Laplacians of graphs and Cheeger inequalities

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Abstract

We define the Laplacian for a general graph and then examine several isoperimetric inequalities which relate the eigenvalues of the Laplacian to a number of graphs invariants such as vertex or edge expansions and the isoperimetric dimension of a graph.

1 Introduction

The study of eigenvalues of graphs has a long history, dated back to the early days of graph theory. In fact, the birth of graph theory was closely associated with the investigation of molecules by chemists [4] while eigenvalues play an important role. There have been a great deal of literature in using matrix theory and algebraic techniques to study the adjacency matrices of graphs. There are a number of excellent books and survey articles on spectra of graphs, such as Biggs [3], Cvetković, Doob and Sachs [10] and Seidel [18].

The major approaches in spectral graph theory before 1980's were essentially "algebraic" with emphasis on the symmetries of strongly regular graphs. In contrast, the advances and breakthroughs in the past ten years are often "geometric". For example, the successful constructions of expander graphs [14, 15] take advantages of the relationship between eigenvalues and isoperimetric properties of graphs. Roughly speaking, isoperimetric properties concern the sizes of the neighborhood of a set of vertices. Here, "size" refers to some appropriate measure on graphs such as the (weighted) number of vertices or edges. The term "expander" or "expansion" usually means that the sizes of the neighborhood of a subset can be lower bounded as a function of the size of the subset. Such isoperimetric properties often provide the foundation for many recent developments, ranging from the fast convergence of Markov chains, efficient approximation algorithms, randomized or derandomized algorithms, amplifying random bits, complexity lower bounds and building efficient communication networks.

A major isoperimetric inequality concerning edge expansion is the Cheeger inequality, which is the discrete analog of its continuous counterpart [6, 7, 11] in studying the Laplace operators of Riemannian manifolds. An analogous version of the Cheeger inequality concerning vertex expansion was established by N. Alon [1]. We remark that many existing isoperimetric inequalities require the graphs to be regular (i.e., all vertices have the same number of neighbors.) Although for some problems this restriction is unimportant, it is crucial to extend these isoperimetric inequalities to all graphs. For example, eigenfunctions are useful in identifying the "bottleneck" or "separator" the graphs [9] and such "divide-and-conquer" methods often require iterative applications. It is easy to see that a subgraph of a regular graph is not necessarily regular.

In the next section, we will define the Laplacian of a graph and point out its natural correspondence to the continuous cases. Some basic facts about the eigenvalues of the Laplacian of graph will be discussed. In Section 3, we will give a proof of the Cheeger inequality for general graphs which relates the (dominant) eigenvalues of the Laplacian to edge expansion of graphs. The isoperimetric inequality for general graphs for vertex expansion is given in Section 4. We generalize these isoperimetric inequalities to weighted graphs in Section 5. In Section 6, we discuss bounds for eigenvalues using isoperimetric dimensions and Sobolev inequalities.

2 The Laplacian of a graph

In a graph G, let d_v denote the degree of the vertex v. We first define Laplacian for graphs without loops and multiple edges. (The general weighted case will be treated later in Section 5.) The matrix L is defined as follows:

$$L(u,v) = \begin{cases} d_v & \text{if } u = v \\ -1 & \text{if } u \text{ and } v \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

Let S denote the diagonal matrix with the (v,v)-th entry having value $\frac{1}{\sqrt{d_v}}$. The Laplacian of G is defined to be

$$\mathcal{L} = SLS.$$

In other words, we have

$$\mathcal{L}(u,v) = \begin{cases} 1 & \text{if } u = v \\ -\frac{1}{\sqrt{d_u d_v}} & \text{if } u \text{ and } v \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

The eigenvalues of $\mathcal{L}$ are denoted by $0 = \lambda_0 \le \lambda_1 \le \cdots \le \lambda_{n-1}$. When G is k-regular, it is easy to see that

$$\mathcal{L} = I - \frac{1}{k}A$$

where A is the adjacency matrix of G.

Let h denote a function which assigns to each vertex v of G a complex value $h(v)$. Then

$$\begin{aligned} \frac{\langle h, \mathcal{L}h\rangle}{\langle h, h\rangle} &= \frac{\langle h, SLSh\rangle}{\langle h, h\rangle} \\ &= \frac{\langle f, Lf\rangle}{\langle S^{-1}f, S^{-1}f\rangle} \\ &= \frac{\displaystyle\sum_{u\sim v}(f(u)-f(v))^2}{\displaystyle\sum_v d_v f(v)^2} \qquad (1) \end{aligned}$$

where $h = S^{-1}f$.

Let $\mathbf{1}$ denote the constant function which assumes value 1 on each vertex. Then $S^{-1}\mathbf{1}$ is an eigenfunction of $\mathcal{L}$ with eigenvalue 0. Also,

$$\begin{aligned} \lambda_G = \lambda_1 &= \min_{f\perp S^{-2}\mathbf{1}} \frac{\displaystyle\sum_{u\sim v}(f(u)-f(v))^2}{\displaystyle\sum_v d_v f(v)^2} \qquad (2) \\ &= \min_f \max_t \frac{\displaystyle\sum_{u\sim v}(f(u)-f(v))^2}{\displaystyle\sum_v d_v (f(v)-t)^2} \qquad (3) \end{aligned}$$

As we can see, the above formulation for λ_G corresponds in a natural way to the eigenvalues for Riemannian manifolds:

$$\lambda_M = \inf \frac{\displaystyle\int_M \|\nabla f\|^2 dM}{\displaystyle\int_M \|f\|^2 dM}$$

For two vertex-disjoint subsets, say, A and B, of V, let $E(A, B)$ denote the set of edges with one vertex in A and one vertex in B. For a subset $X \subset V$, we define

$$h_G(X) = \frac{|E(X, \bar{X})|}{\min(\sum_{x \in X} d_x, \sum_{y \in \bar{X}} d_y)} \tag{4}$$

where $\bar{X}$ denotes the complement of X. The *Cheeger constant* h_G of a graph G is defined to be

$$h_G = \min_X h_G(X) \tag{5}$$

If we adapt the terminology of differential geometry by viewing a graph as a discretization of a manifold, then $E(X, \bar{X})$ corresponds to the *boundary* of X and $\sum_{x \in X} d_x = vol(X)$ is regarded as the *volume* of X.

For a subset X of vertices of V, we consider

$$N(X) = \{v \notin X : v \sim u \in X\}.$$

We define

$$g_G(X) = \frac{vol(N(X))}{\min(vol(X), vol(\bar{X}))} \tag{6}$$

and

$$g_G = \min_X g_G(X) \tag{7}$$

For regular graphs, we have

$$g_G(X) = \frac{|N(X)|}{\min(|X|, |\bar{X}|)}.$$

We define, for a graph G (not necessarily regular)

$$\bar{g}_G(X) = \frac{|N(X)|}{\min(|X|, |\bar{X}|)}$$

and

$$\bar{g}_G = \min_X \bar{g}_G(X).$$

We note that both g_G and $\bar{g}_G$ concern the vertex expansion of a graph and are useful in some applications.

Lemma 1 **(i)**

$$\sum_i \lambda_i = n$$

(ii) *For a graph G on n vertices,*

$$\lambda_1 \le \frac{n}{n-1}.$$

The equality holds if and only if G is the complete graph on n vertices.

(iii) *For a graph which is not a complete graph, we have $\lambda_1 \le 1$.*

(iv) *If G is connected, then $\lambda_1 > 0$. If $\lambda_i = 0$, G has at least $i+1$ connected components.*

(v)

$$\lambda_i \le 2.$$

The equality holds when G is bipartite.

Proof: (i) follows from considering the trace of $\mathcal{L}$. To see (ii), we consider the following function, for a vertex v_0 in G with the minimum degree,

$$f_1(v) = \begin{cases} 1 & \text{if } v = v_0 \\ 0 & \text{otherwise} \end{cases}$$

By taking $t = d_{v_0}/vol(V)$, we obtain (ii) using (3).

Suppose G contains two nonadjacent vertices a and b, and consider

$$f_2(v) = \begin{cases} d_b & \text{if } v = a \\ -d_a & \text{if } v = b \\ 0 & \text{if } v \ne a, b. \end{cases}$$

(iii) then follows from (2).

(iv) follows from the fact that the union of two disjoint graphs have eigenvalues the union of the eigenvalues.

(v) can be seen as follows:

$$\lambda_i \le \max_f \frac{\sum_{\substack{x,y \\ x \sim y}} (f(x) - f(y))^2}{\sum_x f^2(x) d_x} \le 2.$$

The equality holds when $f(x) = -f(y)$ for every edge $\{x, y\}$ in G. Therefore G is bipartite. On the other hand, If G is bipartitie, we can so choose the function f to make $\lambda_{n-1} = 2$. □

Lemma 2

$$2h_G \geq \lambda_G.$$

PROOF: We choose f based on an optimum cut C which achieves h_G and separates the graph G into two parts, A and B:

$$f(v) = \begin{cases} \dfrac{1}{vol(A)} & \text{if } v \text{ is in } A \\ -\dfrac{1}{vol(B)} & \text{if } v \text{ is in } B \end{cases}$$

By substituting f into (2), we have the following:

$$\begin{aligned} \lambda_G &\leq |C|(1/vol(A) + 1/vol(B)) \\ &\leq \frac{2|C|}{\min(vol(A), vol(B))} \\ &= 2h_G \end{aligned}$$

□

Lemma 3 *Let f denote the eigenfunction achieving λ_G in (2). Then for any vertex $x \in V$, we have*

$$\frac{1}{d_x} \sum_{y, y \sim x} (f(x) - f(y)) = \lambda_G f(x).$$

The proofs follows a variational principle and will be omitted.

Remarks on Laplacians and random walks

One of the most common models for random walks on graphs uses the rule of assigning the weight of a vertex to all its neighbors with equal probability. This stochastic process can be described by the matrix P satisfying

$$Pf(v) = \sum_{\substack{u \\ u \sim v}} \frac{1}{d_u} f(u)$$

for any $f : V(G) \to R$.

It is easy to check that

$$P = I - S\mathcal{L}S^{-1}.$$

Therefore, the Laplacian and its eigenvalues have direct implications for random walks on graphs. Further discussions on Laplacians for weighted graphs will be included in Section 5.

3 The Cheeger inequality for general graphs

In the previous section, we derive a simple lower bound for the cheeger constant by eigenvalues of the Laplacian. In this section, we will give a relatively short proof for an inequality in the other direction so that we have the so-called *Cheeger inequality*

$$2h_G \geq \lambda_2 \geq \frac{h_G^2}{2}$$

The above inequality has appeared in many papers [13], [7], and can be traced back to the paper by Polya and Szego [17]. The Cheeger inequality has been very useful in many applications of random walk type problems for bounding the eigenvalues of the graph.

Theorem 1 *For a general graph G,*

$$\lambda_G \geq \frac{h_G^2}{2}.$$

Proof: We consider an eigenfunction h of $\mathcal{L}$ with eigenvalue λ_G. Let $g = Sh$ and we order vertices of G according to g. That is, relabel the vertices so that $g(v_i) \leq g(v_{i+1})$. Let m denote the smallest value such that

$$\sum_{g(v)<m} d_v \geq \sum_{g(u)\geq m} d_u$$

We define $f(v) = g(v) - m$ and we denote by p the index satisfying $f(v_p) = 0$.

For each i, $1 \leq i \leq |V|$, we consider the cut $C_i = \{\ \{v_j, v_k\} \in E(G)\ :\ 1 \leq j \leq i < k \leq n\ \}$. We define α to be

$$\alpha = \min_{1\leq i\leq\ |V|} \frac{|C_i|}{\min(\sum_{j\leq p} d_j, \sum_{j>p} d_j)}$$

It is clear that $\alpha \geq h_G$. We have

$$
\begin{aligned}
\lambda_G &= \frac{\sum_v \sum_{u\sim v}(f(v)-f(u))f(v)}{\sum_v f^2(v)d_v} \\
&= \frac{(\sum_{u\sim v}(f(u)-f(v))^2)(\sum_{u\sim v}(f(u)+f(v))^2)}{\sum_v f^2(v)d_v(2\sum_v f^2(v)d_v - \sum_{u\sim v}(f(u)-f(v))^2)} \\
&\geq \frac{(\sum_{u\sim v}|f^2(u)-f^2(v)|)^2}{(2-\lambda)\sum_v f^2(v)d_v)^2} \\
&\geq \frac{(\sum_i |f^2(v_i)-f^2(v_{i+1})|\;|C_i|)^2}{(2-\lambda)(\sum_v f^2(v)d_v)^2} \\
&\geq \frac{(\sum_{i<p}(f^2(v_i)-f^2(v_{i+1}))\alpha\sum_{j\leq i}d_j)^2 + (\sum_{i>p}(f^2(v_{i+1})-f^2(v_i))\alpha\sum_{j>i}d_j)^2}{(2-\lambda)(\sum_v f^2(v)d_v)^2} \\
&\geq \frac{\alpha^2(\sum_v f^2(v)d_v)^2}{(2-\lambda)(\sum_v d_v f^2(v))^2} \\
&\geq \frac{\alpha^2}{2-\lambda} \geq \frac{h_G^2}{2-\lambda} \geq \frac{h_G^2}{2}
\end{aligned}
$$

This completes the proof of Theorem 1. □

In fact the proceeding proof yields a slightly stronger result:

Corollary 1 *For a general graph G, we have*

$$\lambda_G \geq 1 - \sqrt{1 - h_G^2/4}$$

4 The isoperimetric inequality for vertex expansion

We first derive a lower bound for g_G in terms of eigenvalues. The following result is an improvement and generalization of isoperimetric inequalities obtained by Tanner [19] and Alon and Milman [2].

Theorem 2 *For any subset X of the vertex set of a graph G, we have*

$$\frac{vol(N(X))}{vol(X)} \geq \frac{1-(1-\lambda')^2}{(1-\lambda')^2 + \frac{vol(X)}{vol(\bar{X})}}$$

where

$$\lambda' = \begin{cases} \lambda_1 & \text{if } 1-\lambda_1 \geq \lambda_{n-1}-1 \\ \dfrac{2\lambda_1}{\lambda_1+\lambda_{n-1}} & \text{otherwise} \end{cases}$$

PROOF: For $X \subset V(G)$, we define

$$f_X(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{otherwise} \end{cases}$$

Let a_i denote the Fourier coefficients of $S^{-1}f_X$, i.e.,

$$S^{-1}f_X = \sum_{i=0}^{n-1} a_i \phi_i$$

where ϕ_i's are eigenfunctions of $\mathcal{L}$ and

$$a_0 = \frac{vol(X)}{\sqrt{vol(V)}}\ ,\ \sum_{1}^{n-1} a_i^2 = \frac{vol(X)\ vol(\bar{X})}{vol(V)}$$

First, we consider the case of $1-\lambda_1 \geq \lambda_{n-1}-1$. For $Y = V - X - N(X)$, we have

$$0 = \langle S^{-1}f_Y,\, (I-\mathcal{L})S^{-1}f_X\rangle = \sum_{0}^{n-1}(1-\lambda_i)a_i b_i$$

where b_i's are Fourier coefficients of $S^{-1}f_Y$. Therefore, we have

$$a_0 b_0 \leq |1-\lambda_1| \sqrt{\sum_{i\neq 0} a_i^2 \sum_{i\neq 0} b_i^2}$$

By substituting for a_0, b_0, we have

$$vol(X)\ vol(Y) \leq |1-\lambda_1| \sqrt{vol(X)\ vol(\bar{X}) vol(Y)\ vol(\bar{Y})}$$

The above inequality can be simplified using the fact that $Y = V - X - N(X)$. We have and we obtain:

$$\frac{vol(N(X))}{vol(X)} \geq \frac{1-(1-\lambda_1)^2}{(1-\lambda_1)^2 + \frac{vol(X)}{vol(\bar{X})}}$$

Suppose $1 - \lambda_1 < \lambda_{n-1} - 1$. We then consider

$$0 = \langle S^{-1} f_Y,\, (I - c\mathcal{L}) S^{-1} f_X \rangle = \sum_{0}^{n-1} (1 - c\lambda_i) a_i b_i$$

where $c = 2/(\lambda_1 + \lambda_{n-1})$. Since $1 - c\lambda_1 = c\lambda_n - 1$, by a similar argument as above we have

$$\frac{vol(N(X))}{vol(X)} \geq \frac{1-(1-\lambda')^2}{(1-\lambda')^2 + \frac{vol(X)}{vol(\bar{X})}}$$

□

As an immediate consequence of Theorem 2 (by using the fact that $(1-\lambda')^2 \leq 1$), we have

Corollary 2 *For any $X \subset V$, we have*

$$\frac{vol(N(X))}{vol(X)} \geq (1-(1-\lambda')^2)(1 - \frac{vol(X)}{vol(V)})$$

In particular, for regular graphs we have

Corollary 3 *For any subset X of vertices in a regular graph, we have*

$$\frac{|N(X)|}{|X|} \geq (1-(1-\lambda')^2)(1 - \frac{|X|}{|V|})$$

Corollary 4

$$2 g_G \geq 1 - (1-\lambda')^2$$

For a general graph G, the eigenvalue λ_G can sometimes be much smaller than $g_G^2/2$. One such example is by joining two complete subgraphs by a matching. Suppose n is the total number of vertices. The eigenvalues λ_G is no more than $8/n^2$,but g_G is large. Still, it is desirable to have a lower bound of λ_G in terms of g_G. Here we give the following proof which is quite similar to the arguments given by Alon [1] for regular graphs.

Theorem 3 *For a graph G,*

$$\lambda_G \geq \frac{g_G^2}{2d(2+2g_G+g_G^2)}$$

where d denotes the maximum degree of G.

PROOF: We follow the definition in the proof of Theorem 1 and we define

$$X = \{v : f(v) \geq 0\}$$

We have

$$\begin{aligned}
\lambda_G &= \frac{\sum_{v\in X}\sum_{u\sim v}(f(v)-f(u))f(v)}{\sum_{v\in X} d_v f^2(v)} \\
&\geq \frac{\sum_{u\sim v, u,v\in X}(f(v)-f(u))^2 + \sum_{u\sim v, v\in X, u\notin X} f(v)(f(v)-f(u))}{\sum_{v\in X} d_v f^2(v)} \\
&\geq \frac{\sum_{u\sim v}(f_+(u)-f_+(v))^2}{\sum_{v} d_v f_+^2(v)}
\end{aligned}$$

where

$$f_+(v) = \begin{cases} f(v) & \text{if } f(v)\geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Now we use the max-flow min-cut theorem [12] as follows. Consider the network with node set $\{s,t\}\cup X\cup Y$ where s is the source, t is the sink, and Y is a copy of $V(G)$. The directed edges and their capacities are given by:

- For every u in X, the directed edge (s,u) has capacity $(1+g_G)d_u$.

- For every $u \in X, v \in Y$ and $\{u,v\} \in E$, there is a directed edges (u,v) with capacity d_v.
- For every $v \in Y$, the directed edge (v,t) has capacity d_v,

It is easy to check that this network has min-cut of size $(1+g_G)vol(X)$. By the max-flow min-cut theorem, there exists a flow function $F(u,v)$ for all directed edges in the network so that $F(u,v)$ is bounded above by the capacity of (u,v) and for each fixed $x \in X$ and $y \in Y$, we have

$$\begin{aligned} \sum_v F(x,v) &= (1+g_G)d_x \\ \sum_v F(v,y) &\leq d_y \end{aligned}$$

Then,

$$\begin{aligned} \sum_{\{u,v\}\in E} F^2(u,v)(f_+(u)+f_+(v))^2 &\leq 2\sum_{\{u,v\}\in E} F^2(u,v)(f_+^2(u)+f_+^2(v)) \\ &= 2\sum_v f_+^2(v)(\sum_{\substack{u \\ \{u,v\}\in E}} F^2(u,v) + \sum_{\substack{v \\ \{u,v\}\in E}} F^2(v,u)) \\ &\leq 2(1+(1+g_G)^2)\sum_v f_+^2(v)d_v^2 \\ &\leq 2d(2+2g_G+g_G^2)\sum_v f_+^2(v)d_v \end{aligned}$$

Also,

$$\begin{aligned} \sum_{\{u,v\}\in E} F(u,v)(f_+^2(u)-f_+^2(v)) &= \sum_u f_+^2(u)(\sum_{\substack{v \\ \{u,v\}\in E}} F(u,v) - \sum_{\substack{v \\ \{u,v\}\in E}} F(v,u)) \\ &\geq g_G \sum_v f_+^2(v)d_v \end{aligned}$$

Combining the above facts, we have

$$\begin{aligned}
\lambda_G &= \frac{\sum\limits_{\{u,v\}\in E} (f_+(u) - f_+(v))^2}{\sum\limits_v f_+^2(v) d_v} \\
&= \frac{\sum\limits_{\{u,v\}\in E} (f_+(u) - f_+(v))^2 \sum\limits_{\{u,v\}\in E} F^2(u,v)(f_+(u) + f_+(v))^2}{\sum_v f_+^2(v) d_v \sum_{\{u,v\}\in E} F^2(u,v)(f_+(u) + f_+(v))^2} \\
&\geq \frac{(\sum_{\{u,v\}\in E} \mid F(u,v)(f_+^2(u) - f_+^2(v)) \mid)^2}{\sum_v f_+^2(v) d_v \;\, 2d(2 + 2g_G + g_G^2) \sum\limits_v f_+^2(v) d_v} \\
&\geq \frac{1}{2d(2 + 2g_G + g_G^2)} \left(\frac{(\sum_{\{u,v\}\in E} F(u,v)(f_+^2(u) - f_+^2(v))}{\sum\limits_v f_+^2(v) d_v} \right)^2 \\
&\geq \frac{g_G^2}{2d(2 + 2g_G + g_G^2)}
\end{aligned}$$

as desired. □

5 Laplacians for weighted graphs

A weighted undirected graph G_π with loops allowed has associated with it a weight function $\pi : V \times V \to \mathbf{R}^+ \cup \{\mathbf{0}\}$ satisfying

$$\pi(u,v) = \pi(v,u)$$

and

$$\pi(u,v) = 0 \text{ if } \{u,v\} \notin E(G) \ .$$

The definitions and results in previous sections can be easily generalized as follows:

1. d_v, the degree of a vertex v of G_π is defined by $d_v = \sum\limits_u \pi(v,u)$
2. For a subset $X \subset V$, the volume of X is denoted by

$$vol(X) = \sum_{v\in X} d_v.$$

3. The Laplacian $\mathcal{L}$ of G_π,

$$\mathcal{L}(u,v) = \begin{cases} 1 - \dfrac{\pi(v,v)}{d_v} & \text{if } u = v \\ -\dfrac{\pi(u,v)}{\sqrt{d_u d_v}} & \text{if } u \neq v \end{cases}$$

Let $\lambda_0 = 0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$ denote the eigenvalues of $\mathcal{L}$. Then

$$\lambda_G = \lambda_1 = \min_f \max_t \frac{\sum_{u,v} (f(u) - f(v))^2 \pi(u,v)}{\sum_v d_v (f(v) - t)^2}$$

Using the above definitions, all statements hold including Theorems 1-3 and Lemmas 1,2. Also, irreducible reversible Markov chains can be represented by weighted graphs and the previous isoperimetric inequalities can be used to derive results on the rate of convergence of Markov chains.

6 The isoperimetric dimension

In the previous sections, we deal with the first eigenvalue $\lambda_1 = \lambda_G$. As it turns out, all the eigenvalues λ_i are related to the following graph invariant:

We say that a graph G has *isoperimetric dimension* δ with an *isoperimetric constant* c_δ if for every subset X of $V(G)$, the number of edges between X and the complement $\bar{X}$ of X, denoted by $|E(X,\bar{X})|$, satisfies

$$|E(X,\bar{X})| \geq c_\delta (vol(X))^{\frac{\delta-1}{\delta}} \tag{8}$$

where $vol(X) \leq vol(\bar{X})$ and c_δ is a constant depending only on δ.

For weighted graphs, we take $|E(X,\bar{X})|$ to be the sum of all $\pi(u,v)$ where u ranges over all vertices in X and v ranges over all vertices not in X.

We note the Cheeger constant can be viewed as a special case of the isoperimetric constant c_δ with $\delta = \infty$.

In a recent paper [9], it is proved that

$$\sum_{i \neq 0} e^{-\lambda_i t} \leq c \frac{vol(G)}{t^{\delta/2}} \tag{9}$$

and

$$\lambda_k \geq c' \left(\frac{k}{(vol(G))} \right)^{\frac{2}{\delta}} \tag{10}$$

for suitable constants c and c' which depend only on δ.

The proofs use the following discrete versions of the Sobolev inequalities.

For any function $f : V(G) \to \mathbf{R}$, we have

(i)

$$\sum_{u \sim v} |f(u) - f(v)| \geq c_\delta \frac{\delta - 1}{\delta} \min_m (\sum_v |f(v) - m|^{\frac{\delta}{\delta-1}} d_v)^{\frac{\delta-1}{\delta}}$$

(ii) For $\delta > 2$,

$$\sum_{u \sim v} |f(u) - f(v)|^2 \geq c_\delta \frac{(\delta - 1)^2}{2\delta^2} \min_m (\sum_v |f(v) - m|^{\alpha} d_v)^{\frac{2}{\alpha}}$$

where $\alpha = \frac{2\delta}{\delta-2}$, and $u \sim v$ means that u and v are adjacent in G.

The proofs will not be included here (see [9]). The techniques have similar flavor as the methods for estimating eigenvalues of Riemannian manifolds which can be traced back to the work of Nash [20]. In a sense, a graph can be viewed as a discretization of a Riemannian manifold in $\mathbf{R}^n$ where n is roughly equal to δ. The eigenvalue bound in (10) is an analogue of the Polya's conjecture [17] for Dirichlet eigenvalues of regular domains M in $\mathbf{R}^n$:

$$\lambda_k \geq \frac{2\pi}{w_n} \left(\frac{k}{volM} \right)^{2/n}$$

where w_n is the volume of the unit disc in $\mathbf{R}^n$.

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