$ is equal to the numerical radius of $\mathcal{Z}$. Namely,*

$$\|\mathcal{Z}\| \quad = \quad \sup_{\mathbf{F}} \frac{|\langle \mathbf{F}, \mathcal{Z}\mathbf{F} \rangle|}{\|\mathbf{F}\|^2}$$

where $\mathbf{F} : V(G) \to \mathbb{C}$.

(b)

$$\|\mathcal{Z}\| \quad = \quad \sup_{\mathbf{f},\mathbf{g}} \frac{|\langle \mathbf{f}, \mathcal{Z}\mathbf{g} \rangle|}{\|\mathbf{f}\| \ \|\mathbf{g}\|}$$

where $\mathbf{f}, \mathbf{g} : V(G) \to \mathbb{R}$. In addition, the $\mathbf{f}$ and $\mathbf{g}$ achieving the supremum have $\mathbf{f}, \mathbf{g}$ orthogonal and $\|f\| = \|g\|$.

(c) *If $\mathcal{P} = \mathbf{I} - \mathcal{L} + \mathcal{Z}$ as in Section 2, then*

$$\|\mathcal{Z}\| = \sup_{\mathbf{z} \in \mathbb{C}^n} \frac{\mathrm{Im}(\langle \mathbf{z}, \mathcal{P}\mathbf{z} \rangle)}{\langle \mathbf{z}, \mathbf{z} \rangle}$$

where $\mathrm{Im}(\cdot)$ indicates the imaginary part.

Proof. From Lemma 1, we see that $i\mathcal{Z}$ is Hermitian and therefore the spectral radius of $i\mathcal{Z}$ is achieved by an eigenfunction F. Thus, for some real θ,

$$\|\mathcal{Z}\| = \|i\mathcal{Z}\| = \frac{\|\mathbf{F}\mathcal{Z}\|}{\|\mathbf{F}\|} = |\theta| \tag{4}$$

where

$$\mathbf{F}\mathcal{Z} = i\theta\mathbf{F}.$$

This proves (a). Without loss of generality, we may assume $\theta \neq 0$. Suppose $\mathbf{F} = \mathbf{f} + i\mathbf{g}$ where $\mathbf{f}, \mathbf{g}$ are real-valued functions. By substitution into (4), we have

$$\mathbf{f}\mathcal{Z} = -\theta\mathbf{g}$$
$$\mathbf{g}\mathcal{Z} = \theta\mathbf{f}$$

Since $\theta\|f\| = |\langle \mathbf{f}\mathcal{Z}, \mathbf{g}\rangle| = \theta\|\mathbf{g}\|$, we have $\|\mathbf{f}\| = \|\mathbf{g}\|$. Also, from Lemma 1 (v), $0 = \langle f\mathcal{Z}, f\rangle = \theta\langle \mathbf{f}, \mathbf{g}\rangle$. This implies $\langle \mathbf{f}, \mathbf{g}\rangle = 0$ and (b) is proved.

Note that $i\mathcal{Z}$ is Hermitian, so we have $\|\mathcal{Z}\| =$

$$\|i\mathcal{Z}\| = \sup_{\mathbf{z}\in\mathbb{C}^n} \frac{i\langle \mathbf{z}, \mathcal{Z}\mathbf{z}\rangle}{\langle \mathbf{z}, \mathbf{z}\rangle} = \sup_{\mathbf{z}\in\mathbb{C}^n} \frac{\mathrm{Im}\langle \mathbf{z}, \mathcal{Z}\mathbf{z}\rangle}{\langle \mathbf{z}, \mathbf{z}\rangle}.$$

Also since $(\mathbf{I} - \mathcal{L})$ is Hermitian, $\mathrm{Im}\langle \mathbf{z}, (\mathbf{I} - \mathcal{L})\mathbf{z}\rangle = 0$, so we have

$$= \sup_{\mathbf{z}\in\mathbb{C}^n} \frac{\mathrm{Im}\langle \mathbf{z}, (\mathbf{I} - \mathcal{L} + \mathcal{Z})\mathbf{z}\rangle}{\langle \mathbf{z}, \mathbf{z}\rangle} = \sup_{\mathbf{z}\in\mathbb{C}^n} \frac{\mathrm{Im}\langle \mathbf{z}, \mathcal{P}\mathbf{z}\rangle}{\langle \mathbf{z}, \mathbf{z}\rangle}.$$

This proves (c). □

Finally, we establish relationships between $\mathcal{P}$, $\mathcal{L}$, $\mathcal{Z}$.

Fact 2.

(i) $\sigma_i(\mathbf{I} - \mathcal{P}) \geq \lambda_i$

(ii) $\sigma_i(\mathbf{I} - \mathcal{P}) \geq \|\mathcal{Z}\|$

(iii) Either $\sigma_1(\mathcal{P}) \leq 2\overline{\lambda}$ or $\sigma_i(\mathcal{P}) \leq 2\|\mathcal{Z}\|$

This fact is a simple application of the triangle inequality. As a result, there is a strong relationship between $\mathcal{P}$ and $\mathbf{I} - \mathcal{L}$ and also $\mathcal{P}$ and $\mathcal{Z}$. However, it is emphasized that there is, in fact, no such relationship between $\mathbf{I} - \mathcal{L}$ and $\mathcal{Z}$ in the following theorem:

Theorem 1. *There exists a strongly connected aperiodic directed graph for which $\sigma_1(\mathbf{I}-\mathcal{L}) < \|\mathcal{Z}\|$, and also, there exists a graph for which $\sigma_1(\mathbf{I}-\mathcal{L}) > \|\mathcal{Z}\|$.*

Here, we omit a formal proof. A construction of a graph satisfying the first inequality can be found in the proof for Theorem 6 whose eigenvalues can be bounded using Theorems 2 and 3. The second inequality is satisfied by any non-bipartite strongly connected undirected graph.

Lastly, we provide the following inequality:

Lemma 3.

$$\sigma_{n-2}(\mathbf{I}-\mathcal{P}) \le \lambda_1 + \|\mathcal{Z}\|$$

Proof.

$$\begin{aligned}
\sigma_{n-2}(\mathbf{I}-\mathcal{P}) &= \inf_{\mathbf{x}\perp\phi^{1/2}} \frac{\|(\mathbf{I}-\mathcal{P})\mathbf{x}\|}{\|x\|} \\
&= \inf_{\mathbf{x}\perp\phi^{1/2}} \frac{\|(\mathcal{L}-\mathcal{Z})\mathbf{x}\|}{\|\mathbf{x}\|} \\
&\le \inf_{\mathbf{x}\perp\phi^{1/2}} \frac{\|\mathcal{L}x\|}{\|\mathbf{x}\|} + \sup_{\mathbf{x}\perp\phi^{1/2}} \frac{\|\mathcal{Z}\mathbf{x}\|}{\|\mathbf{x}\|} \\
&= \lambda_1 + \|\mathcal{Z}\|
\end{aligned}$$

□

4. The discrepancy between a set and its complement and applications of $\mathcal{L}$

Recall that for a given subset S of vertices in G, the *flow* leaving S is

$$f_\phi(S,\overline{S}) = \sum_{u\in S, v\notin S} f_\phi(u,v).$$

Let χ_S denote the characteristic function with $\chi_S(v) = 1$ if v is in S and 0 otherwise. We first state a useful fact concening $f_\phi(S,\overline{S})$.

Lemma 4. *In a directed graph G, for any subset S of vertices we have*

$$f_\phi(S,\overline{S}) = f_\phi(\overline{S},S).$$

Proof. Let $\mathbf{1}$ denote the all 1's vector. We have

$$\begin{aligned}
f_\phi(S,\overline{S}) &= \langle \chi_S, \Phi\mathbf{P}\chi_{\overline{S}}\rangle \\
&= \langle \chi_S, \Phi\mathbf{P}(\mathbf{1}-\chi_S)\rangle \\
&= \langle \chi_S, \phi\rangle - \langle \chi_S, \Phi\mathbf{P}\chi_S\rangle
\end{aligned}$$

since $P\mathbf{1} = \mathbf{1}$. On the other hand, we have

$$\begin{aligned}
f_\phi(\overline{S},S) &= \langle (\mathbf{1}-\chi_S), \Phi\mathbf{P}\chi_S\rangle \\
&= \langle \chi_S, \phi\rangle - \langle \chi_S\Phi\mathbf{P}\chi_S\rangle.
\end{aligned}$$

□

In the case of undirected graphs, we have a discrepancy inequality from [7]:

Lemma 5. *[7] Let $G^* = (V^*, E^*)$ be a weighted undirected graph. Let $w(u,v)$ denote the weight of the edge $\{u,v\}$, $E^*(S,T)$ denote the sum of edge weights between S and T, and $\mathrm{vol}(S) = E^*(S, G^*)$. Then, for any subsets $S, T \subset V^*$,*

$$\left| E^*(S,T) - \frac{\mathrm{vol}(S)\mathrm{vol}(T)}{\mathrm{vol}(G^*)} \right| \leq \overline{\lambda} \sqrt{\mathrm{vol}(S)\mathrm{vol}(\overline{S})\mathrm{vol}(T)\mathrm{vol}(\overline{T})}.$$

where $\overline{\lambda}$ is the spectral gap of the weighted normalized Laplacian matrix of G^*

We use this lemma in the case of directed graphs for the following:

Lemma 6. *Let G be a strongly connected, aperiodic directed graph with a unique stationary distribution ϕ. Let $\phi(S)$ denote the stationary distribution on S, $f_\phi(S, \overline{S})$ denote the flow from S to its complement under ϕ. Then, for any subset S of vertices, we have*

$$\left| f_\phi(S, \overline{S}) - \phi(S)\phi(\overline{S}) \right| \leq \overline{\lambda}\ \phi(S)\phi(\overline{S})$$

where $\overline{\lambda} = \max_{i \neq 0} |1 - \lambda_i|$ and $\lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_{n-1}$ are eigenvalues of the Laplacian $\mathcal{L}$ of G.

Proof. For a given directed graph G, we define an associated undirected weighted graph G^* with edge weight $w(u,v)$ defined as follows:

$$w(u,v) = \frac{\phi(u)P(u,v) + \phi(v)P(v,u)}{2}.$$

Observe that in G^*, the weighted degree w_v of a vertex v is just $\phi(v)$; therefore, for any set S, $\mathrm{vol}(S) = \phi(S)$. Hence, $\mathcal{L}(G) = \mathcal{L}(G^*)$ where $\mathcal{L}(G^*)$ is the normalized Laplacian of G^* as defined in [8]. As a consequence of Lemma 4, we have

$$f_\phi(S, \overline{S}) = \frac{f_\phi(S, \overline{S}) + f_\phi(\overline{S}, S)}{2} = E^*(S, \overline{S}).$$

As a result, we can apply S into both sets in the statement of Lemma 5. □

Hence, analogous to $\mathrm{disc}(G)$ we define $\mathrm{disc}^*(G)$ to be the discrepancy of G^* as in the proof above. Specifically, $\mathrm{disc}^*(G)$ is the smallest α such that for all subsets $S, T \subset V(G)$,

$$\left| \frac{f_\phi(S,T) + f_\phi(T,S)}{2} - \phi(S)\phi(T) \right| \leq \alpha \sqrt{\phi(S)\phi(T)}$$

A simple application of the definition of $\mathrm{disc}(G)$ and the triangle inequality gives the following fact:

Fact 3.

$$\mathrm{disc}^*(G) \leq \mathrm{disc}(G)$$

Lemma 7. *(based upon [4]) Let* $\overline{\lambda} = \max_{i \neq 0} |1 - \lambda_i|$ *where* λ_i *are the non-zero eigenvalues of* $\mathcal{L}$*. Then, for* $\mathrm{disc}^*(G) < 1/4$*,*

$$\overline{\lambda} \leq -170 \ \mathrm{disc}^*(G) \log \left(\mathrm{disc}^*(G) \right).$$

A proof of this fact follows from the techniques in [4] which we use in Theorem A. The constant 170 follows similarly.

In addition, we can use the usual Cheeger's inequality for undirected graphs to give bounds for $f_\phi(S, \overline{S})$ as well. For a subset S of vertices, we denote by $h^*(S)$ the *Cheeger ratio* of S defined by

$$h^*(S) = \frac{f_\phi(S, \overline{S})}{\min\{\phi(S), \phi(\overline{S})\}}.$$

The *Cheeger constant* of a graph, $h^*(G)$, is defined by

$$h^*_G = \min_S h^*(S).$$

Lemma 8. *[7] [8] Let* G *be a directed graph with a unique stationary distribution,* ϕ*. Let* $\phi(S)$ *denote the stationary distribution on* S*,* $f_\phi(S, \overline{S})$ *denote the flow from* S *to its complement under* ϕ*. Then, for any subset* S *of vertices, we have*

$$\frac{\lambda_1}{2} \leq h^*_G \leq f_\phi(S, \overline{S})$$

where h^* *denotes the Cheeger constant of* G^**.*

For a proof see [7] or [8].

In [12], Li and Zhang gave the following lower bound for the Cheeger constant as follows:

$$\sigma_{n-2}(\mathbf{I} - \mathcal{P}) \leq 2h^*_G \Big(1 + \frac{\|\mathcal{Z}\|}{\lambda_1} \Big)$$

where $\sigma_{n-2}(\mathbf{I} - \mathcal{P})$ is the smallest non-zero singular value of $\mathbf{I} - \mathcal{P}$.

Here, we give an improvement without using λ_1.

Lemma 9.

$$\sigma_{n-2}(\mathbf{I} - \mathcal{P}) \leq 2h^*_G \Big(1 + \frac{\|\mathcal{Z}\|}{2h^*_G} \Big)$$

Proof. By Lemma 3, we have $\sigma_{n-2}(\mathbf{I} - \mathcal{P}) \leq \lambda_1 + \|\mathcal{Z}\|$. By applying Chung's bound of $\lambda_1 \leq 2h^*_G$ in Lemma 8, we have $\sigma_{n-2}(\mathbf{I} - \mathcal{P}) \leq 2h^*_G + \|\mathcal{Z}\|$, or equivalently, $\sigma_{n-2}(\mathbf{I} - \mathcal{P}) \leq 2h^*_G(1 + \frac{\|\mathcal{Z}\|}{2h^*_G})$.

We remark that this is an improvement in two ways. First, it provides an inequality with fewer parameters, but secondly, since $\lambda_1 \leq 2h_G$, the second term in Theorem 9 is strictly less than that in [12].

5. Bounds for disc(S,T) and disc(G)

In this section, we present bounds for discrepancy, $\text{disc}(S,T)$ and $\text{disc}(G)$, in terms of the singular values $\sigma_1(\mathcal{P})$.

An upper bound for $\text{disc}(S,T)$ in terms of $\sigma_1(\mathcal{P})$ is a natural extension of the case for undirected graphs under the normalized Laplacian (see [7]).

However, the lower bound has a rich history. In the case of undirected graphs, Chung in [7] asked whether the discrepancy could be bounded from below using the spectral gap. This question has been answered on several fronts. Bilu and Linial [4] showed that for d-regular graphs, $\lambda_2 \leq c_1 \,\text{disc}(G)(1+\log(d/\text{disc}(G)))$ for some constant c_1. Bollobás and Nikiforov [5] extended this result for non-regular graphs. This technique has been used by Butler [6] to show that for directed graphs the discrepancy (when measured by in- and out-degrees) is bounded below by the second largest singular value of $\mathbf{D}_{out}^{-1/2}\mathbf{A}\mathbf{D}_{in}^{-1/2}$ to within a logarithmic factor where $\mathbf{D}_{out}$ and $\mathbf{D}_{in}$ are the diagonal degree matrices and $\mathbf{A}$ is the adjacency matrix.

First, we have the following fact analogous to Fact 2 which bounds $\text{disc}(G)$ above and below:

Fact 4.

(i) $\text{disc}^*(G) \leq \text{disc}(G)$

(ii) $\text{disc}'(G) \leq \text{disc}(G)$

(iii) $\text{disc}(G) \leq \text{disc}^*(G) + \text{disc}'(G)$

The proofs of these items are applications of the triangle inequality and the definitions of the various forms of discrepancy.

We now present similar upper and lower bounds for $\text{disc}(S,T)$ and $\text{disc}(G)$ in terms of $\sigma_1(\mathcal{P})$.

Theorem 2. *Let G be a directed graph whose associated random walk has a unique stationary distribution, ϕ. Let $f_\phi(S,T)$ denote the flow from S to T under ϕ. Then, for any subset of vertices $S \subset V(G)$, we have*

$$| \, f_\phi(S,T) - \phi(S)\phi(T) \, | \leq \sigma_1(\mathcal{P})\sqrt{\phi(S)\phi(\overline{S})\phi(T)\phi(\overline{T})}$$

where $\sigma_1(\mathcal{P})$ is the second largest singular value of $\mathcal{P}$, the normalized transition probability matrix of G. Therefore, we have

$$\mathrm{disc}(G) \leq \sigma_1(\mathcal{P}).$$

In the other direction, we have for $\mathrm{disc}(G) < 1/4$

$$\sigma_1(\mathcal{P}) \leq -340\ \mathrm{disc}(G) \log \mathrm{disc}(G).$$

Proof. The first part of the theorem follows from the fact that

$$\begin{aligned} & \mid f_\phi(S,T) - \phi(S)\phi(T) \mid \\ = & \mid \langle \chi_S, \Phi \mathbf{P} \chi_T \rangle - \phi(S)\phi(T) \mid \\ = & \mid \langle \Phi^{1/2}\chi_S, (\Phi^{1/2}\mathbf{P}\Phi^{-1/2})\Phi^{1/2}\chi_T \rangle - \phi(S)\phi(T) \mid \\ = & \mid \langle \Phi^{1/2}\chi_S, (\mathcal{P} - \mathbf{P}_0)\Phi^{1/2}\chi_T \rangle \mid \end{aligned}$$

where $\mathbf{P}_0 = (\Phi^{1/2}\mathbf{1})(\mathbf{1}^*\Phi^{1/2})$ is the projection to the (left and right) eigenspace generated by the unit eigenvector $\Phi^{1/2}\mathbf{1}$ associated with eigenvalue 1 of $\mathcal{P}$. Then, we have

$$\begin{aligned} & \mid f_\phi(S,T) - \phi(S)\phi(T) \mid \\ \leq & \|\mathcal{P} - \mathbf{P}_0\| \cdot \|\Phi^{1/2}\chi_S\| \cdot \|\Phi^{1/2}\chi_T\| \\ \leq & \sigma_1(\mathcal{P})\phi(S)\phi(T). \end{aligned}$$

In addition, notice that

$$\begin{aligned} & \mid f_\phi(S,T) - \phi(S)\phi(T) \mid \\ = & \mid \phi(S) - f_\phi(S,\bar{T}) - \phi(S)\phi(T) \mid \\ = & \mid \phi(S) - \phi(S)\phi(T) - f_\phi(S,\bar{T}) \mid \\ = & \mid \phi(S)(1 - \phi(T)) - f_\phi(S,\bar{T}) \mid \\ = & \mid \phi(S)\phi(\bar{T}) - f_\phi(S,\bar{T}) \mid \\ = & \mid f_\phi(S,\bar{T}) - \phi(S)\phi(\bar{T}) \mid . \end{aligned}$$

By reiterating the same steps for S and $\bar{S}$, we have:

$$\mid f_\phi(S,T) - \phi(S)\phi(T) \mid = \mid f_\phi(\bar{S},\bar{T}) - \phi(\bar{S})\phi(\bar{T}) \mid .$$

Hence, altogether:

$$\begin{aligned} & \mid f_\phi(S,T) - \phi(S)\phi(T) \mid \\ = & \sqrt{\mid f_\phi(S,T) - \phi(S)\phi(T) \mid \quad \mid f_\phi(\bar{S},\bar{T}) - \phi(\bar{S})\phi(\bar{T}) \mid} \\ \leq & \sigma_1(\mathcal{P})\sqrt{\phi(S)\phi(\bar{S})\phi(T)\phi(\bar{T})}. \end{aligned}$$

For the remaining portion, observe that by Fact 2 (iii) either $\sigma_1(\mathcal{P}) \leq 2\overline{\lambda}$ or $\sigma_1(\mathcal{P}) \leq 2\|\mathcal{Z}\|$. Hence by Fact 2 and Theorem 3 we have either $\sigma_1(\mathcal{P}) \leq -340 \text{ disc}'(G)\log(\text{disc}'(G))$ or $\sigma_1(\mathcal{P}) \leq -340 \text{ disc}^*(G)\log(\text{disc}^*(G))$. Finally, since $x \log x$ is monotonically increasing on $(0, 1/4)$ and $\text{disc}^*(G) \leq \text{disc}(G)$ and $\text{disc}'(G) \leq \text{disc}(G)$ by Lemma 4 above, we have $\sigma_1(\mathcal{P}) \leq -340 \,\text{disc}(G)\log(\text{disc}(G))$ which completes the proof. □

6. Bounds for $\text{disc}'(A, B)$ and $\text{disc}'(G)$

In this section, we give bounds for $\text{disc}'(A, B)$ and $\text{disc}'(G)$ in terms of $\|\mathcal{Z}\|$. Before we continue, we will establish basic facts regarding $\text{disc}'(A, B)$, $\text{disc}'(G)$, and $\mathcal{Z}$ which we will use to establish our results.

Fact 5.

- *For any two subsets $A, B \subset V$,*

$$\text{disc}'(A, B) = \left| \frac{f_\phi(A, B) - f_\phi(B, A)}{2} \right| = \text{disc}'(B, A).$$

- *For any directed graph G,*

$$\text{disc}'(G) \leq \text{disc}(G)$$

where the second item is an application of the triangle inequality.

Theorem 3. *Let G be a directed graph with a unique stationary distribution, ϕ. Let $f_\phi(S, T)$ denote the flow between S and T under ϕ. Then for any two subsets of vertices A, B,*

$$\left| \frac{f_\phi(A, B) - f_\phi(B, A)}{2} \right| \leq \|\mathcal{Z}\| \sqrt{\phi(A)\phi(B)\phi(\overline{A})\phi(\overline{B})}$$

where $\mathcal{Z}$ is the asymmetric part of $\mathcal{P}$. Therefore, we have

$$\text{disc}'(G) \leq \|\mathcal{Z}\|.$$

In the other direction, we have for $\text{disc}'(G) < 1/4$

$$\|\mathcal{Z}\| \leq -170 \text{ disc}'(G)\log(\text{disc}'(G)). \tag{1}$$

Proof. Observe that

$$\begin{aligned}
\frac{f_\phi(A, B) - f_\phi(B, A)}{2} &= (\Phi^{1/2}\chi_A)^*(\frac{\mathcal{P} - \mathcal{P}^*}{2})(\Phi^{1/2}\chi_B) \\
&= (\Phi^{1/2}\chi_A)^*\mathcal{Z}(\Phi^{1/2}\chi_B).
\end{aligned}$$

Since $\Phi^{1/2}\mathbf{1}$ is a left and right eigenvector of $\mathcal{Z}$ with eigenvalue 0, we have:

$$\begin{aligned}\left|\frac{f_\phi(A,B)-f_\phi(B,A)}{2}\right| &= |\,(\Phi^{1/2}\chi_A)^*\mathcal{Z}(\Phi^{1/2}\chi_B)\,| \\ &= |\,(\Phi^{1/2}\chi_A-\phi(A)\Phi^{1/2}\mathbf{1})^*\mathcal{Z}(\Phi^{1/2}\chi_B-\phi(B)\Phi^{1/2}\mathbf{1})\,| \\ &\leq \|\mathcal{Z}\|\sqrt{\phi(A)\phi(\overline{A})\phi(B)\phi(\overline{B})}.\end{aligned}$$

To prove (1), we need the following fact:

Theorem A: Let $\mathbf{M}$ be a $n\times n$ nonnegative matrix such that there exists a unique diagonal matrix $\mathbf{B}$ with non-negative entries with $\mathbf{M1}=\mathbf{B1}$, $\mathbf{1}^*\mathbf{M}=\mathbf{1}^*\mathbf{B}$, and $\|\mathbf{B}\|\leq 1$. Suppose there is an $\alpha<1/4$ such that for all subsets $S,T\subset\{1,2\dots,n\}$

$$|\langle\chi_S,(\mathbf{M}-\mathbf{M}^*)\chi_T\rangle|\leq\alpha\sqrt{\langle\chi_S,\mathbf{M1}\rangle\langle\mathbf{1},\mathbf{M}\chi_T\rangle}$$

where χ_S and χ_T are the indicating vectors on the sets S and T respectively.

Then,

$$\|\mathbf{B}^{-1/2}(\mathbf{M}-\mathbf{M}^*)\mathbf{B}^{-1/2}\|\leq -340\alpha\log\alpha$$

The proof of Theorem A follows in a similar fashion to [4] and [6] and is relegated to the appendix.

The proof of Theorem 3 is completed by applying $\mathbf{M}=\mathbf{\Phi}P$ and $\mathbf{B}=\mathbf{\Phi}$. □

With regards to Theorem A, we emphasize that not every matrix M has such a B, and further, this additional condition is imperative for the result to hold.

7. Non-normalized bounds for $\mathrm{disc}'(S,T)$

By optimizing $\sqrt{\phi(S)\phi(\overline{S})\phi(T)\phi(\overline{T})}$ from Theorem 3 to be $\frac{1}{4}$ one might suspect that $\mathrm{disc}'(S,T)\leq\frac{1}{4}\|\mathcal{Z}\|$ is the optimal non-normalized bound for the skew-discrepancy between two subsets. We show for two disjoint subsets that the $\frac{1}{4}$ factor can be improved to $\frac{1}{\sqrt{27}}$. We then continue to show that this is the best bound.

Before we continue, we need a lemma:

Lemma 10. *Let G be a strongly-connected aperiodic directed graph. Let A,B,C be three sets which partition the vertices. Then, $f_\phi(A,B)-f_\phi(B,A)=f_\phi(B,C)-f_\phi(C,B)=f_\phi(C,A)-f_\phi(A,C)$.*

Proof. We begin by applying Lemma 4.

$$\begin{aligned} f_\phi(A,\overline{A}) &= f_\phi(\overline{A},A) \\ f_\phi(A,B\cup C) &= f_\phi(B\cup C,A) \\ f_\phi(A,B)+f_\phi(A,C) &= f_\phi(B,A)+f_\phi(C,A) \\ f_\phi(A,B)-f_\phi(B,A) &= f_\phi(C,A)-f_\phi(A,C). \end{aligned}$$

The remaining equality results from permuting the roles of A, B and C. □

Theorem 4. *Let G be a strongly connected, aperiodic directed graph whose associated random walk has a unique stationary distribution, ϕ. Then, for any two disjoint subsets of vertices S, T,*

$$\mathrm{disc}'(S,T) \le \frac{1}{\sqrt{27}}\|\mathcal{Z}\|$$

where $\|\mathcal{Z}\|$ is the norm of the skew-symmetric part of $\mathcal{P}$.

Proof. Let C denote the complement of $A \cup B$.

Set $g(x) = (\chi_A + \omega\chi_B + \omega^2\chi_C)\Phi^{1/2}$ where χ denotes the characteristic function and $\omega = e^{2\pi i/3}$. We then apply Lemma 2 to $\mathbf{g}$:

$$\begin{aligned} \|\mathcal{Z}\| &\ge Im(\langle \mathbf{g}, \Phi\mathbf{P}\mathbf{g}\rangle) \\ &= Im\big[f_\phi(A,A)+f_\phi(B,B)+f_\phi(C,C) \\ &\quad +\omega(f_\phi(A,B)+f_\phi(B,C)+f_\phi(C,A)) \\ &\quad +\omega^2(f_\phi(A,C)+f_\phi(C,B)+f_\phi(B,A))\big] \\ &= \frac{\sqrt{3}}{2}\big[f_\phi(A,B)+f_\phi(B,C)+f_\phi(C,A) \\ &\quad -f_\phi(B,A)-f_\phi(C,B)-f_\phi(A,C)\big]. \end{aligned}$$

By applying Lemma 10 to the right-hand side, we have

$$\|\mathcal{Z}\| \ge \sqrt{27}\ \frac{f_\phi(A,B)-f_\phi(B,A)}{2} = \sqrt{27}\ \mathrm{disc}'(A,B).$$

By solving for $\mathrm{disc}'(A,B)$, we have

$$\mathrm{disc}'(A,B) \le \frac{1}{\sqrt{27}}\|\mathcal{Z}\|.$$

□

Now that we have proven upper bounds for $\text{disc}'(S,T)$ when S and T are disjoint, one might want to know how good these bounds are. We now construct a sequence of graphs to show that these bounds are asymptotically sharp, and hence, these are the best bounds possible.

Theorem 5. *The bound in Theorem 4 is asymptotically sharp. That is, there exists a sequence of directed graphs G_n each with a stationary distribution which has subsets A_n and B_n such that as $n \to \infty$, $\text{disc}'(A_n, B_n) - \frac{1}{\sqrt{27}}\|\mathcal{Z}(G_n)\| \to 0$.*

Ideally, we would like to choose $\overrightarrow{C}_3$, the directed cycle on 3 vertices. However, $\overrightarrow{C}_3$ does not have a unique stationary distribution, and hence, cannot be considered. Instead, we construct a sequence of graphs with three parts that approaches a 3 cycle.

Proof. Let G_n be a graph on $3n$ vertices and whose vertices are partitioned into 3 equal parts A_n, B_n, C_n such that every vertex in A_n has an arc to every vertex in B_n; and likewise, every vertex in B_n to every vertex in C_n; and also C_n to A_n. Finally, choose one arc from $a \in A_n$ to $c \in C_n$ and add the arc in the reverse direction to make the associated random walk on the graph ergodic.

By symmetry, since all of the vertices in $A_n \backslash \{a\}$ have the same neighborhood, they have the same value for ϕ. The same applies for $C_n \backslash \{c\}$. Therefore, we can calculate the stationary distribution by considering the quotient walk given by combining all of the vertices in $A_n \backslash \{a\}$ into one vertex, as well as all the vertices $C_n \backslash \{c\}$ and also B_n.

A straightforward matrix calculation has that $\phi(u) = \frac{1}{3n} + o(n^{-1})$ for any $u \in V(G)$. Hence, we see that $\text{disc}'(A_n, B_n) \to \frac{1}{6}$ as $n \to \infty$. Also, we have

$$\begin{aligned} 2\|\mathcal{Z}(G_n)\| &= \|\Phi^{1/2}\mathbf{P}\Phi^{-1/2} - \Phi^{-1/2}\mathbf{P}^*\Phi^{1/2}\| \\ &= \left(\sqrt{3n} + o(n^{1/2})\right)\left(\sqrt{\frac{1}{3n}} + o(n^{-1/2})\right)\|\mathbf{P} - \mathbf{P}^*\|. \end{aligned}$$

Since G is nearly an n-edge cover of the the directed 3-cycle except for a change made by a matrix with spectral norm of at most $2/n$, so $\|P - P^*\| = \sqrt{3} + o(1)$. Hence, we have

$$\|\mathcal{Z}(G_n)\| = \frac{\sqrt{3}}{2}(1 + o(1)).$$

Altogether, as $n \to \infty$, $\|\mathcal{Z}(G_n)\| \to \frac{\sqrt{3}}{2}$ and $\text{disc}'(A_n, B_n) \to \frac{1}{6}$. ∎

We remark that the above construction of an "almost cycle" yields a counterexample to Theorem 4 regarding the "digraph gap" in [12].

8. Constructions and Applications

In this section, we provide constructions to illustrate applications of the results in the previous sections.

Thus far we have shown that various types of discrepancies are controlled by corresponding eigenvalues and spectral norms. However, one may ask how these different variations of discrepancies are related to one another. We have shown in Lemma 2, that $\mathrm{disc}'(G) \le \mathrm{disc}(G)$. However, it is natural to ask whether or not $\mathrm{disc}'(G)$ is controlled, at all, by the discrepancy of the underlying graph. In the following constructions we show that $\mathrm{disc}'(G)$ is not related to the discrepancy of the underlying graph.

Theorem 6. *There exists a sequence of directed graphs G_n on n vertices with underlying graphs G_n^* such that the ratio $\frac{\mathrm{disc}'(G_n)}{\mathrm{disc}(G_n^*)}$ goes to infinity when n increases.*

Proof. Let us only consider odd $n \ge 5$. Let G_n be a tournament on n vertices indexed by the elements of $\mathbb{Z}_n^+, \{0, \ldots n-1\}$ with $(u, v) \in E(G)$ if and only if the representative of $u - v$ in $\{0, \ldots n-1\}$ in less than $n/2$. This creates a tournament as the representatives of $u - v$ and $v - u$ must add in $\mathbb{Z}$ to n, hence exactly one of them must be less than $n/2$.

By symmetry, the stationary distribution $\phi(u) = 1/n$ for all vertices u in both G_n and G_n^*

First, we compute an upper bound for $\mathrm{disc}(G_n')$. Since $G_n^* = K_n$, the complete graph on n vertices and the eigenvalues for $\mathcal{P}(K_n) = \mathbf{I} - \mathcal{L}(K_n)$ are 1 with multiplicity 1 and $\frac{-1}{k-1}$ with multiplicity $k-1$, we can apply the first inequality of Theorem 2 and conclude $\mathrm{disc}(G_n^*) \le \frac{1}{n-1}$

Next, we provide a lower bound on the skew-discrepancy of G_n. Consider the two subsets of vertices $X, Y \subset V(G)$ where $X = \{0, \ldots, n-1\} \cap [0, n/4]$ and $Y = \{0, \ldots, n-1\} \cap (n/4, n/2]$. Observe that by the construction of G all of the arcs between X and Y in fact go from X to Y and none go from Y to X. Notice that for each vertex $x \in X$, $1/2 + o(1)$ of its out-neighborhood is contained in Y. Hence $f_\phi(X, Y) = 1/8 + o(1)$ and $\mathrm{disc}'(X, Y) = 1/16 + o(1)$. Also, observe that X and Y each contain $1/4 + o(1)$ of the vertices, so $\phi(X) = 1/4 + o(1)$ and $\phi(Y) = 1/4 + o(1)$. We have that

$$\mathrm{disc}'(G_n) \ge \frac{1/16 + o(1)}{1/4 + o(1)} = 1/4 + o(1).$$

Altogether,

$$\frac{\mathrm{disc}'(G_n)}{\mathrm{disc}(G_n')} \ge \frac{n}{4}(1 + o(1))$$

□

It would seem natural that a directed graph where all of the edges are strictly one-directional would yield a high skew-discrepancy with low discrepancy. However, it turns out this is not always the case.

In our next construction, we show that a tournament (where every edge is strictly one-directional) can, in fact, have small skew-discrepancy.

In this construction, we generate a *random tournament* on n vertices, denoted by $R(n)$, under the following process:

- For every pair of vertices u and v include the arc (u, v) with probability 1/2.
- If the arc (u, v) is not included, include the arc (v, u).
- The determination regarding whether (u, v) or (v, u) is included as a arc is independent of any other pair.

While we cannot use a random tournament to construct an explicit example, we show that the graph $R(n)$ has desired properties with probability tending to 1 as n goes to infinity. When this occurs, we say that event occurs *asymptotically almost surely.*

We will apply the following lemmas to construct our example:

Lemma 11. *Fix $\rho \in (0, 1)$. Let G be a directed graph on n vertices such that the in-degree and out-degree of every vertex is $n\rho(1+o(1))$, then $\phi(u) = \frac{1}{n}(1+o(1))$ for every vertex u.*

Lemma 12. *Asymptotically almost surely, every vertex of $R(n)$ has in-degree and out-degree equal to $\frac{n}{2}(1 + o(1))$.*

Lemma 13. *Fix $\epsilon \in (0, 1)$. Then, asymptotically almost surely, for all pairs of subsets $A, B \subset V(R(n))$ such that $|A|, |B| \geq \varepsilon n$, $|E(A, B)| = \frac{1}{2}|A||B|(1 + o(1))$.*

The proofs of the these lemmas above are simple applications of the Chernoff bound and are omitted. For details, see [2, 9].

Theorem 7. *Asymptotically almost surely, a random tournament $R(n)$ has skew-discrepancy $\mathrm{disc}'(R(n)) = o(1)$.*

Proof. We provide an upper bound for $|f_\phi(A, B) - f_\phi(B, A)|$. By Lemma 11 together with Lemma 12, we have that $\phi(u) = \frac{1}{n}(1 + o(1))$ for all vertices u. Further, since asymptotically almost surely the degree of each vertex has out-degree of $\frac{n}{2}(1 + o(1))$, the flow along each arc is $\frac{2}{n^2}(1 + o(1))$. Hence, by applying Lemma 13, we have almost surely that for all sets A and B, $f_\phi(A, B) = \frac{|A||B|}{n^2}(1 + o(1))$. In particular, asymptotically almost surely,

$$
\begin{aligned}
disc'(G) &= \sup_{A,B} \frac{\left|f_\phi(A,B) - f_\phi(B,A)\right|}{\sqrt{\phi(A)\phi(B)}} \\
&\le \sup_{A,B} \frac{n|A||B|o(1)}{n^2\sqrt{|A||B|}} \\
&= \sup_{A,B} \frac{\sqrt{|A||B|}o(1)}{n} \\
&= o(1).
\end{aligned}
$$

□

Finally, by applying Theorem 3 with the previous theorem we have the following:

Corollary 1. *Asymptotically almost surely,* $\|\mathcal{Z}(R(n))\| = o(1)$.

That is, almost surely, the eigenvalues of $\mathcal{Z}$ are bounded away from 1. This type of result might ordinarily require analytical arguments in random matrices. However, using the tools as developed in Theorem 3, it comes immediately.

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Appendix A. Proof of Theorem A

Here, we give a complete proof of Theorem A.

Theorem 8 (Theorem A)**.** *Let* $\mathbf{M}$ *be a* $n \times n$ *nonnegative matrix, such that there exists a unique diagonal matrix* $\mathbf{B}$ *with* $(\mathbf{M1})^* = (\mathbf{B1})^* = \mathbf{1}^*\mathbf{M} = \mathbf{1}^*\mathbf{B}$. *with* $\|\mathbf{B}\| \leq 1$ *Define* $\Delta = \mathbf{M} - \mathbf{M}^*$ *to be twice the skew-symmetric part of* $\mathbf{M}$. *In addition, suppose there is an* $\alpha < 1/4$ *such that for all subsets* $S, T \subset \{1, 2 \ldots, n\}$

$$\left|\langle \chi_S, \Delta\chi_T \rangle\right| \leq \alpha\sqrt{(\langle \chi_S^* \mathbf{M1} \rangle)(\langle \mathbf{1}, \mathbf{M}\chi_T)}$$

where χ_S *and* χ_T *are the indicating vectors on the sets* S *and* T *respectively. Then we have,*

$$\|\mathbf{B}^{-1/2}\Delta\mathbf{B}^{-1/2}\| \leq -340\ \alpha \log \alpha$$

where $\|\mathbf{B}^{-1/2}\Delta\mathbf{B}^{-1/2}\|$ *is the largest singular value of* $\mathbf{B}^{-1/2}\Delta\mathbf{B}^{-1/2}$

We remark that the condition regarding the normalizing diagonal matrix $\mathbf{B}$ is not met by all matrices.

In order to prove Theorem A, we need the following lemma:

Lemma 14. *Let Δ be a skew-symmetric $n \times n$ real matrix and let ζ be a positive real number with $1 - 2\zeta - \zeta^2 > 0$. Let $\mathbf{x} = \mathbf{y} + i\mathbf{z}$ be a eigenvector with associated with eigenvalue $i\|\Delta\|$ such that $\mathbf{y}, \mathbf{z}$ are real vectors with $\|\mathbf{y}\| = \|\mathbf{z}\| = 1$. Suppose for some diagonal matrices $\mathbf{R}, \mathbf{S}$ satisfying $\|\mathbf{Ry}\| \leq 1$, $\|\mathbf{Sz}\| \leq 1$, $\|\mathbf{y} - \mathbf{Ry}\| \leq \zeta$, and $\|\mathbf{z} - \mathbf{Sz}\| \leq \zeta$ then $\|\Delta\| \leq \frac{1}{1-2\zeta-\zeta^2}\mathbf{y}^*\mathbf{R}\Delta\mathbf{Sz}$.*

Proof. By Lemma 2, $\|\Delta\| = \mathbf{y}^*\Delta\mathbf{z} = (\mathbf{y} - \mathbf{Ry} + \mathbf{Ry})^*\Delta(\mathbf{z} - \mathbf{Sz} + \mathbf{Sz}) = (\mathbf{y} - \mathbf{Ry})^*\Delta(\mathbf{z} - \mathbf{Sz}) + (\mathbf{Ry})^*\Delta(\mathbf{z} - \mathbf{Sz}) + (\mathbf{y} - \mathbf{Ry})^*\Delta(\mathbf{Sz}) + (\mathbf{Ry})^*\Delta(\mathbf{Sz})$. So, by applying the triangle inequality, $\frac{\|\Delta\|}{2} \leq \zeta^2\|\Delta\| + 2\zeta\|\Delta\| + \mathbf{y}^*\mathbf{R}\Delta\mathbf{Sz}$. Altogether, $\|\Delta\| \leq \frac{1}{1-2\zeta-\zeta^2} y^*\mathbf{R}\Delta\mathbf{Sz}$ □

We are ready to prove Theorem A:

Proof. For $\mathbf{M}$, let $\mathbf{B}^{1/2}(bx + i\mathbf{y})$ be an eigenvector for the skew-symmetric matrix $(\mathbf{B}^{-1/2}(\mathbf{M} - \mathbf{M}^*)\mathbf{B}^{-1/2})$ where $\mathbf{x}, \mathbf{y}$ are strictly real vectors with $\|\mathbf{B}^{1/2}\mathbf{x}\| = 1$ and $\|\mathbf{B}^{1/2}\mathbf{y}\| = 1$ by Lemma 2.

Let $\beta = 0.696574$. (This choice of β is simply made by optimizing a subsequent formula). Construct $\mathbf{x}'$ by rounding each component of $\mathbf{x}$ toward 0 to the first power of β. In particular, $\mathbf{x}' = \sum_s \mathbf{x}^{(s)}\beta^s$ where $\mathbf{x}^{(s)}$ is a vector whose components are either -1, 0, or 1. Further, each component is non-zero for at most one $\mathbf{x}^{(s)}$. Repeat the process for $\mathbf{y}$ and $\mathbf{y}'$.

We now consider

$$\left|\mathbf{x}'^*\mathbf{B}^{1/2}(\mathbf{B}^{-1/2}(\mathbf{M} - \mathbf{M}^*)\mathbf{B}^{-1/2})\mathbf{B}^{1/2}\mathbf{y}'| = |\mathbf{x}^*(\mathbf{M} - \mathbf{M}^*)\mathbf{y}'\right|$$

$$= \left|\sum_s \sum_t (\mathbf{x}^{(s)})^*(\mathbf{M} - \mathbf{M}^*)(\mathbf{y}^{(t)})\beta^{s+t}\right|$$

By the triangle inequality:

$$\leq \sum_s \sum_t \left|(\mathbf{x}^{(s)})^*(\mathbf{M} - \mathbf{M}^*)\mathbf{y}^{(t)}\beta^{s+t}\right|$$

Now, we will divide the sum into three parts, one where s is sufficiently greater then t, one where they are roughly equal, and the last one where t is sufficiently greater then s:

$$\begin{aligned} = & \sum_{s,t,s>t+\tau} \left|(\mathbf{x}^{(s)})^*(\mathbf{M}-\mathbf{M}^*)\mathbf{y}^{(t)}\beta^{s+t}\right| \\ + & \sum_{s,t,|s-t|\leq\tau} \left|(\mathbf{x}^{(s)})^*(\mathbf{M}-\mathbf{M}^*)\mathbf{y}^{(t)}\beta^{s+t}\right| \\ + & \sum_{s,t,t>s+\tau} \left|(\mathbf{x}^{(s)})^*(\mathbf{M}-\mathbf{M}^*)\mathbf{y}^{(t)}\beta^{s+t}\right| \end{aligned}$$

Where

$$\tau = \frac{-\log\beta - 2W\left(\frac{\log\beta}{2\alpha\sqrt{\beta}}\right)}{2\log\beta}$$

and $W(\cdot)$ is the Lambert Product-Log function (i.e., $z = W(z)\exp[W(z)]$).

To bound the middle term, we apply the orthogonality of the $\mathbf{x}^{(s)}$ and $\mathbf{y}^{(t)}$. After which, we have,

$$\sum_{s,t,|s-t|\leq\tau} \left|(\mathbf{x}^{(s)})^*(\mathbf{M}-\mathbf{M}^*)\mathbf{y}^{(t)}\beta^{s+t}\right|$$

$$\leq \sum_{s,t,|s-t|\leq\tau} 4\alpha\sqrt{|(\mathbf{x}^{(s)})|^*\mathbf{M}\mathbf{1})(\mathbf{1}^*\mathbf{M}|\mathbf{y}^{(t)}|)}\beta^{s+t}$$

Where $|\cdot|$ represents the absolute value entry-wise.

By the arithmeitc-geometric mean inequality,

$$\leq \sum_{s,t,|s-t|\leq\tau} 2\alpha\left(|\mathbf{x}^{(s)}|^*\mathbf{M}\mathbf{1}\right) + \left(\mathbf{1}^*\mathbf{M}|\mathbf{y}^{(t)}|\right)\beta^{s+t}$$

By considering the highest factor of β, and independence of sums,

$$\leq 2(2\tau+1)\alpha\left(\sum_s \left|\mathbf{x}^{(s)}\right|^* \mathbf{M}\mathbf{1}\beta^{2s}\right) + \left(\sum_t \mathbf{1}^*\mathbf{M}\left|\mathbf{y}^{(t)}\right|\ \beta^{2t}\right)$$

However, by the choice of $\mathbf{x}'$, $\sum_s |\mathbf{x}^{(s)}|^*\mathbf{M}\mathbf{1}\beta^{2s} \leq \|\mathbf{B}^{1/2}\|_2^2 \leq 1$

Hence, altogether, the middle term is bounded by $2\alpha(2\tau+1)$.

As for the two other sums, we have

$$\sum_{s,t,s>t+\tau} \left|\mathbf{x}^{(s)H}(\mathbf{M}-\mathbf{M}^*)\mathbf{y}^{(t)}\beta^{s+t}\right|$$

$$\sum_{s,t,t>s+\tau} \left| \mathbf{x}^{(s)H} (\mathbf{M} - \mathbf{M}^*) \mathbf{y}^{(t)} \beta^{s+t} \right|$$

Without loss of generality, we will bound the first sum. The second sum is bounded similarly.

Applying the triangle inequality along with the non-negativity of $\mathbf{M}$ and $\mathbf{B}$ (for emphasis, this is where the non-negativity of $\mathbf{M}$ is essential):

$$\leq \sum_{s,t,s>t+\tau} \left| \mathbf{x}^{(s)} \right|^* (\mathbf{M} + \mathbf{M}^*) \left| \mathbf{y}^{(t)} \right| \ \beta^{s+t}$$

Replacing β^s with the highest possible exponent $\beta^{t+\tau}$

$$\leq \sum_{s,t} \left| \mathbf{x}^{(s)} \right|^* (\mathbf{M} + \mathbf{M}^*) \left| \mathbf{y}^{(t)} \right| \ \beta^{2t+\tau}$$

By independence of sums,

$$\leq \sum_{s} \left| \mathbf{x}^{(s)} \right|^* \sum_{t} (\mathbf{M} + \mathbf{M}^*) |\mathbf{y}^{(t)}| \ \beta^{2t+\tau}$$

By definition of $|\mathbf{x}'(s)|$, $\sum_s |\mathbf{x}^{(s)}| = \mathbf{1}$, so we have,

$$\leq \sum_{t} \mathbf{1}^* (\mathbf{M} + \mathbf{M}^*) |\mathbf{y}^{(t)}| \ \beta^{2t+\tau}$$

Since $(\mathbf{1}^*\mathbf{B})^* = (\mathbf{1}^*\mathbf{M})^* = \mathbf{M}\mathbf{1} = \mathbf{B}\mathbf{1}$

$$\leq \mathbf{1}^* \left(\sum_{t} 2\mathbf{B} |\mathbf{y}^{(t)}| \beta^{2t} \right) \beta^{\tau}$$

And finally, since $\sum_t \mathbf{B}|\mathbf{y}^{(t)}|\beta^{2t} = \mathbf{B}|\mathbf{y}'|^2 \leq 1$, we have

$$\leq 2\|\mathbf{B}\|^2 \beta^{\tau} \leq 2\beta^{\tau}$$

Hence, the sum of the three sums is bounded by: $2\alpha(2\tau + 1) + 4\beta^{\tau}$.

By choice of τ these two terms are equal. Hence, $\mathbf{x}'^* \mathbf{B}^{1/2} (\mathbf{B}^{-1/2} (\mathbf{M} - \mathbf{M}^*) \mathbf{B}^{-1/2}) \mathbf{B}^{1/2} \mathbf{y}' \leq 8\alpha(2\tau + 1)$. So by Lemma 14,

$$
\begin{aligned}
\|\mathbf{B}^{-1/2}\mathbf{\Delta}\mathbf{B}^{-1/2}\| &\leq \frac{-16\alpha W\left(-\frac{\log\beta}{4\alpha\sqrt{\beta}}\right)}{[1-2(1-\beta)-(1-\beta)^2]\log\beta} \\
&\leq \frac{16\sqrt{\beta}\alpha\log\alpha}{[1-2(1-\beta)-(1-\beta)^2](\log\beta)^2}
\end{aligned}
$$

where the last inequality follows by comparing $-\alpha W\left(-\frac{\log\beta}{4\alpha\sqrt{\beta}}\right)$ against $-\alpha\log\alpha$ for $\alpha \leq 1/4$

By choice of $\beta = 0.696574$, an evaluation yields

$$
\|(\mathbf{B}^{-1/2}(\mathbf{M}-\mathbf{M}^*)\mathbf{B}^{-1/2})\| \leq -340\ \alpha\log\alpha.
$$

□