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Chordal completions of grids and planar graphs

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ABSTRACT. It is common to model a finite probability space with a graph where nodes correspond to events and edges indicate dependent pairs of events. This paper is an extended abstract of a full length article in which we study chordal completions of graphs which are related to models in which marginal and conditional probabilities can be efficiently computed.

1. Introduction

Increasingly, artificial intelligence has turned to the theory of Bayesian statistics to provide a solid theoretical foundation and a source of useful algorithms for reasoning about the world in conditions of uncertain and incomplete information. This is true both in familiar high-level applications, such as medical expert systems, and in low-level applications such as speech recognition and computer vision (see [**8**] for an overview, [**6**] for medical applications, [**9**] for speech, [**5**] for vision). However, all serious applications demand probability spaces with thousands of random variables, and some simplification is required before you can even write down probability distributions in such spaces. The reason this approach is even partially tractable is that one assumes there are many pairs of random variables which are conditionally independent, given various other variables. One extremely useful way to describe this sort of probability space is based on graph theory: one assumes that a graph G is given, whose vertices $V(G)$ correspond to the random variables in the application, and whose edges $E(G)$ denote pairs of variables which *directly* affect each other. What this means is that if $v, w \in V(G), S \subset V(G)$, and every path from v to w crosses S, then the corresponding variables X_v, X_w are conditionally independent given X_S. As is well-known, this assumption implies that the probability distribution has the

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Gibbs form:

$$\Pr(\vec{X}_v) = \frac{e^{-\sum_C E_C(\{X_w\}_{w\in C})}}{Z}$$

where C runs over the *cliques* of G containing the vertex v, E_C is a measure of the likelihood of the simultaneous values of the variables in the clique C, and Z is a normalizing constant.

A typical problem in this setting is to find the maximum likelihood estimate of the variables $\vec{X}_v$, i.e. the minimum of the so-called energy

$$E(\vec{X}_v) = \sum_C E_C(\{X_w\}_{w\in C}).$$

Unfortunately, minimizing such complex functions of huge numbers of variables is not an easy task. One situation in which the minimum can be quickly and accurately computed is that studied in dynamic programming [**1**]. This is the case where the variables can be ordered in such a way that X_k is conditionally independent of all but a few of the previous X_l's, given the values of these few. A Markov chain is the simplest example of this, and this approach, under the name of the Viterbi algorithm, dominates research in speech recognition. However, it has turned out that modifications of the dynamic programming perspective are much more widely applicable [**6**]. In [**6**], the authors propose using a *chordal completion* $\hat{G}$: this is a chordal graph with the same vertex set $V(G)$ and edge set containing $E(G)$. Recall that a chordal graph is a graph in which all cycles of length at least 4 contain chords, sometimes called a triangulated graph. (For graph-theoretical terminology, the reader is referred to [**2**].) If the cliques in $\hat{G}$ are not too large, one can carry out a variant of dynamic programming for Gibbs fields based on G, and compute essentially all marginal and conditional probabilities of interest.

2. Grid Graphs

In computer vision, one seeks to analyze a two-dimensional signal, finding first edges and areas of homogeneous texture, secondly using these to segment the domain of the signal and thirdly identifying particular regions as resulting from the play of light and shadow on known types of objects such as faces. The random variables that arise in this analysis are firstly I_{ij}, the light intensity measured by a receptor at a position (i, j) of the camera's or eye's focal plane, secondly "line processes" l_{ij} indicating an edge separating adjacent "pixels" (i, j) and $(i, j+1)$ or $(i+1, j)$, and many higher level variables. What interests us is that the measured variables are parametrized by points of a lattice, and that the structures which one calculates are found by examining *local* interactions of these variables. In fact, even a high level variable like the presence of a face is linked to local areas of the image, rather than the whole image, because a face will usually be a subset of the image domain and its presence is more or less independent of the scene in the background. What this means is that the cliques of the graph involve local areas in the lattice, and do not require long range interaction of the pixel values I_{ij}. The simplest example of such a graph is the simple n by n grid, which we denote by L_n: it has vertex set

$$V(L_n) = \{(i,j) : 0 \leq i,j \leq n\}$$

and edges joining:

$$(i,j) \text{ to } (i+1,j), 0 \leq i < n, 0 \leq j \leq n,$$
$$(i,j) \text{ to } (i,j+1), 0 \leq i \leq n, 0 \leq j < n.$$

What we would like to know is how big are the chordal completions of graphs of this sort: how many edges do they have and what are their degrees? We prove the following theorems for grid graphs:

THEOREM 1. *A chordal completion of the n by n grid L_n must contain at least $c\,n^2 \log n$ edges for some constant c. Furthermore, we construct a chordal completion of L_n with $(7.75)n^2 \log n$ edges.*

We note that throughout this paper, all logarithms are to the base 2. By using results on the treewidth of a graph[**10**], we prove

THEOREM 2. *A chordal completion of the n by n grid must contain a vertex of degree cn for some absolute constant c.*

The above theorems can be generalized to all planar graphs by using the planar separator theorems [**4, 7**].

THEOREM 3. *A planar graph on n vertices has a chordal completion with $cn \log n$ edges for some absolute constant c.*

It is also known that there is an $O(n \log n)$ algorithm for constructing the chordal completion of a planar graph.

How good are these bounds? Compared to random graphs, they look quite good: P. Erdös first raised the question how large is a chordal completion of a random graph on n vertices with edge density k/n for some fixed k (or random k-regular graphs). It turns out that chordal completions of random graphs must contain cn^2 edges for some constant c. The reader is referred to [**2**] for models of random graphs or random regular graphs. Unfortunately for the application to computer vision, the lower bounds on the size of the chordal completions of grids are still too big to make the use of dynamic programming or its variants practical in vision: typical values of n are 100 or more, and probability tables for the values of 100 random variables are quite impossible. However, the construction given in the full paper for a chordal completion of L_n is strongly reminiscent of the approach to vision problems called "pyramid algorithms" [**11**], e.g. wavelet expansions [**3**]. This link is interesting to explore.

References

1. R. Bellman, *Dynamic Programming*, Princeton Univ. Press (1957).
2. B. Bollobás, Random Graphs, *Academic Press,* New York (1987)
3. P. Burt and E. Adelson, The Laplacian pyramid as a compact image code, *IEEE Trans. on Comm.*, **31** (1983), 532-540.
4. F. R. K. Chung, Improved separators for planar graphs, in *Graph Theory, Combinatorics, and Applications*, Y. Alavi et al eds., (1991), 265-270.
5. R. Geman, Random Fields and Inverse Problems in Imaging, Springer Lecture Notes in Mathematics.
6. Lauritzen and Spiegelhalter, Local computations with probabilities on graphical structures and their applications to expert systems, *J.Royal Soc., Part B*, **50** (1988) 157-224.
7. R. J. Lipton and R. E. Tarjan, A separator theorem for planar graphs, *SIAM J. Appl. Math.* **36** (1979) 177-189
8. J. Pearl, *Probabilistic reasoning in intelligent systems: networks of plausible inference*, Morgan Kaufmann (1988).
9. L. Rabiner, A Tutorial on Hidden Markov Models and Selected Applications in Speech Recognition, *Proc. of the IEEE*, **77** (1989), 257-285.
10. N. Robertson and P. D. Seymour, Graph minors II, Algorithmic aspects of tree-width, *J. Algorithms* **7**(1986), 309-322
11. A. Rosenfeld ed., *Multiresolution Image Processing and Analysis*, Springer Verlag, 1984.
12. J. R. Walter, Representation of rigid cycle graphs, Ph. D. Thesis, Wayne State University 1972

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