CHORDAL COMPLETIONS OF PLANAR GRAPHS

F. R. K. CHUNG AND DAVID MUMFORD

Abstract. We prove that every planar graph on n vertices is contained in a chordal graph with at most $cn \log n$ edges for some absolute constant c and this is best possible to within a constant factor.

1. Introduction

A graph is said to be *chordal* if every cycle with at least 4 vertices always contains a chord. (A chordal graph is sometimes called a triangulated graph in the literature. For graph-theoretical terminology, the reader is referred to [4].) A *chordal completion* of a graph G is a chordal graph with the same vertex set as G which contains all edges of G. The problem of interest is to find for a given graph a chordal completion with as few edges as possible. Also, it is desirable to have a chordal completion with small maximum degree. These problems are motivated by applications in computer vision and artificial intelligence (for further discussion see Section 6).

We will prove the following:

Theorem 1. *Every planar graph on n vertices has a chordal completion with $cn \log n$ edges for some absolute constant c.*

We remark that there is an $O(nlogn)$ algorithm for constructing a chordal completion of a planar graph. Also, we note that Theorem 1 is best possible to within a constant factor by considering the n by n grid graph, denoted by L_n, which has vertex set

$$V(L_n) = \{(i, j) : 0 \leq i, j \leq n\}$$

and edges joining:

$$(i, j) \text{ to } (i + 1, j), 0 \leq i < n, 0 \leq j \leq n,$$
$$(i, j) \text{ to } (i, j + 1), 0 \leq i \leq n, 0 \leq j < n.$$

Theorem 2. *A chordal completion of the n by n grid L_n must contain at least $c\, n^2 \log n$ edges for some constant c. Furthermore, there is a chordal completion of L_n with at most $7.75n^2 \log n$ edges.*

Date:

Throughout this paper, all logarithms are to the base 2. The proof of Theorem 2 is due to Paul Seymour.

Theorem 3. *A chordal completion of the n by n grid graph must contain a clique of size cn for some absolute constant c.*

Consequently, a chordal completion of the n by n grid graph must contain a vertex of degree at least cn.

This paper is organized as follows: Planar separator theorems are used to prove Theorem 1 in Section 2. Section 3 contains the proofs of Theorem 2. Theorem 3 is proved in Section 4 by using the treewidth of graphs. We extend and generalize Theorem 1 to several families of graphs and random graphs in Section 5. Motivations and applications are discussed in Section 6.

2. Separators and chordal completions

In a graph G, a subset S of vertices is said to be a *separator* if $V(G)-S$ can be partitioned into two parts A and B satisfying the following conditions:

(i): There is no edge joining a vertex in A and a vertex in B.
(ii): The ratio of $|A|$ and $|B|$ is between $\frac{1}{2}$ and 2.

The separator S is also called a bisector if (ii) is replaced by:

(ii'): $|A| = |B|$.

Tarjan and Lipton [12] proved the following theorem:

Theorem A *A planar graph on n vertices has a bisector of size at most $C_0\sqrt{n}$.*

The constant C_0 in the original bisector theorem was $2\sqrt{2}/(1-\sqrt{2/3}) \approx 15.413$. It was later [7] improved to $3\sqrt{6} \approx 7.348$. Also, there is a linear algorithm for finding a bisector of size $O(\sqrt{n})$ in a planar graph.

To prove Theorem 1, we need the following lemma:

Lemma *Let G be a graph and suppose its vertices are subdivided into three disjoint parts $V(G) = V_1 \cup S \cup V_2$. Assume:*
(i) S is a clique,
(ii) V_1 and V_2 are not connected by any edge,
(iii) $V_k \cup S$ is chordal for $k = 1, 2$.
Then G is chordal.

The proof of the lemma is immediate: take a cycle C in G. If $C \subset (V_1 \cup S)$, C has a chord; if $C \subset (V_2 \cup S)$, C has a chord; if neither holds, C contains vertices $v_1 \in V_1, v_2 \in V_2$, and because V_1 and V_2 are not connected directly, C must cross S twice - hence it has a chord.

Proof of Theorem 1:

We shall use repeatedly the process of taking a graph G and a subset $S \subset V(G)$ and adding all edges between vertices in S, making S into a clique. We will denote the enlarged graph by $G_{/S}$. Using this concept, what we will actually show is the following stronger result:

(*): If G is a planar graph with n vertices and $H \subset V(G)$ has m vertices in it, then $G_{/H}$ has a chordal completion $\hat{G}_{/H}$ with at most:

$$f(n,m) = C_1 n \log(n) + C_2 m\sqrt{n} + C_3 m^2$$

edges. Here C_1, C_2 and C_3 are universal constants.

To prove (*), we use the bisector theorem on G, writing $V(G) = V_1 \cup S \cup V_2$, where $|V_1| = |V_2|$ and $|S| \leq C_0\sqrt{n}$ and S separates V_1 from V_2. Let $G_k = V_k \cup S$, $H_k = (H \cap V_k) \cup S$ and $m_k = |H \cap V_k|$ for $k = 1, 2$. Now use chordal completions $(\hat{G_k})_{/H_k}$ of $(G_k)_{/H_k}$ satisfying (*) and construct $\hat{G}_{/H}$ by:

$$E(\hat{G}_{/H}) = E((\hat{G_1})_{/H_1}) \cup E((\hat{G_2})_{/H_2}) \cup ((H \cap V_1) \times (H \cap V_2)).$$

This graph is chordal: in fact, the subgraph of $\hat{G}_{/H}$ on the vertices:

$$(V_1 - H \cap V_1) \cup (H_1) \cup (H \cap V_2)$$

is chordal by the lemma applied to the three subsets in parentheses. Likewise, the subgraph on

$$(H \cap V_1) \cup (H_2) \cup (V_2 - H_2 \cap V_2)$$

is chordal. Apply the lemma again for the three-way decomposition

$$V(G) = (V_1 - H \cap V_1) \cup (S \cup H) \cup (V_2 - H \cap V_2)$$

and the full graph $\hat{G}_{/H}$ is chordal. Using this completion of $G_{/H}$, we obtain the bound:

$$f(n,m) \leq f(\frac{n}{2} + C_0\sqrt{n}, m_1 + C_0\sqrt{n}) + f(\frac{n}{2} + C_0\sqrt{n}, m_2 + C_0\sqrt{n}) + m_1 m_2.$$

For any n, we can always make the theorem hold for graphs G with $n \leq n_0$ by making C_1 sufficiently large. Assuming this has been done, we can make an induction for $n \geq n_0$, hence we may assume (*) holds for the two smaller graphs. We assume, in particular, that $n_0 \geq 864$ so that the constant $C_0' = (1 + (2C_0/\sqrt{n_0}) \leq 3/2$. Moreover take $C_3 = 0.5$. Then substituting and simplifying:

$$f(n,m) \leq C_1 n \log(n) + 6\sqrt{6}C_1\sqrt{n}\log(n) + \left(\frac{\sqrt{3}C_2}{2} + 3\sqrt{6}\right)\sqrt{n}m + 0.5m^2 + \left(\log(.75)C_1 + 9\sqrt{2}C_2 + 54\right)n$$

In order that the right hand side is bounded by $C_1 n \log(n) + C_2 m\sqrt{n} + 0.5m^2$, we first take $C_2 \geq 12\sqrt{6} + 18\sqrt{2}$, e.g. $C_2 = 55$, and we get:

$$f(n,m) \leq C_1 n \log(n) + C_2\sqrt{n}m + 0.5m^2 + \left(\left(\log(.75) + 6\sqrt{6}\frac{\log(n_0)}{\sqrt{n_0}}\right)C_1 + 9\sqrt{2}C_2 + 54\right)n.$$

If n_0 is big enough, the coeffecient of C_1 is negative, hence if C_1 is large enough, the last term can be dropped and the estimate now follows by induction.

This completes the proof of Theorem 1.

3. The chordal completions of grid graphs

Although the construction of a chordal graph for the grid graph follows from the proof of Theorem 1, we here derive a slightly improved upper bound because grid graphs are useful for various applications (see Section 6).

We consider the $a \times b$ grid graph, denoted by L_{ab}, with vertices:

$$V(L_{ab}) = \{(i,j) : 0 \leq i \leq a, 0 \leq j \leq b\}$$

and edges joining:

$$(i,j) \text{ to } (i+1,j), 0 \leq i < a, 0 \leq j \leq b,$$
$$(i,j) \text{ to } (i,j+1), 0 \leq i \leq a, 0 \leq j < b.$$

The boundary of L_{ab} consists of all vertices (i,j) with $i = 0, a$ or $j = 0, b$.

Instead of considering the chordal completion of L_{ab}, we examine the chordal completion of the graph $\bar{L}_{ab}$, which is formed from L_{ab} by making the boundary of L_{ab} into a clique, i.e. by adding all edges joining any two vertices in the boundary of L_{ab}. Clearly, a chordal completion of $\bar{L}_{ab}$ is also a chordal completion of L_{ab}.

We now construct $\hat{L}_{ab}$, which will later be proved to be a chordal completion of $\bar{L}_{ab}$, by combining the chordal completions of two subgraphs. $f(a,b)$ will denote the number of edges in $\hat{L}_{ab}$. We assume without loss of generality that $a \leq b$, and begin by dividing L_{ab} along the "b-axis" into two grid graphs with sizes as equal as possible. The graph $\hat{L}_{ab}$ includes all edges which appear in the chordal completions of the two halves of $\bar{L}_{ab}$ plus the edges in

$\bar{L}_{ab}$. It is easy to check that, starting from the edges in chordal completions of the smaller graphs, adding the edges connecting the boundaries of the two pieces and subtracting the edges counted twice in their common boundary, the number of edges of $\hat{L}_{ab}$, denoted by $f(a,b)$ satisfies

$$f(a,b) \leq f(a, \lfloor \frac{b}{2} \rfloor) + f(a, \lceil \frac{b}{2} \rceil) + (a+b-1)^2 - (b \bmod 2) - \binom{a+1}{2}.$$

To construct the completions of the two pieces, now repeat this construction, but now subdividing the two pieces $L_{a,\lfloor b/2 \rfloor}$ and $L_{a,\lceil b/2 \rceil}$ along the "a-axis". With a little calculation, this gives us:

$$\begin{aligned} f(a,b) &\leq f(\lfloor \frac{a}{2} \rfloor, \lfloor \frac{b}{2} \rfloor) + f(\lfloor \frac{a}{2} \rfloor, \lceil \frac{b}{2} \rceil) + f(\lceil \frac{a}{2} \rceil, \lfloor \frac{b}{2} \rfloor) + f(\lceil \frac{a}{2} \rceil, \lceil \frac{b}{2} \rceil) \\ &\quad +3(a-1)^2 + 4b(a-1) + \frac{3}{2}b^2 - \binom{a+1}{2} - \binom{\lfloor \frac{b}{2} \rfloor + 1}{2} - \binom{\lceil \frac{b}{2} \rceil + 1}{2} \\ &\leq f(\lfloor \frac{a}{2} \rfloor, \lfloor \frac{b}{2} \rfloor) + f(\lfloor \frac{a}{2} \rfloor, \lceil \frac{b}{2} \rceil) + f(\lceil \frac{a}{2} \rceil, \lfloor \frac{b}{2} \rfloor) + f(\lceil \frac{a}{2} \rceil, \lceil \frac{b}{2} \rceil) \\ &\quad + \frac{5}{2}a^2 + 4ab + \frac{5}{4}b^2 - \frac{13}{2}a - \frac{9}{2}b + 3 \end{aligned}$$

The construction continues in this way, subdividing alternately along the a and b axes, until $a = 1$. In this case, the chordal completion is just one clique on $2b+2$ vertices, so

$$f(1,b) = 2b^2 + 3b + 1.$$

It is now easy to prove by induction that

$$f(a,b) \leq \left(\frac{5}{2}a^2 + 4ab + \frac{5}{4}b^2 \right) \cdot (\log(a) + 2).$$

(handling the four cases a, b even/odd cases carefully.) In particular, for $n = a = b$, we have

$$f(n) = f(n,n) \leq 7.75n^2(\log(n) + 2).$$

It remains to prove by induction that $\hat{L}_{ab}$ is chordal. Suppose $\hat{L}_{a'b'}$ is chordal for all $a' < a$ and $b' < b$. By construction, $\hat{L}_{ab}$ is the union of the two such smaller chordal graphs, which we denote Q_1 and Q_2, plus edges making the original boundary B of L_{ab} into a clique. Note that in fact $B^* = B \cup (Q_1 \cap Q_2)$ is a clique in $\hat{L}_{ab}$. Now let C denote a cycle in $\hat{L}_{ab}$. If the vertices of C are entirely contained in B^*, it has a chord. If not, C contains a vertex in the interior of one of the two subgraphs Q_1 or Q_2: suppose it contains $v \in \text{Int}(Q_1)$. If C is entirely contained in Q_1, it again has a chord since Q_1 is chordal. If not, then it contains some vertex in $Q_2 - Q_1$. Let w_1 and w_2 be the last vertices to the left and right of v on C which are in Q_1. These vertices must be on the boundary of Q_1, hence form a chord of C in $\hat{L}_{ab}$. Thus every cycle C has a chord.

To complete the proof of Theorem 2, we consider a chordal completion of L_n, which is denoted by $\hat{L}_n$. We want to show $\hat{L}_n$ has at least $cn^2 \log n$ edges. In L_n, there are n^2 four cycles each must contain a chord in $\hat{L}_n$ and therefore there is a trivial lower bound of n^2 for the number $e(\hat{L}_n)$ of edges in $\hat{L}_n$. An edge in $\hat{L}_n$ is said to be *k-long* if it joins two vertices (i, j) and (i', j') where $|i - i'| = k$ or $|j - j'| = k$. We need the following useful fact:

(*): Suppose a cycle C is the boundary of a k-square (isomorphic to L_k) in L_n. Then C contains a chord in $\hat{L}_n$ which is k-long.

Proof of (*): Consider the graph R consisting of C and its chords. Let C' denote the minimum cycle in R with the property that C' contains a vertex from each side of C. If C' contains only three vertices, it is easy to see that C contains a k-long chord. Suppose C' contains at least four vertices. Since $\hat{L}_n$ is chordal, C' must contain a chord $\{x, y\}$. Suppose the chord $\{x, y\}$ is not k-long, vertices x and y must belong to adjacent sides of C. We can then form a shorter cycle using the edge $\{x, y\}$ and parts of C', from y to x. This contradicts the fact that C' is minimal. Therefore $\{x, y\}$ must be k-long and (*) is proved.

Now there are $(n - k + 1)^2$ different k-squares in L_n. Each k-square contains a chord k-long. On the other hand, each edge which is k-long can be a chord of at most $k + 1$ k-squares. Therefore in $\hat{L}_n$ there are at least $\frac{(n-k+1)^2}{k+1}$ edges which are k-long. We then have

$$e(\hat{L}_n) \geq \sum_{k=1}^{n} \frac{(n - k + 1)^2}{k + 1} \geq n^2 \log n - 2n^2$$

Theorem 2 is proved.

4. The maximum degree in chordal completions

This result is an easy translation of the theorems of Robertson and Seymour on graph minors [16]. Before we proceed to prove Theorem 3, we first state a well-known characterization of chordal graphs (see [5, 8, 18]).

Theorem B *For a chordal graph G, there is a tree H so that each vertex of G corresponds to a subtree in H and two vertices are adjacent in G if and only if the corresponding subtrees in H have nontrivial intersection.*

The *treewidth* of a graph G is defined as follows: A family $\mathcal{F}$ of subsets of vertices is said to be a tree-covering of G if $\mathcal{F}$ satisfies the following conditions:

(i): The union of subsets F in $\mathcal{F}$ is the vertex set $V(G)$.

(ii): Members of $\mathcal{F}$ are indexed by vertices of some tree T.

(iii): Suppose a vertex v is in both F_a and F_b where $F_a, F_b \in \mathcal{F}$. Then v is in every F_c if c is in the path of T joining a and b.

(iv): If u is adjacent to v in G, then there is a F_c in $\mathcal{F}$ containing both u and v.

For each tree-covering $\mathcal{F}$ of G, the width of $\mathcal{F}$ is just the maximum size of F in $\mathcal{F}$. The tree-width of G is defined to be the minimum width over all tree-coverings of G. The following result can be found in [16].

Theorem C *The n by n grid graph has treewidth at least cn.*

We are now ready to prove Theorem 3.

Proof of Theorem 3:

Let $\hat{L}_n$ denote a chordal completion of the grid graph L_n. By Theorem B, there is a tree H so that each vertex v of $\hat{L}_n$ corresponds to a subtree T_v in H. For each vertex w of H, we define F_w to be the set of all v in $\hat{L}_n$ so that T_v contains w. Two vertices u and v are adjacent in $\hat{L}_n$ if and only if F_u and F_v have nontrivial intersection. It is straightforward to verify that $\{F_w : w \in H\}$ satisfies (i) to (iv) and is a tree-covering of $\hat{L}_n$. By Theorem C, one of the F_w's must contain at least cn vertices in $\hat{L}_n$. In addition, any two vertices u and v in F_w correspond to trees T_u and T_v with nontrivial intersection (including w). Therefore, by Theorem B, u and v are adjacent in $\hat{L}_n$ and $\hat{L}_n$ has a clique of size at least cn. Theorem 3 is proved.

5. Chordal completions for other types of graphs

A family $\mathcal{F}$ of graphs is called a *closed* family if for every G in $\mathcal{F}$, by removing one vertex and its incident edges the remaining graph is also in $\mathcal{F}$. A family $\mathcal{F}$ of graphs is said to have a *bisector function* f if $\mathcal{F}$ is closed and every graph G in $\mathcal{F}$ with n vertices has a bisector $f(n)$. For example, the family of planar graphs has a bisector function $3\sqrt{6}\sqrt{n}$. We can now generalize Theorem 1 as follows.

Theorem 4. Suppose a family $\mathcal{F}$ of graphs has a bisector function $f(n) = cn^\alpha$ for fixed positive values c and α. Then every graph in $\mathcal{F}$ on n vertices has a chordal completion with $c_1 n \log n$ edges if $\alpha = 1/2$; with $c_2 n^{2\alpha}$ edges if $\alpha > 1/2$; and with $c_3 n$ edges if $\alpha < 1/2$ where constants c_1, c_2, c_3 depend only on c.

Proof: The proof is quite similar to that of Theorem 1 except that the number $e(n)$ of edges satisfies the following recurrence inequality

$$e(n) \leq 2e(n/2 + cn^\alpha) + c^2 n^{2\alpha}.$$

Of special interest is the case for graphs with bounded genus:

Theorem 5. A graph of genus g has a chordal completion with $c\sqrt{g}n\log n$ edges.

Proof: This follows from the above theorem and the fact that a graph of genus g has a separator of size $c\sqrt{gn}(see[10])$.

Theorem 6. A graph with n vertices and with no K_h-minor has a choral completion with $ch^{3/2}n\log n$ edges.

Theorem 7 is a direct consequence of the fact that a graph with no minor K_h has a separator of size $h^{3/2}\sqrt{n}$ (see [1]).

P. Erdös first raised the question how large is a chordal completion of a random graph on n vertices with edge density k/n for some fixed k (or random k-regular graphs). The reader is referred to [4] for models of random graphs or random regular graphs. For example, "A random graph has property P" means that "a graph on n nodes satisfies property P with probability approaches 1 when n approaches infinity." In particular, we consider random graphs with edge density k/n (i.e., each unordered pair is an edge with probability k/n).

Theorem 7. A chordal completion of a random graph with edge density k/n for fixed k has cn^2 edges. In fact it contains a complete subgraph on $c'n$ vertices.

Proof: This follows from the following two facts:

Fact 1. A random graph with edge density k/n for fixed k has a separator at least c_1n.

Fact 2. A graph with treewidth k has a separator of size k.

Fact 1 can be easily proved by standard probabilistic methods [2] and Fact 2 can be found in [15]. These two facts imply that a random graph with edge density k/n has treewidth c_2n and therefore by Theorems A and B, its chordal completion contains a complete subgraph of $c_2n^2/2$ edges.

6. Motivations and applications

Increasingly, artificial intelligence has turned to the theory of Bayesian statistics to provide a solid theoretical foundation and a source of useful algorithms for reasoning about the world in conditions of uncertain and incomplete information. This is true both in familiar high-level applications, such as medical expert systems, and in low-level applications such as speech recognition and computer vision (see [13] for an overview, [11] for medical applications, [14] for speech, [9] for vision). However, all serious applications demand probability spaces with thousands of random variables, and some simplification is required before you can even write down probability distributions in such spaces. The reason this approach is even partially tractable is that one assumes there are many pairs of random variables which are conditionally independent, given various other variables. One extremely useful way to describe this sort of probability space is based on graph theory: one assumes that a graph G is given, whose vertices $V(G)$ correspond to the random variables in the application, and whose edges $E(G)$ denote pairs of variables which *directly* affect each other. What this means is that if $v, w \in V(G), S \subset V(G)$, and every path from v to w crosses S, then the corresponding variables X_v, X_w are conditionally independent given X_S. As is well-known, this assumption implies that the probability distribution has the Gibbs form:

$$\Pr(\vec{X}) = \frac{e^{-\sum_C E_C(\{X_w\}_{w\in C})}}{Z}$$

where C runs over the *cliques* of G, E_C is a measure of the likelihood of the simultaneous values of the variables in the clique C, and Z is a normalizing constant.

A typical problem in this setting is to find the maximum likelihood estimate of the variables $\vec{X}$, i.e. the minimum of the so-called energy

$$E(\vec{X}) = \sum_C E_C(\{X_w\}_{w\in C}).$$

Unfortunately, minimizing such complex functions of huge numbers of variables is not an easy task. One situation in which the minimum can be quickly and accurately computed is that studied in dynamic programming [3]. This is the case where the variables can be ordered in such a way that X_k is conditionally independent of all but a few of the previous X_l's, given the values of these few. A Markov chain is the simplest example of this, and this approach, under the name of the Viterbi algorithm, dominates research in speech recognition. However, it has turned out that modifications of the dynamic programming perspective are much more widely applicable [11]. In [11], the authors propose using a *chordal completion* $\hat{G}$ of a given graph G. *If* the cliques in $\hat{G}$ are not too large, one can carry out a variant of dynamic programming for Gibbs fields based on G, and compute essentially all marginal and conditional probabilities of interest.

In computer vision, one seeks to analyze a two-dimensional signal, finding first edges and areas of homogeneous texture, secondly using these to segment the domain of the signal and thirdly identifying particular regions as resulting from the play of light and shadow on known types of objects such as faces. The random variables that arise in this analysis are firstly I_{ij}, the light intensity measured by a receptor at a position (i, j) of the camera's or eye's focal plane, secondly "line processes" l_{ij} indicating an edge separating adjacent "pixels" (i, j) and $(i, j+1)$ or $(i+1, j)$, and many higher level variables. What interests us is that the measured variables are parametrized by points of a lattice, and that the structures which one calculates are found by examining *local* interactions of these variables. In fact, even a high level variable like the presence of a face is linked to local areas of the image, rather than the whole image, because a face will usually be a subset of the image domain and its presence is more or less independent of the scene in the background. What this means is that the cliques of the graph involve local areas in the lattice, and do not require long range interaction of the pixel values I_{ij}. The simplest example of such a graph is the simple n by n grid graph L_n. In order to apply a dynamic programming like algorithm to computing the ML estimate, we would like to know is how big are the chordal completions of graphs of this sort: how many edges do they have and what are their degrees?

In previous sections, we answer these problems for grid graphs and planar graphs. Unfortunately for the application to computer vision, the lower bounds on the size of the chordal completions of grids are still too big to make the use of dynamic programming or its variants practical in vision: typical values of n are 100 or more, and probability tables for the values of 100 random variables are quite impractical. However, the construction given above for a chordal completion of L_n is strongly reminiscent of the approach to vision problems called "pyramid algorithms" [17], e.g. wavelet expansions [6]. This link is interesting to explore.

References

[1] Noga Alon, Paul Seymour and Robin Thomas, A separator theorem for nonplanar graphs, *J. of AMS*, 3 (1990) 801-808.

[2] Noga Alon and Joel H. Spencer, The Probabilistic Method, *John Wiley & Sons*, New York, (1991)

[3] R. Bellman, *Dynamic Programming*, Princeton Univ. Press (1957).

[4] B. Bollobás, Random Graphs, *Academic Press,* New York (1987)

[5] P. Buneman, A characterization of rigid circle graphs, *Discrete Math.* 9 (1974) 202-212

[6] P. Burt and E. Adelson, The Laplacian pyramid as a compact image code, *IEEE Trans. on Comm.*, **31** (1983), 532-540.

[7] F. R. K. Chung, Improved separators for planar graphs, in *Graph Theory, Combinatorics, and Applications*, Y. Alavi et al eds., (1991), 265-270.

[8] F. Gavril, The intersection graphs of subtrees in trees are exactly the chordal graphs, *J. Combinatorial Theory B***16** (1974) 47-56

[9] R. Geman, Random Fields and Inverse Problems in Imaging, Springer Lecture Notes in Mathematics.

[10] J. R. Gilbert, J. P. Hutchinson, and R. E. Tarjan, A separator theorem for graphs with bounded genus, *J. Algorithms 5* (1984) 391-407.

[11] Lauritzen and Spiegelhalter, Local computations with probabilities on graphical structures and their applications to expert systems, *J.Royal Soc., Part B*, **50** (1988) 157-224.

[12] R. J. Lipton and R. E. Tarjan, A separator theorem for planar graphs, *SIAM J. Appl. Math.* **36** (1979) 177-189.
[13] J. Pearl, *Probabilistic reasoning in intelligent systems: networks of plausible inference*, Morgan Kaufmann (1988).
[14] L. Rabiner, A Tutorial on Hidden Markov Models and Selected Applications in Speech Recognition, *Proc. of the IEEE*, **77** (1989), 257-285.
[15] N. Robertson and P. D. Seymour, Graph minors II, Algorithmic aspects of tree-width, *J. Algorithms* **7**(1986), 309-322
[16] N. Robertson and P. D. Seymour, Graph minor X. Obstructions to tree-decomposition, *J. Combinatorial Theory B* **52** (1991), 153-190.
[17] A. Rosenfeld ed., *Multiresolution Image Processing and Analysis*, Springer Verlag, 1984.
[18] J. R. Walter, Representation of rigid cycle graphs, Ph. D. Thesis, Wayne State University 1972

Bell Communications Research, Morristown, New Jersey 07960

Department of Mathematics, Harvard University, Cambridge, Massachussetts 02138