Inversion-descent polynomials for restricted permutations

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Abstract

We derive generating functions for a variety of distributions of joint permutation statistics all of which involve a bound on the *maximum drop size* of a permutation π, i.e., $\max\{i-\pi(i)\}$. Our main result treats the case for the joint distribution of the number of inversions, the number of descents and the maximum drop size of permutations on $[n] = \{1, 2, \ldots, n\}$. A special case of this (ignoring the number of inversions) connects with earlier work of Claesson, Dukes and the authors on descent polynomials for permutations with bounded drop size. In that paper, the desired numbers of permutations were given by sampling the coefficients of certain polynomials Q_k. We find a natural interpretation of *all* the coefficients of the Q_k in terms of a restricted version of Eulerian numbers.

1 Introduction

There is an extensive literature on various statistics for S_n, the set of all permutations of $\{1, 2, \ldots, n\}$ (e.g., see [1, 3, 4, 7, 8, 12, 13, 14, 16, 17, 18, 19, 20]). For a permutation π in S_n, we say that π has a *drop* at i if $\pi(i) < i$, and the *drop size* is $i - \pi(i)$. We say that π has a *descent* at i if $\pi(i+1) < \pi(i)$. One of the earliest results [12] in permutation statistics asserts that the number of permutations in S_n with k drops equals the number of permutations with k descents. Other statistics for a permutation π include the number of inversions of π (i.e., $|\{(i,j) : i < j, \pi(i) > \pi(j)\}|$), and the major index of π (i.e, the sum of the indices i at which a descent of π occurs). Many of these papers study the distribution of the above statistics and their q-analogs as well as the distribution of various multivariate statistics.

In this paper, we examine joint statistics of permutations with the additional constraint on the maximum drop size. Enumeration problems of permutations with bounded maximum drop size arise in the study of juggling patterns as well as certain sorting algorithms. In [3], the descent polynomials with bounded

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maximum drop size were studied. In this paper we extend the methods to examine the joint statistics of inversions, descents and maximum drop size. The derivation of the generating functions of such combined statistics of permutations involves an interplay of q-nomial coefficients and various modified versions of Eulerian numbers.

An outline of the paper is as follows. In Section 2, we will present our main result dealing with the joint distribution of descents and inversions over permutations with bounded drop size. In Section 3, we specialize this result by ignoring inversions. This relates to earlier work of Claesson, Dukes and the authors [3] on the same subject. In Section 4, we will show how to interpret all the coefficients in the polynomials arising in [3] in terms of counting certain restricted permutations. Finally, in Section 5, we will make some general comments and suggest a number of open problems.

2 Inversions, descents and maxdrop

We begin by listing some of the standard terminology we will be using. With $[n] = \{1, 2, \ldots, n\}$, we let S_n denote the set of $n!$ permutations on $[n]$. We let $\mathrm{DES}(\pi) = \{i \in [n] : \pi \text{ has a descent at } i\}$ and we set $\mathrm{des}(\pi) = |\,\mathrm{DES}(\pi)|$. Also, we define $\mathrm{maxdrop}(\pi) = \max\{i - \pi(i)\}$. Finally, we let $\mathrm{inv}(\pi)$ denote the number of inversions of $\pi \in S_n$, i.e., $\mathrm{inv}(\pi) = |\{i < j : \pi(i) > \pi(j)\}$.

For a formal parameter q, we use the standard definitions for Gaussian coefficients:

$$\begin{aligned}
[n]_q &= 1 + \cdots + q^{n-1}, \\
[n]_q! &= [n]_q [n-1]_q \cdots [1]_q, \\
\begin{bmatrix} a \\ b \end{bmatrix}_q &= \frac{[a]_q!}{[b]_q! [a-b]_q!}, \\
\mathrm{Exp}_q(z) &= \sum_{n \geq 0} q^{\binom{n}{2}} \frac{z^n}{[n]_q!}.
\end{aligned}$$

If we define $A_n^{\mathrm{inv},\mathrm{des}}(q, y)$ by

$$A_n^{\mathrm{inv},\mathrm{des}}(q, y) = \sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} y^{\mathrm{des}(\pi)}$$

then a classic result of Stanley [17] shows that

$$\sum_{n \geq 0} A_n^{\mathrm{inv},\mathrm{des}}(q, y) \frac{z^n}{[n]_q!} = \frac{1 - y}{\mathrm{Exp}_q(z(y-1)) - y}. \tag{2.1}$$

Our first result can be thought of as a variant of (2.1) using ordinary generating functions rather than exponential generating functions where we include a restriction on the maxdrop of the permutations as well. To state it, we first need several definitions. For a power series $P(z) = \sum_{n\geq 0} p(n)z^n$, the notation $[z^{\leq t}]P(z)$ denotes the truncated sum $\sum_{n\leq t} p(n)z^n$, while $[z^{\geq t}]P(z)$ denotes the sum $\sum_{n\geq t} p(n)z^n$ and $[z^t]P(z)$ denotes the single term $p(t)z^t$.

We define

$$B_{n,k}(q,y) = \sum_{\pi\in S_{n,k}} q^{\mathrm{inv}(\pi)} y^{\mathrm{des}(\pi)}$$

where $S_{n,k} = \{\pi \in S_n : \mathrm{maxdrop}(\pi) \leq k\}$.

Theorem 2.1. *For $k \geq 1$, the generating function for $B_{n,k}$ satisfies*

$$\mathbf{B}_k = \mathbf{B}_k(q,y,z) = \sum_{n\geq 0} B_{n,k}(q,y)z^n = \frac{F_k}{G_k},$$

where

$$G_k = G_k(q,y,z) = 1 - \sum_{j=1}^{k+1} \begin{bmatrix} k+1 \\ j \end{bmatrix}_q q^{\binom{j}{2}} (y-1)^{j-1} z^j,$$

$$\mathbf{A}^{\mathrm{inv,des}} = \mathbf{A}^{\mathrm{inv,des}}(q,y,z) = \sum_{n\geq 0} A_n^{\mathrm{inv,des}}(q,y)z^n,$$

$$F_k = F_k(q,y,z) = [z^{\leq k}](\mathbf{A}^{\mathrm{inv,des}} \cdot G_k).$$

Note that $\mathbf{A}^{\mathrm{inv,des}}$ is not the usual power series of Stanley for inversions and descents.

For example, for $k = 1$, we have

$$\mathbf{B}_1(q,y,z) = \frac{1-qz}{1-(1+q)z-q(y-1)z^2}.$$

Proof. We will consider

$$\begin{aligned} B_{n,k}(q,y+1) &= \sum_{\pi\in S_{n,k}} q^{\mathrm{inv}(\pi)}(y+1)^{\mathrm{des}(\pi)} \\ &= \sum_{\pi\in S_{n,k}} \sum_{T\subseteq[n]} I(T,\mathrm{DES}(\pi)) q^{\mathrm{inv}(\pi)} y^{|T|} \end{aligned}$$

where

$$I(T,S) = \begin{cases} 1 & \text{if } T \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

For $\pi \in S_n$, define

$$t(\pi) = \max\{i : \pi \text{ has descents at } n-i, n-i+1, n-i+2, \ldots, \text{ and } n-1\},$$

and define $t(\pi) = 0$ if $\pi(n-1) < \pi(n)$. Thus, we have

$$\pi(n-t(\pi)) > \pi(n-t(\pi)+1) > \cdots > \pi(n),$$

for $t(\pi) > 0$. Now, for π with $\text{maxdrop}(\pi) \le k$, we have $\pi(n) \ge n-k$. Therefore,

$$\pi(n - t(\pi) + j) \ge n - t(\pi) + j - k \text{ for } 0 \le j \le t(\pi).$$

Hence, we can write

$$\begin{aligned} B_{n,k}(q, y+1) &= \sum_{\pi \in S_{n,k}} q^{\text{inv}(\pi)} (y+1)^{\text{des}(\pi)} \\ &= \sum_{\pi \in S_{n,k}} \sum_{S \subseteq [n]} I(S, \text{DES}(\pi)) q^{\text{inv}(\pi)} y^{|S|} \\ &= \sum_{i=1}^{k+1} \sum_{\substack{\pi \in S_{n,k} \\ t(\pi)=i-1}} \sum_{T \subseteq [n-i]} I(T, \text{DES}(\pi)) q^{\text{inv}(\pi)} y^{|T|+i-1}. \end{aligned} \tag{2.2}$$

For $\pi \in S_n$, let $\tilde{\pi}$ denote the "reduced" permutation on $[n - t(\pi) - 1]$. That is, the images $\tilde{\pi}(j), j \in [n - t(\pi) - 1]$, have the same relative order as the images $\pi(j), j \in [n - t(\pi) - 1]$, (so the number of descents and inversions of π and $\tilde{\pi}$ on this interval are the same). Note that for $\pi \in S_n$, the number inversions occurring at position i (i.e., the number of $u < i$ with $\pi(u) > \pi(i)$) is exactly $n - \pi(i) - |\{j : \pi(j) > \pi(i) \text{ for } j > i\}|$. For example, for $i = n$, the number of inversions occurring at position n is just $n - \pi(n)$.

Continuing the proof, we have

$$\begin{aligned} [z^{\ge k+1}]\mathbf{B}_k(q, y+1, z) &= \sum_{n \ge k+1} B_{n,k}(q, y+1) z^n \\ &= \sum_{n \ge k+1} \sum_{i=1}^{k+1} \sum_{\substack{\pi \in S_{n,k} \\ t(\pi)=i-1}} \sum_{T \subseteq [n-i]} I(T, \text{DES}(\tilde{\pi})) q^{\text{inv}(\pi)} y^{|T|+i-1} z^n \\ &= \sum_{n \ge k+1} \sum_{i=1}^{k+1} \left(\sum_{\tilde{\pi} \in S_{n-i,k}} \sum_{T \subseteq [n-i]} I(T, \text{DES}(\tilde{\pi})) q^{\text{inv}(\tilde{\pi})} y^{|T|} z^{n-i} \right) \\ &\qquad \times \left(\sum_{a_1 < a_2 < \ldots < a_i} q^{\sum_{j=1}^i a_j} y^{i-1} z^i \right) \end{aligned}$$

where $0 \le a_1 < a_2 < \ldots < a_i \le k$ are defined by the identification

$$\{\pi(n - t(\pi)), \pi(n - t(\pi) - 1), \ldots, \pi(n)\} = \{n - a_1, n - a_2, \ldots, n - a_i\}.$$

Thus, we have

$$\begin{aligned} &[z^{\ge k+1}]\mathbf{B}_k(q, y+1, z) \\ &= \sum_{n \ge k+1} \sum_{i=1}^{k+1} B_{n-i,k}(q, y+1) z^{n-i} \left(\sum_{a_1 < a_2 < \ldots < a_i} q^{\sum_{j=1}^i a_j} y^{i-1} z^i \right) \\ &= [z^{\ge k+1}] \left(y^{-1} \sum_{i=1}^{k+1} \mathbf{B}_{n-i}(q, y+1, z) \cdot [z^i] \Big(\prod_{j=0}^{k} (1 + q^j y z) - 1 \Big) \right) \end{aligned}$$

Comparing the coefficients of z^n for $n \geq k+1$, we can conclude that

$$[z^{\geq k+1}]\mathbf{B}_k(q, y+1, z) = [z^{\geq k+1}]\bigg(y^{-1}\mathbf{B}_k(q, y+1, z)\Big(\prod_{j=0}^{k}(1+q^j yz) - 1\Big)\bigg).$$

Consequently, we have

$$[z^{\geq k+1}]\bigg(\mathbf{B}_k(q, y+1, z)\Big(1 - y^{-1}\Big(\prod_{j=0}^{k}(1+q^j yz) - 1\Big)\Big)\bigg) = 0$$

or, equivalently,

$$[z^{\geq k+1}]\bigg(\mathbf{B}_k(q, y, z)\Big(1 - (y-1)^{-1}\Big(\prod_{j=0}^{k}(1+q^j (y-1)z) - 1\Big)\Big)\bigg) = 0$$

which can be written as

$$[z^{\geq k+1}]\Big(\mathbf{B}_k(q, y, z)G_k(q, y, z)\Big) = 0 \tag{2.3}$$

by choosing

$$G_k(q, y, z) = 1 - (y-1)^{-1}\Big(\prod_{j=0}^{k}(1+q^j (y-1)z) - 1\Big). \tag{2.4}$$

We now set

$$F_k(q, y, z) = [z^{\leq k}]\Big(\mathbf{B}_k(q, y, z)G_k(q, y, z)\Big). \tag{2.5}$$

Since F_k only has powers of z at most k, we have

$$[z^{\leq k}]\mathbf{B}_k(q, y, z) = [z^{\leq k}]\mathbf{A}^{\mathrm{inv},\mathrm{des}}(q, y, z)$$

because permutations on $[n]$ cannot have drops of size k or larger when $n \leq k$. Thus, (2.5) can be written as

$$F_k(q, y, z) = [z^{\leq k}]\Big(\mathbf{A}^{\mathrm{inv},\mathrm{des}}(q, y, z)G_k(q, y, z)\Big). \tag{2.6}$$

From (2.3), we have

$$\mathbf{B}_k(q, y, z)G_k(q, y, z) = F_k(q, y, z)$$

and we conclude that

$$\mathbf{B}_k = \frac{F_k}{G_k}.$$

Finally, we can transform G_k into the desired form using the following standard q-binomial theorem (e.g., see [9]):

$$\prod_{j=0}^{n-1}(1+q^j t) = \sum_{i=0}^{n} q^{\binom{i}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_q t^i.$$

Applying (2.4) transforms G_k to

$$G_k(q,y,k) = 1-(y-1)^{-1}\Big(\prod_{j=0}^{k}(1+q^j(y-1)z)-1\Big)$$
$$= 1-\sum_{j=1}^{k+1}\begin{bmatrix}k+1\\ j\end{bmatrix}_q q^{\binom{j}{2}}(y-1)^{j-1}z^j$$

as desired. This proves the theorem. □

Let us now specialize Theorem 2.1 by setting $y=1$. In this case we have

$$G_k|_{y=1} = G_k(q,1,z) = 1-z[k+1]_q.$$

Also,

$$A_n^{\text{inv,des}}(q,y)|_{y=1} = \sum_{\pi\in S_n} q^{\text{inv}(\pi)} = [n]_q!$$

since

$$H(n) = \sum_{\pi\in S_n} q^{\text{inv}(\pi)} = (1+q+q^2+\ldots+q^{n-1})H(n-1) = [n]_q!. \tag{2.7}$$

Consequently,

$$(\mathbf{A}^{\text{inv,des}}\cdot G_k)|_{y=1} = \Big(\sum_{n\geq 0}[n]_q!z^n\Big)(1-[k+1]_qz).$$

This implies that for $1\leq j\leq k$, the negative of the coefficient of z^j is

$$[k+1]_q[j-1]_q! - [j]_q! = ([k+1]_q-[j]_q)[j-1]_q!$$
$$= q^j[k+1-j]_q[j-1]_q!.$$

Plugging these expressions into Theorem 2.1 with $y=1$ yields

Corollary 2.2. *The generating function for inversions and maxdrop is given by*

$$\mathbf{H}_k(z) = \sum_{\substack{n\geq 0\\ \pi\in S_{n,k}}} q^{\text{inv}(\pi)}z^n = \frac{1-\sum_{j=1}^{k} q^j[k+1-j]_q[j-1]_q!\,z^j}{1-[k+1]_qz}. \tag{2.8}$$

For example, for $k=1$, (2.8) yields

$$\mathbf{H}_1(z) = \sum_{\substack{n\geq 0\\ \pi\in S_{n,1}}} q^{\text{inv}(\pi)}z^n = \frac{1-qz}{1-(1+q)z}$$
$$= 1+\sum_{n\geq 0}(1+q)^n z^{n+1}$$

which implies that for $n \geq 1$, the number of $\pi \in S_n$ with j inversions and $\mathrm{maxdrop}(\pi) \leq 1$ is just $\binom{n-1}{j}$.

We can give an alternative proof of Corollary 2.2 as follows. Let us think of the term q^i in the multiplier below as being associated with the choice of $\pi(n)$ with $n - \pi(n) = i$. For $n > k$, we can write

$$\begin{aligned} H_k(n) &= \sum_{\pi \in S_{n,k}} q^{\mathrm{inv}(\pi)} \\ &= (1 + q + q^2 + \cdots + q^k) H_k(n-1) \quad (\text{since } \pi(n) \geq n-k) \\ &= [k+1]_q^{n-k-1} [k+1]_q!. \end{aligned}$$

This implies that the generating function $\mathbf{H}_k(z)$ for the $H_k(n)$ is given by

$$\begin{aligned} \mathbf{H}_k(z) &= \sum_{n \geq 0} H_k(n) z^n \\ &= \sum_{m \leq k} [m]_q!\, z^m + \frac{[k+1]_q!\, z^{k+1}}{1 - [k+1]_q z} \\ &= \frac{1 - \sum_{j=1}^{k} q^j [k+1-j]_q [j-1]_q!\, z^j}{1 - [k+1]_q z} \end{aligned}$$

which proves (2.8).

3 Descents and maxdrop

In this section, we specialize Theorem 2.1 by ignoring inversions. This in fact was the main focus of an earlier paper of Claesson, Dukes and the authors [3]. We first need a few definitions.

We will let $\left\langle {n \atop k} \right\rangle$ denote the usual Eulerian number [10]. It is a standard fact that $\left\langle {n \atop k} \right\rangle$ enumerates the number of permutations in S_n which have k descents (and also which have k drops). The n^{th} Eulerian polynomial $E_n(y)$ is defined by

$$E_n(y) = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle y^k = \sum_{\pi \in S_n} y^{\mathrm{des}(\pi)}.$$

Thus, we have

$$\mathbf{A}^{\mathrm{inv,des}}(q, y, z)|_{q=1} = \mathbf{A}^{\mathrm{inv,des}}(1, y, z) = \sum_{\substack{n \geq 0 \\ \pi \in S_n}} y^{\mathrm{des}(\pi)} z^n = \sum_{n \geq 0} E_n(y) z^n.$$

On the other hand,

$$\begin{aligned} G_k(q, y, z)|_{q=1} = G_k(1, y, z) &= 1 - \sum_{j=1}^{k+1} \binom{k+1}{j} (y-1)^{j-1} z^j \\ &= \frac{y - (1 + (y-1)z)^{k+1}}{y-1}. \end{aligned}$$

Consequently,

$$(\mathbf{A}^{\mathrm{inv,des}} \cdot G_k)|_{q=1} = \left(\sum_{n\geq 0} E_n(y) z^n\right) \cdot \left(1 - \sum_{j=1}^{k+1} \binom{k+1}{j} (y-1)^{j-1} z^j\right)$$

from which it follows (after a modest computation) that

$$\begin{aligned} F_k|_{q=1} &= [z^{\leq k}](\mathbf{A}^{\mathrm{inv,des}} \cdot G_k)|_{q=1} \\ &= 1 + \sum_{t=1}^{k} \left(E_t(y) - \sum_{j=1}^{t} \binom{k+1}{j} (y-1)^{j-1} E_{t-1}(y)\right) z^t \end{aligned}$$

and so, we have the generating function (also see [3])

$$\sum_{\substack{n\geq 0 \\ \pi\in S_{n,k}}} y^{\mathrm{des}(\pi)} z^n = \frac{1 + \sum_{t=1}^{k} \left(E_t(y) - \sum_{j=1}^{t} \binom{k+1}{j} (y-1)^{j-1} E_{t-1}(y)\right) z^t}{1 - \sum_{j=1}^{k+1} \binom{k+1}{j} (y-1)^{j-1} z^j}.$$

For example, when $k = 1$, this becomes

$$\begin{aligned} \sum_{\substack{n\geq 0 \\ \pi\in S_{n,1}}} y^{\mathrm{des}(\pi)} z^n &= \frac{1-z}{1-2z-(y-1)z^2} \\ &= 1 + z + (1+y)z^2 + (1+3y)z^3 + (1+6y+y^2)z^4 + \ldots \\ &= \sum_{n\geq 0} \sum_{i\geq 0} \binom{n}{2i} y^i z^n. \end{aligned} \tag{3.1}$$

Thus, if we let $\left\langle {n \atop i} \right\rangle_{[k]}$ denote the number of $\pi \in S_n$ which have i descents and $\mathrm{maxdrop}(\pi) \leq k$, then $\left\langle {n \atop i} \right\rangle_{[1]}$ is just the binomial coefficient $\binom{n}{2i}$ (there is a nice bijective proof of this fact that the reader may like to find!).

As it happens, Anders Claesson and Mark Dukes [5] earlier had come across these permutations in their work on a class of sorting algorithms, and they noticed that the same type of restricted permutations arose in the analysis of certain juggling patterns [2]. In addition to seeing that $\left\langle {n \atop i} \right\rangle_{[1]}$ was just the coefficient of u^{2i} in the polynomial $(1+u)^n$, computation suggested that $\left\langle {n \atop i} \right\rangle_{[2]}$ was the coefficient of u^{3i} in the polynomial

$$(1 + u + 2u^2 + u^3 + u^4)(1 + u + u^2)^{n-2},$$

and even further, that $\left\langle {n \atop i} \right\rangle_{[3]}$ was the coefficient of u^{4i} in the polynomial

$$(1 + u + 2u^2 + 4u^3 + 4u^4 + 4u^5 + 4u^6 + 2u^7 + u^8 + u^9)(1 + u + u^2 + u^3)^{n-3}!$$

Following these clues, Claesson, Dukes and the authors [3] were able to confirm these conjectures with the following general theorem.

Theorem 3.1. *Let $\left\langle {n \atop i} \right\rangle_{[k]}$ denote the number of $\pi \in S_n$ with i descents and* maxdrop$(\pi) \leq k$*. If $n \geq k$, then $\left\langle {n \atop i} \right\rangle_{[k]}$ is equal to the coefficient of $u^{(k+1)i}$ in the polynomial*

$$P_k(u)(1 + u + \ldots + u^k)^{n-k}$$

where

$$P_k(u) = \sum_{j=0}^{k} E_{k-j}(u^{k+1})(u^{k+1} - 1)^j \sum_{s=j}^{k} \binom{s}{j} u^{-s}.$$

For $n \leq k$, $\left\langle {n \atop i} \right\rangle_{[k]} = \left\langle {n \atop i} \right\rangle$ (the usual Eulerian number) is equal to the coefficient of $u^{(k+1)i}$ in the polynomial $P_k(u)$.

The first few polynomials $P_k(u)$ are given in Table 1.

Table 1:

k	$P_k(u)$
0	1
1	$\mathbf{1} + u$
2	$\mathbf{1} + u + 2u^2 + \boldsymbol{u^3} + u^4$
3	$\mathbf{1} + u + 2u^2 + 4u^3 + \boldsymbol{4u^4} + 4u^5 + 4u^6 + 2u^7 + \boldsymbol{u^8} + u^9$
4	$\mathbf{1} + u + 2u^2 + 4u^3 + 8u^4 + \boldsymbol{11u^5} + 11u^6 + 14u^7 + 16u^8 +$
	$+14u^9 + \boldsymbol{11u^{10}} + 11u^{11} + 8u^{12} + 4u^{13} + 2u^{14} + \boldsymbol{u^{15}} + u^{16}$

In Table 1, we have indicated in bold the coefficients in the $P_k(u)$ which are guaranteed by the theorem to be Eulerian numbers. However, we had no idea at that time what the *other* coefficients of $P_k(u)$ might mean, if anything. Of course, since they are positive integers, one could suspect that they did have a nice interpretation. It turns out that this suspicion was correct. This will be the topic in the next section.

4 Interpreting all the coefficients of $P_k(u)$.

It will be convenient to introduce the polynomials $Q_k(u) = u^k P_k(u)$ for $k \geq 0$. Thus, $Q_k(u)$ is given explicitly as

$$Q_k(u) = \sum_{i=0}^{k} E_{k-i}(u^{k+1})(u^{k+1} - 1)^j \sum_{s=i}^{k} \binom{s}{i} u^{k-s}.$$

We show the first few $Q_k(u)$ in Table 2.

We show the same $Q_k(u)$ as in Table 2 but this time with the coefficients arranged in a $(k+1)\times(k+1)$ array C_k. The (i,j) entry $C_k(i,j)$ of C_k corresponds to the coefficient of $u^{(k+1)i+j}$ for $0 \leq i, j \leq k$. Thus, we can write

Table 2:

k	$Q_k(u)$
0	1
1	$u+u^2$
2	$u^2+u^3+2u^4+u^5+u^6$
3	$u^3+u^4+2u^5+4u^6+4u^7+4u^8+4u^9+2u^{10}+u^{11}+u^{12}$
4	$u^4+u^5+2u^6+4u^7+8u^8+11u^9+11u^{10}+14u^{11}+16u^{12}$
	$+14u^{13}+11u^{14}++11u^{15}+8u^{16}+4u^{17}+2u^{18}+u^{19}+u^{20}$

Table 3:

C_0

$i \backslash j$	0
0	1

C_1

$i \backslash j$	0	1
0	0	1
1	1	0

C_2

$i \backslash j$	0	1	2
0	0	0	1
1	1	2	1
2	1	0	0

C_3

$i \backslash j$	0	1	2	3
0	0	0	0	1
1	1	2	4	4
2	4	4	2	1
3	1	0	0	0

C_4

$i \backslash j$	0	1	2	3	4
0	0	0	0	0	1
1	1	2	4	8	11
2	11	14	16	14	11
3	11	8	4	2	1
4	1	0	0	0	0

$$Q_k(u) = \sum_{0\le i,j\le k} C_k(i,j)u^{(k+1)i+j}$$

We now introduce a "stretched" polynomial $\bar{Q}_k(u)$ defined by

$$\bar{Q}_k(u) = \sum_{j=0}^{k} E_{k-j}(u^{k+2})(u^{k+2}-1)^j \sum_{s=j}^{k} u^{k+1-s}$$

It follows (see [3]) that $\bar{Q}_k(u)$ can also be written as

$$\bar{Q}_k(u) = \sum_{0\le i,j\le k} C_k(i,j)u^{(k+2)i+j+1}$$

Thus, $\bar{Q}_k(u)$ differs from $Q_k(u)$ in that 0's are inserted in positions corresponding to $u^{(k+2)i}$, for $0 \leq i \leq k+1$. For example,

$$\bar{Q}_0(u) = u,$$
$$\bar{Q}_1(u) = u^2 + u^4,$$
$$\bar{Q}_2(u) = u^3 + u^5 + 2u^6 + u^7 + u^9,$$
$$\bar{Q}_3(u) = u^4 + u^6 + 2u^7 + 4u^8 + 4u^9 + 4u^{11} + 4u^{12} + 2u^{13} + u^{14} + u^{16}.$$

Representing the coefficients of $\bar{Q}_k(u)$ in an array $\bar{C}_k = \big(\bar{C}_k(i,j)\big)$, for $0 \leq i \leq k$ and $0 \leq j \leq k+1$, we see that $\bar{C}_k$ is formed from C_k by adding an initial column of 0's as shown in Table 4. The following key fact relating Q_{k+1} to $\bar{Q}_k$

Table 4:

	0	1
0	0	1

$\bar{C}_0$

	0	1	2
0	0	0	1
1	0	1	0

$\bar{C}_1$

	0	1	2	3
0	0	0	0	1
1	0	1	2	1
2	0	1	0	0

$\bar{C}_2$

	0	1	2	3	4
0	0	0	0	0	1
1	0	1	2	4	4
2	0	4	4	2	1
3	0	1	0	0	0

$\bar{C}_3$

	0	1	2	3	4	5
0	0	0	0	0	0	1
1	0	1	2	4	8	11
2	0	11	14	16	14	11
3	0	11	8	4	2	1
4	0	1	0	0	0	0

$\bar{C}_4$

was proved in [3]:

Theorem 4.1 ([3]).

$$Q_{k+1} = \bar{Q}_k \cdot (1 + u + \cdots + u^{k+1}). \tag{4.1}$$

Note that the symmetry and unimodality of the coefficients of $Q_k(u)$ and C_k follow from this result (applied recursively). In particular, we have $C_k(i,j) = C_k(k-i, k-j)$ for $0 \leq i, j \leq k$.

As we noted earlier, when $n = k$, the condition that $\text{maxdrop}(\pi) \leq k$ for $\pi \in S_n$ is automatically satisfied. In this case $C_k(i,k)$, the coefficient of $u^{(k+1)i+k}$ in $Q_k(u)$ is just the Eulerian number $\left\langle {k \atop i} \right\rangle = \left\langle {k \atop i} \right\rangle_{[k]}$.

Let us define the "restricted" Eulerian number $\left\langle {m \atop i} \right\rangle^j$ for $1 \leq j \leq m$, to be the number of $\pi \in S_m$ with $\text{des}(\pi) = i$ and with $\pi(m) = j$. It is clear for example that $\left\langle {m+1 \atop i} \right\rangle^{m+1} = \left\langle {m \atop i} \right\rangle$ since the only possible descents in any $\pi \in S_{m+1}$ with $\pi(m+1) = m+1$ occur at places i for $1 \le i \le m-1$. Thus, the entries $C_k(i,k)$ forming the right-most column of C_k can be replaced by the restricted Eulerian number $\left\langle {k+1 \atop i} \right\rangle^{k+1}$.

It turns out that *all* the entries of C_k can be expressed as restricted Eulerian numbers.

Theorem 4.2. *For $k \ge 0$,*

$$Q_k(u) = \sum_{0 \le i,j \le k} \left\langle {k+1 \atop i} \right\rangle^{j+1} u^{(k+1)i+j}. \tag{4.2}$$

Proof. We will proceed by induction on k, using (4.1). For $k = 0$, Equation (4.2) certainly holds, since $Q_0(u) = 1$ and $\left\langle {1 \atop 0} \right\rangle^1 = 1$. (Check for $k = 1$ if you are nervous about just using the case $k = 0$!). Assume that (4.2) holds for some $k \ge 0$. Notice by (4.1) that each coefficient of $Q_{k+1}(u)$ is a sum of $k+2$ consecutive coefficients of $\bar{Q}_k(u)$. However, each block of $k+2$ consecutive coefficients of $\bar{Q}_k(u)$ contains exactly one of the 0's in the first column of the corresponding array $\bar{C}_k$. Hence, each coefficient of $Q_{k+1}(u)$ will be a sum of $k+1$ consecutive coefficients of $Q_k(u)$.

There are two cases.

(a) The $(k+2)$-block of coefficients of $\bar{Q}_k(u)$ starts with one of the left-hand 0's , i.e., with $\bar{C}_k(i,0)$. Then the sum of the entries is

$$\sum_{j=0}^{k+1} \bar{C}_k(i,j) = \sum_{j=1}^{k+1} \left\langle {k+1 \atop i} \right\rangle^{j}$$

by the induction hypothesis.

However, each $\pi \in S_{k+2}$ with $\mathrm{des}(\pi) = i+1$ and $\pi(k+2) = 1$ corresponds to a unique $\pi' \in S_{k+1}$ with $\mathrm{des}(\pi') = i$ by defining $\pi'(t) = \pi(t) - 1$ for $1 \le t \le k+1$. (The additional descent in π occurs at the place $k+1$.) Thus,

$$\sum_{j=1}^{k+1} \left\langle {k+1 \atop i} \right\rangle^{j} = \left\langle {k+2 \atop i+1} \right\rangle^{1}$$

which is what we need.

(b) The $(k+2)$-block of coefficients of $\bar{Q}_k(u)$ starts with $\bar{C}_k(i,r)$ for some r, where $1 \le r \le k+1$. Thus, the coefficient sum is now

$$\sum_{j=r}^{k+1} \left\langle {k+1 \atop i} \right\rangle^{j} + \sum_{j+1}^{r-1} \left\langle {k+1 \atop i+1} \right\rangle^{j}.$$

However, we can argue as before that each $\pi \in S_{k+2}$ with $\mathrm{des}(\pi) = k+1$ and $\pi(k+2) = r$ corresponds to a unique $\pi' \in S_{k+1}$ counted in one of the two sums. Namely, define

$$\pi'(s) = \begin{cases} \pi(s) & \text{if } \pi(s) < r, \\ \pi(s) - 1 & \text{if } \pi(s) > r. \end{cases}$$

It is easy to check that $\mathrm{des}(\pi') = \mathrm{des}(\pi)$ if $\pi(k+1) < r$ and $\mathrm{des}(\pi') = \mathrm{des}(\pi) - 1$ if $\pi(k+1) > r$. This implies that

$$\sum_{j=r}^{k+1} \left\langle {k+1 \atop i} \right\rangle^{j} + \sum_{j=1}^{r-1} \left\langle {k+1 \atop i+1} \right\rangle^{j} = \left\langle {k+2 \atop i+1} \right\rangle^{r}.$$

Since these arguments hold for all $0 \le i \le k$, our induction is complete, and the theorem is proved. □

We next deal with the case when the $\mathrm{maxdrop}(\pi) \le k$ condition comes into play.

Theorem 4.3.

$$Q_k(u)(1+u+u^2+\cdots+u^k)^{n-k} = \sum_{0 \le i \le n} \sum_{0 \le j \le k} \left\langle {n+1 \atop i} \right\rangle_{[k]}^{n+1-k+j} u^{(k+1)i+j}. \quad (4.3)$$

Proof. We will proceed by induction on $n \ge k$. Equation (4.3) holds for $n = k$ by Theorem 4.2. Suppose (4.3) holds for some $n \ge k$ (where $k \ge 0$ is fixed). The coefficients of the powers of u in the product $Q_k(u)(1 + u + \ldots + u^k)$ are sums of $k + 1$ consecutive coefficients of $Q_k(u)$. Again, these are two cases.

(a) The coefficient sum is

$$\sum_{j=0}^{k} \left\langle {n \atop i} \right\rangle_{[k]}^{n-k+j}.$$

In this case, it is not hard to see that each π counted by $\left\langle {n+1 \atop i} \right\rangle_{[k]}^{n+1}$ is in fact counted by exactly one of the terms $\left\langle {n \atop i} \right\rangle_{[k]}^{n-k+j}$ in the sum so that we have

$$\sum_{j=0}^{k} \left\langle {n \atop i} \right\rangle_{[k]}^{n-k+j} = \left\langle {n+1 \atop i} \right\rangle_{[k]}^{n+1}.$$

(b) The coefficient sum is

$$\sum_{j=r}^{k} \left\langle {n \atop i} \right\rangle_{[k]}^{n-k+j} + \sum_{j=0}^{r-1} \left\langle {n \atop i+1} \right\rangle_{[k]}^{n-k+j},$$

for some r, $1 \le r \le k$. We claim this sum is equal to

$$\left\langle {n+1 \atop i+1} \right\rangle_{[k]}^{n-k+r}.$$

As before, for $\pi \in S_{n+1}$ with $\mathrm{des}(\pi) = i+1, \pi(n+1) = n-k+r$ and $\mathrm{maxdrop}(\pi) \le k$, we define $\pi' \in S_n$ by

$$\pi'(s) = \begin{cases} \pi(s) & \text{if } \pi(s) < n-k+r, \\ \pi(s)-1 & \text{if } \pi(s) > n-k+r. \end{cases}$$

It is easy to check that if $\pi(n) < n-k+r$ then $\mathrm{des}(\pi') = \mathrm{des}(\pi)$ and $\mathrm{maxdrop}(\pi') \le k$, so that this π' is represented by the term $\left\langle {n \atop i+1} \right\rangle_{[k]}^{\pi(n)}$ in the second sum. On the other hand, if $\pi(n) > n-k+r$ then $\mathrm{des}(\pi') = \mathrm{des}(\pi) - 1$ and $\mathrm{maxdrop}(\pi) \le k$, so that this π' is represented by the term $\left\langle {n \atop i} \right\rangle_{[k]}^{\pi(n)}$ in the first sum. Since these maps are invertible then we have for all $i, 0 \le i \le n$,

$$\sum_{j=r}^{k} \left\langle {n \atop i} \right\rangle_{[k]}^{n-k+j} + \sum_{j=0}^{r-1} \left\langle {n \atop i+1} \right\rangle_{[k]}^{n-k+j} = \left\langle {n+1 \atop i+1} \right\rangle_{[k]}^{n-k+dr},$$

as desired. This completes the proof of Theorem 4.3. □

5 Concluding remarks

The fact that $Q_k(u)$ and $R_{k,n}(u) = Q_k(u)(1+u+\ldots+u^k)^{n-k}$ are symmetric raises some interesting bijection questions. For example, since $C_k(i,j) = C_k(k-i, k-j)$, then we have

$$\left\langle {k+1 \atop i} \right\rangle^{j+1} = \left\langle {k+1 \atop k-i} \right\rangle^{k-j+1}, 0 \le i, j \le k.$$

This is not hard to see bijectively by associating each permutation $\pi \in S_n$ with the unique permutation $\sigma \in S_n$ given by $\sigma(t) = k+2-\pi(t)$, for $t = 1, \ldots, k+1$. More interesting is the symmetry of $R_{n,k}(u)$. Since $Q_k(u)$ has degree $k(k+1)$ in u, then $R_{n,k}(u)$ has degree $k(k+1) + (n-k)k = (n+1)k$ in u. The first nonzero term of $R_{n,k}(u)$ is u^k (which has a coefficient $\left\langle {n+1 \atop 0} \right\rangle_{[k]}^{n+1} = 1$). Also, the last nonzero term of $R_{n,k}(u)$ is $u^{(n+1)k}$, which also has a coefficient 1. By Theorem 4.3, this coefficient of $u^{(n+1)k}$ is $\left\langle {n+1 \atop i} \right\rangle_{[k]}^{n+1-k+j}$ where $(n+1)k = (k+1)i + j$, $0 \le j \le k$. That is, if $j \equiv (n+1)k \pmod{(k+1)}$, $0 \le j \le k$, and $i = \lfloor \frac{(n+1)k}{k+1} \rfloor$ then

$$\left\langle {n+1 \atop i} \right\rangle_{[k]}^{n+1-k+j} = 1.$$

More generally, the coefficients of u^r and $u^{r'}$ must be equal where $r + r' = (n+2)k$. Thus, for $r = (k+1)i + j$, $r' = (k+1)i' + j'$, $0 \le i', j' \le k$ and $r + r' = (n+2)k$, we have

$$\left\langle {n+1 \atop i} \right\rangle_{[k]}^{n+1-k+j} = \left\langle {n+1 \atop i'} \right\rangle_{[k]}^{n+1-k+j'}. \tag{5.1}$$

For example, for $k=2, n=4, r=4=3\cdot 1+1$ and $r'=8=3\cdot 2+2$, we have $\left\langle {5 \atop 1} \right\rangle_{[2]}^{4} = \left\langle {5 \atop 2} \right\rangle_{[2]}^{5} = 7$. We list the corresponding permutations in Table 5.

Table 5:

1	2	3	4	5		1	2	3	4	5
1	2	3	5	4		3	2	1	4	5
1	2	5	3	4		4	2	1	3	5
1	3	5	2	4		2	1	4	3	5
1	5	2	3	4	$\longleftrightarrow$	3	1	4	2	5
2	5	1	3	4		1	4	3	2	5
3	5	1	2	4		4	3	1	2	5
5	1	2	3	4		4	1	3	2	5
		$\left\langle {5 \atop 1} \right\rangle_{[2]}^{4}$						$\left\langle {5 \atop 2} \right\rangle_{[2]}^{5}$		

Is there an obvious bijection which proves (5.1)? Even for the simple case for the two sets of permutations shown in Table 5, it is not clear what the correspondence should be!

It seems to us that the maxdrop statistic can be combined with other standard permutation statistics to produce interesting results, e.g., such as in the recent paper of Hyatt and Remmel [11]. More generally, we believe that there should be many similar results for analogs to maxdrop such as the maximum descent (maxdes), the maximum value of the number of inversions (maxinv), the maximum value of the major index (maxmaj), etc., (e.g., see [20]). These have not yet been explored but we hope to return to some of these questions in the future.

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References

[1] D. Beck and J. B. Remmel, Permutation enumeration of the symmetric group and the combinatorics of symmetric functions, *J. Combin. Theory* A **72** (1995), 1–49.

[2] F. Chung and R. Graham, Primitive juggling sequences, *Amer. Math. Monthly* **115** (2008), 185–194.

[3] F. Chung, A. Claesson, M. Dukes and R. Graham, Descent polynomials for permutations with bounded drop size, *European J. Combin.*, **31**, (2010), 1853–1867.

[4] R. J. Clarke, E. Steingrímsson and J. Zeng, New Euler-Mahonian statistics on permutations and words, *Adv. in Appl. Math.* **18** (1997), 237–270.

[5] A. Claesson and M. Dukes, personal communication.

[6] L. Euler, Methodus universalis series summandi ulterius promota, *Commentarii academiae scientiarum imperialis Petropolitanae* **8** (1736), 147–158. Reprinted in his *Pera Omnia*, series 1, volume 14, 124–137.

[7] D. Foata and G. Han, q-series in Combinatorics; permutation statistics (Lecture Notes), preliminary edition, 2004.

[8] D. Foata and M.-P. Schützenberger, Major index and inversion number of permutations, *Math. Nachr.* **83** (1978), 143–159.

[9] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge Univ. Press, Cambridge, England, (1990.)

[10] R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics*, Addison-Wesley, 1994.

[11] M. Hyatt and J. Remmel The classification of 231-avoiding permutations by descents and maximum drop (preprint, 2012).

[12] P. A. MacMahon, *Combinatory Analysis*, 2 volumes, Cambridge University Press, London, 1915-1916. Reprinted by Chelsea, New York, 1960.

[13] D. Rawlings, Enumeration of permutations by descents, idescents, imajor index, and basic components, *J. Combin. Theory* A **36** (1984), 1–14.

[14] O. Rodrigues, Note sur les inversions, ou dérangements produits dans les permutations, *J. de Math.* **4** (1839), 236–240.

[15] M. Skandera, An Eulerian partner for inversions, *Sém. Lothar. Combin.* **46** (2001/02), Art. B46d, 19 pp. (electronic).

[16] R. P. Stanley, Ordered structures and partitions, *Memoirs Amer. Math. Soc.* **119** (1972), 107 pp.

[17] R. P. Stanley, Binomial posets, Möbius inversion, and permutation enumeration, *J. Combin. Theory* A **20** (1976), 336–356.

[18] R. Stanley, Enumerative Combinatorics, vol. 1, Wadsworth and Brooks/Cole,Pacific Grove, CA, 2$^{\text{nd}}$ printing, (1996).

[19] J. Shareshian and M. L. Wachs, q-Eulerian polynomials: excedance number and major index, *Electron. Res. Announc. Amer. Math. Soc.* **13** (2007), 33–45.

[20] J. Shareshian and M. L. Wachs, Eulerian quasisymmetric functions, *Adv. in Math.* **225** (2010), 2921–2966.