Oblivious and Adaptive Strategies for the Majority and Plurality Problems

Fan Chung[1] *, Ron Graham[1] **, Jia Mao[1] ***, and Andrew Yao[2]

[1] Department of Computer Science and Engineering, University of California, San Diego
[2] Department of Computer Science, Tsinghua University, China

Abstract. In the well-studied *Majority problem*, we are given a set of n balls colored with two or more colors, and the goal is to use the minimum number of color comparisons to find a ball of the majority color (i.e., a color that occurs for more than $\lceil n/2 \rceil$ times). The *Plurality problem* has exactly the same setting while the goal is to find a ball of the dominant color (i.e., a color that occurs most often). Previous literature regarding this topic dealt mainly with adaptive strategies, whereas in this paper we focus more on the oblivious (i.e., non-adaptive) strategies. Given that our strategies are oblivious, we establish a linear upper bound for the *Majority problem* with arbitrarily many different colors. We then show that the *Plurality problem* is significantly more difficult by establishing quadratic lower and upper bounds. In the end, we also discuss some generalized upper bounds for adaptive strategies in the *k-color Plurality problem.*

1 Introduction

The *2-color Majority problem* was first raised by J. Moore in 1982 in connection with problems in the design of fault-tolerant computer systems. (It appeared in an equivalent setting of finding the majority vote among n processors with minimum number of paired comparisons[14]). In the colored-ball setting, we are given a set of n balls, each of which is colored in one of $k \in Z^+$ possible colors $\phi = \{c_1, c_2, ..., c_k\}$. We can choose any two balls a and b and ask questions of the form "Do a and b have the same color?". Our goal is to identify a ball of the *majority color* (i.e., meaning that this color occurs more than half of the time) or determine there is no majority color, using minimum possible number of questions.

We can view this problem as a game played between two players: **Q**, the Questioner, and **A**, the adversary. **Q**'s role is to ask a sequence of queries $Q(a, b) :=$ "Is $\phi(a) = \phi(b)$?". **A** can answer each such query with the hope of extending the game as long as possible before **Q** can finally identify a ball of the majority color or determine there is no majority. For the case when $k = 2$, a number of proofs were given (see Saks and Werman[15], Alonso, Reingold, Scott[5], and Wiener[17]) showing that $n - w_2(n)$ color comparisons are necessary and sufficient in the worst case, where $w_2(n)$ is the number of 1's in the binary representation of n.

* Research supported in part by NSF Grant DMS 0100472 and ITR 0205061
** Research supported in part by NSF Grant CCR-0310991
*** Research supported in part by NSF Graduate Fellowship

Recently, several variants of this problem were also analyzed by Aigner[1]. One natural generalization is the so-called *Plurality problem* where the goal is just to identify a ball of the *dominant color* (i.e., meaning that this color occurs more often than any other color). This variant seems to be more difficult variant and only recently linear upper and lower bounds were given for $k = 3$ colors[2, 3].

In general, two types of strategies can be considered for **Q**. These are *adaptive* strategies in which each query can depend on the answers given to all previous queries, and *oblivious* (or non-adaptive) strategies in which all the queries must be specified before **A** is required to answer any of them. Clearly, in the oblivious case, **A** has more opportunity to be evasive. To the best of our knowledge, except for the case when $k = 2$[1], very little is known about the bounds for oblivious strategies for other variants of the *Majority problem*.

Let $MO_*(n)$ denote the minimum number of queries needed by **Q** in the *Majority problem* for arbitrarily many colors (i.e., k is not fixed) over all Oblivious strategies, $PO_k(n)$ (resp. $PA_k(n)$) the corresponding minimum in the *Plurality problem* for k colors over all oblivious (resp. adaptive) strategies. In this paper we will establish a linear upper bound for $MO_*(n)$ assuming a majority color exists, quadratic bounds for $PO_k(n)$, and also a generalized linear upper bound for $PA_k(n)$.

2 Oblivious strategy for the Majority problem

Consider the case where the number of possible colors k is unrestricted. In principle, this is a more challenging situation for **Q**. At least the upper bounds we have in this case are weaker than those for $k = 2$ colors[9]. A linear upper bound can be shown assuming the existence of a majority color. We also remark that without such assumption, a quadratic lower bound can be proven to be very close to the worst-case $\binom{n}{2}$ upper bound using similar argument as in the proof of Theorem 2 in Section 3. In fact, this quadratic lower bound can also be extended to the general case where the goal is to identify a ball whose color has appeared more than $t \geq \lceil n/2 \rceil$ times.

Theorem 1. *For all n,*

$$MO_*(n) \leq (1 + o(1))27n.$$

assuming a majority color exists.

Proof. Suppose the majority color is c. Let B denote the set of remaining colors $\{c_1, c_2, \ldots, c_R\}$ where R is unknown. Thus, v is not of majority color if and only if $\phi(v) \in B$. We will specify the queries of **Q** by a graph H on the vertex set V, where an edge $\{v_i, v_j\}$ in H corresponds to the query $Q(v_i, v_j) :=$ "Is $\phi(v_i) = \phi(v_j)$?". The edge is colored *blue* if they are equal, and *red* if they are not equal. By a *valid assignment* ϕ on S we mean a mapping $\phi : S \to \{c, c_1, \ldots, c_R\}$ such that:
(i) $\phi(v_i) = \phi(v_j) \Rightarrow \{v_i, v_j\}$ is blue,
(ii) $\phi(v_i) \neq \phi(v_j) \Rightarrow \{v_i, v_j\}$ is red,
(iii) $|\phi^{-1}(c)| > n/2$.

We are going to use certain special graphs $X^{p,q}$, called Ramanujan graphs, which are defined for any primes p and q congruent to 1 modulo 4 (see [13]).

$X^{p,q}$ has the following properties:

(i) $X^{p,q}$ has $n = \frac{1}{2}q(q^2-1)$ vertices;

(ii) $X^{p,q}$ is regular of degree $p+1$;

(iii) The adjacency matrix of $X^{p,q}$ has the large eigenvalue $\lambda_0 = p+1$ and all other eigenvalues λ_i satisfying $|\lambda_i| \leq 2\sqrt{p}$.

We will use the following discrepancy inequality (see [4][8]) for a d-regular graph $H = H(n)$ with eigenvalues satisfying

$$\max_{i\neq 0} |\lambda_i| \leq \delta.$$

For any subset $X, Y \subseteq V(H)$, the vertex set of H, we have

$$\left| e(X,Y) - \frac{d}{n}|X|\ |Y| \right| \leq \frac{\delta}{n}\sqrt{|X|(n-|X|)|Y|(n-|Y|)} \tag{1}$$

where $e(X,Y)$ denotes the number of edges between X and Y.

Applying (1) to $X^{p,q}$, we obtain for all $X, Y \subseteq V(X^{p,q})$,

$$\left| e(X,Y) - \frac{p+1}{n}|X|\ |Y| \right| \leq \frac{2\sqrt{p}}{n}\sqrt{|X|(n-|X|)|Y|(n-|Y|)} \tag{2}$$

where $n = \frac{1}{2}q(q^2-1) = |V(X^{p,q})|$.

Now construct a Ramanujan graph $X^{p,q}$ on our vertex set $V = \{v_1, \ldots, v_n\}$. Let ϕ be a valid assignment of V to $\{c, c_1, \ldots, c_R\}$ and consider the subgraph G of $X^{p,q}$ induced by $\phi^{-1}(c)$ (the majority-color vertices of $X^{p,q}$ under the mapping ϕ).

Claim: Suppose $p \geq 41$. Then G has a connected component C with size at least $c'n$, where

$$c' > \frac{1}{2} - \frac{8p}{(p-1)^2}$$

Proof: We will use (2) with $X = C$, the largest connected component of G, and $Y = \phi^{-1}(c) \setminus X$. Write $|\phi^{-1}(c)| = \alpha n$ and $|C| = \beta n$. Since $e(X,Y) = 0$ for this choice, then by (2) we have

$$\begin{aligned}
(p+1)^2|X|\ |Y| &\leq 4p(n-|X|)(n-|Y|),\\
(p+1)^2\beta(\alpha-\beta) &\leq 4p(1-\beta)(1-\alpha+\beta),\\
\beta(\alpha-\beta) &\leq \frac{4(1-\alpha)p}{(p-1)^2},
\end{aligned}$$

There two possibilities:

$$\beta \geq \frac{1}{2}\left(\alpha + \sqrt{\alpha^2 - \frac{16(1-\alpha)p}{(p-1)^2}}\ \right) \text{ or } \beta \leq \frac{1}{2}\left(\alpha - \sqrt{\alpha^2 - \frac{16(1-\alpha)p}{(p-1)^2}}\ \right)$$

Subcase (a).

$$\begin{aligned}\beta &\geq \frac{1}{2}\left(\alpha+\sqrt{\alpha^2-\frac{16(1-\alpha)p}{(p-1)^2}}\;\right)\\ &> \frac{1}{4}\left(1+\sqrt{1-\frac{32p}{(p-1)^2}}\;\right) \qquad\qquad \text{since } \alpha \geq 1/2\\ &\geq \frac{1}{2}-\frac{8p}{(p-1)^2} \qquad\qquad \text{since } p \geq 37\end{aligned}$$

as desired.
Subcase (b).

$$\begin{aligned}\beta &\leq \frac{1}{2}\left(\alpha-\sqrt{\alpha^2-\frac{16(1-\alpha)p}{(p-1)^2}}\;\right)\\ &\leq \frac{8(1-\alpha)p}{\alpha(p-1)^2}\end{aligned}$$

Thus, we can choose a subset F of some of the connected components whose union $\cup F$ has size $xn = |\cup F|$ satisfying

$$\frac{\alpha}{2}-\frac{4(1-\alpha)p}{\alpha(p-1)^2} \leq x < \frac{\alpha}{2}+\frac{4(1-\alpha)p}{\alpha(p-1)^2} \tag{3}$$

Now we apply the discrepancy inequality (2) again by choosing $X = \cup F$ and $Y = \phi^{-1}(c) \setminus X$. We have

$$\begin{aligned}(p+1)^2 x(\alpha-x) &\leq 4p(1-x)(1-\alpha+x)\\ \text{or } x(\alpha-x) &\leq \frac{4(1-\alpha)p}{(p-1)^2}.\end{aligned}$$

However, it is easily checked that because of (3) this is not possible for $\alpha \geq 1/2$ and $p \geq 41$. Hence, subcase (b) cannot occur. This proves the claim.

Now we prove Theorem 1. Let $n = \frac{1}{2}q(q^2-1)$. Consider an algorithm $\mathbf{Q}$ specified by a graph $H = X^{p,q}$ where $p \geq 53$. We show that a good element can always be identified after all the queries are answered. Suppose we have an arbitrary blue/red coloring of the edges of $X^{p,q}$ with $p \geq 53$, and $\phi : S \to \{c, c_1, \ldots, c_R\}$ is a valid assignment on $V = V(X^{p,q})$. Consider the connected components formed by the blue edges of $X^{p,q}$. By the Claim there is at least one blue component of size at least $(\frac{1}{2}-\frac{8p}{(p-1)^2})n > \frac{1}{3}n$ (since $p \geq 53$). Call any such blue component *large*.

If there is only one large component then we are done, i.e., every point in it must be good. Since $p \geq 53$, there cannot be three large blue components. So the only remaining case is that we have exactly two large blue components, say S_1 and S_2. Again, if either $S_1 \subseteq \phi^{-1}(c)$ or $S_2 \subseteq \phi^{-1}(c)$ is forced, then we are done. So we can assume there is a valid assignment ϕ_1 with $S_1 \subseteq \phi_1^{-1}(c)$, $S_2 \subseteq \phi_1^{-1}(B)$, and a valid assignment ϕ_2 with $S_2 \subseteq \phi_2^{-1}(c), S_1 \subseteq \phi_2^{-1}(B)$ (where we recall that $B = \{c_1, c_2, \ldots, c_R\}$).

Let us write $S_i' = \phi_i^{-1}(c) \setminus S_i, i = 1, 2$. Clearly we must have $A := S_1' \cap S_2' \neq \emptyset$. Also note that $|A| \leq n - |S_1| - |S_2| < \frac{16p}{(p-1)^2} n$.

Define $B_1 = S_1' \setminus A, B_2 = S_2' \setminus A$. Observe that there can be no edge between A and $S_1 \cup S_2 \cup B_1 \cup B_2$. Now we are going to use (2) again, this time choosing $X = A, Y = S_1 \cup S_2 \cup B_1 \cup B_2$. Note that

$$n > |Y| = |\phi_1^{-1}(c)| - |A| + |\phi_2^{-1}(c)| - |A| > n - 2|A|.$$

Since $e(X, Y) = 0$, we have by (2),

$$\begin{aligned} (p+1)^2 |X|\ |Y| &\leq 4p(n - |X|)(n - |Y|), \\ (p+1)^2 |A|(n - 2|A|) \leq &\quad 4p(n - |A|)2|A|. \end{aligned}$$

However, this implies

$$\begin{aligned} (p+1)^2(n - 2|A|) &\leq 8p(n - |A|), \\ \text{i.e.,} \qquad n((p+1)^2 - 8p) &\leq 2|A|((p+1)^2 - 4p) \\ &\leq 2|A|(p-1)^2 \\ &< 32pn \\ (p+1)^2 - 8p &< 32p \end{aligned}$$

which is impossible for $p \geq 41$.

Setting $p = 53$ (so that $X^{p,q} = X^{53,q}$ is regular of degree $p + 1 = 54$), we see that $X^{53,q}$ has $(1+o(1))27n$ edges. This shows that Theorem 1 holds when $n = \frac{1}{2}q(q^2-1)$ for a prime $q \equiv 1 (\text{mod } 4)$.

If $\frac{1}{2}q_i(q_i^2 - 1) < n < \frac{1}{2}q_{i+1}(q_{i+1}^2 - 1) = n'$ where q_i and q_{i+1} are consecutive primes of the form $1 (\text{mod } 4)$, we can simply augment our initial set V to a slightly larger set V' of size n' by adding $n' - n = \delta(n)$ additional good elements. Standard results from number theory show that $\delta(n) = o(n^{3/5})$, for example. Since the Ramanujan graph query strategy of $\mathbf{Q}$ actually identifies $\Omega(n')$ balls of the majority color c from V' (for fixed p) then it certainly identifies a ball of the majority color of our original set V.

This proves Theorem 1 for all n. □

We remark that the constant 27 can be further reduced by using random sparse graphs and applying concentration estimates from probabilistic graph theory. However, such methods can only deduce the existence of a graph with the desired properties whereas we use an explicit construction (Ramanujan graphs) here.

3 Oblivious strategies for the Plurality problem

The *Plurality problem* generalizes the *Majority problem* where the goal is to identify a ball whose color occurs most often or show that there is no dominant color. When there are only $k = 2$ possible colors, the *Plurality problem* degenerates to the *Majority problem* with two colors, and hence there are tight bounds for both adaptive ($n - w_2(n)$) and oblivious ($n - 2$ for n odd, $n - 3$ for n even) strategies[9].

In general, it seems clear that the *k-color Plurality problem* should take more queries than the corresponding *Majority problem.* But exactly how much more difficult it is compared with the *Majority problem* was not so clear to us at the beginning. Similar arguments using concentration inequalities in random graphs seemed possible for achieving a linear upper bound. In the following section, we will prove the contrary by establishing a quadratic lower bound, even for the case when $k = 3$. Also note that the lower bound would remain quadratic even if we assume the existence of a plurality color through slight modification on the proof of Theorem 2. Intuitively, we can say that this is because the existence of a majority color gives us much more information than the existence of a plurality color.

3.1 Lower bound

Theorem 2. *For the Plurality problem with $k = 3$ colors, the number of queries needed for any oblivious strategy satisfies*

$$PO_3(n) > \frac{n^2}{6} - \frac{3n}{2}$$

Proof. Consider any query graph G with n vertices and at most $\frac{n^2}{6} - \frac{3n}{2}$ edges. Therefore there must exist a vertex v with $deg(v) \leq n/3 - 3$. Denote the neighborhood of v by $N(v)$ (which consists of all vertices adjacent to v in G), and the remaining graph by $H = G \setminus (N(v) \cup \{v\})$. Hence,$H$ has at least $2n/3 + 2$ vertices.

Now split H into three parts H_1, H_2, and X where $|H_1| = |H_2| + 1$ and $|X| \leq 1$. Assign color 1 to all vertices in H_1, color 2 to all vertices in H_2, color 3 to all vertices in $N(v)$ and X, and color 1 or 2 to v. Note that based on either one of the two possible color assignments, all query answers are forced.

Since color 3 cannot possibly be the dominant color, we see that whether color 1 is the dominant color or there is no dominant color (because of a tie) solely depends on the color of v, which the Questioner cannot deduce from the query answers.

This proves the lower bound to the number of queries needed for the oblivious strategy is $\frac{n^2}{6} - \frac{3n}{2}$. □

This quadratic lower bound also applies to all $k \geq 3$ colors for the *Plurality problem* using oblivious strategies, since we don't need to use any additional colors beyond 3 for this argument.

3.2 Upper bound

A trivial upper bound is the maximum number of possible queries we can ask, which is $\binom{n}{2}$. In this section we will improve this by showing that for k colors, we have an upper bound of essentially $(1-1/k)\binom{n}{2}$ on $PO_k(n)$. This follows from the following fact.

Fact. Let $p \geq 1-1/k+\epsilon$ where $\epsilon > 0$. Then for n sufficiently large, a random graph G_p on n vertices almost surely has the property that for any subset S of vertices of size at least n/k, the graph $G_p[S]$ induced by S is connected.

Theorem 3. *For every $\epsilon > 0$*

$$PO_k(n) < (1-1/k+\epsilon)\binom{n}{2}$$

provided $n > n_0(\epsilon)$.

Proof. Given $p \geq 1-1/k+\epsilon$ where $\epsilon > 0$, we consider random graphs G_p for sufficiently large n (i.e. $n > n_0(\epsilon)$). Using the above fact, let $x_1 \in S$. Then almost surely, x_1 has a neighborhood of size at least $(1-1/k+\delta)n$ for some fixed $\delta > 0$. Thus, this neighborhood must intersect S, say in the point x_2. Now, look at the neighborhood of $x_1 \cup x_2$. Almost surely, this neighborhood has size at least $(1-1/k+\delta)n$, and so, hits $S\backslash\{x_1, x_2\}$, say in the point x_3. We can continue this process until we reach the set $X_t = \{x_1, x_2, \ldots, x_t\}$ where $t \approx log(n)$. It is clear (by the usual probabilistic arguments) that we can now continue this argument until the growing set X_t becomes all of S. As a result, using this graph, the Adversary cannot avoid forming a blue clique with all the vertices with the dominant color (since there are more than n/k of them). This proves Theorem 3. □

4 Adaptive strategies for the Plurality problem

Aigner et al.[2, 3] showed linear bounds for adaptive strategies for the *Plurality problem* with $k = 3$ colors. In this section, we first note a linear upper bound for general k in this case, and then strengthen it using a generalized argument.

Theorem 4. *For the Plurality problem with k colors where $k \in Z^+$, the minimum number of queries needed for any adaptive strategy satisfies*

$$PA_k(n) \leq (k-1)n - \frac{k(k-1)}{2}$$

Proof. There are k possible colors for the given n balls. We will use k buckets, each for a different possible color. All buckets are empty initially. The first ball s_1 is put in the first bucket b_1. The second ball is compared against a ball from b_1; if they have the same color, it is put in b_1, otherwise, it is put in a new bucket b_2. Similarly, the i^{th} ball has to be compared against a ball from every non-empty bucket (at most $(i-1) \leq k-1$ many of them). Therefore the number of comparisons is no more than

$$1 + 2 + ... + (k-1) + (k-1)(n-k) = (k-1)n - \frac{k(k-1)}{2}$$

□

In [2], it was proved that $PA_3(n) \leq \frac{5}{3}n - 2$. We will now give a generalized proof for all $k \geq 3$ in this setting. It is also known to us through a very recent communication with the authors of [2] that they have independently come up with a different proof for similar claims which will appear in [3].

Theorem 5. *For the Plurality problem with k colors where $k \in Z^+$, the minimum number of queries needed for any adaptive strategy*

$$PA_k(n) \leq (k - \frac{1}{k} - 1)n - 2$$

Proof. Let us denote the comparison of ball a against b by $(a : b)$, and define a *color class* to be a set of balls having the same color. There are two phases in this game. Given n balls $\{s_1, s_2, ..., s_n\}$, in Phase I the Questioner handles one ball at a time (except for the first query) and keeps a state vector v_i after ball s_i is handled. Each v_i is simply the list of color class cardinalities, in non-decreasing order, $(a_{i1}, a_{i2}, ..., a_{ik})$ where $a_{i1} \geq a_{i2}... \geq a_{ik}$. The Questioner also keeps a representative ball from every non-empty color class for comparisons and updates this list whenever there is a change in the state vector.
Claim: At every state, the Questioner has a strategy such that the total number t_i of comparisons up to v_i (inclusive) satisfies

$$t_i \leq (k-1)a_{i1} + (k-2)\sum_{j=2}^{k-1} a_{ij} + (k-1)a_{ik} - 2$$

Proof: We proceed by induction. After the first comparison, $v_2 = (1, 1, 0, ...)$ or $(2, 0, ...)$, so $t_2 = 1 \leq (k-1) + (k-2) - 2 \leq 2(k-1) - 2$ because $k \geq 3$.

For $2 \leq i \leq n$, let $v_i = (a_{i1}, a_{i2}, ..., a_{ik})$ be the state vector and $\{A_{i1}, A_{i2}, ..., A_{ik}\}$ be the set of corresponding representative balls (some may be null if the color class has cardinality 0). Now, ball s_{i+1} is to be handled as follows:

1. If there are no ties in the entries of v_i (i.e., $a_{is} \neq a_{it}$ for all s, t), we will compare s_{i+1} with the middle representative balls first, namely, the comparison order is

   $$(s_{i+1} : A_{i2}), (s_{i+1} : A_{i3}), ..., (s_{i+1} : A_{i(k-1)}), (s_{i+1} : A_1)$$

   with a total number of no more than $(k-1)$ comparisons. Note whenever the Adversary answers *Yes*, we know to which color class s_{i+1} belongs, and hence, we can skip the remaining comparisons.

2. Otherwise, pick any tie, say $a_{ij} = a_{i(j+1)}$, and compare s_{i+1} with all the other representative balls first, namely, the comparison order is

$$(s_{i+1} : A_{i1}), ..., (s_{i+1} : A_{i(j-1)}), (s_{i+1} : A_{i(j+2)}), ..., (s_{i+1} : A_{ik})$$

with a total number of no more than $(k-2)$ comparisons.

After identifying to which color class s_{i+1} belongs, only one of the numbers in v_i gets incremented by 1 and possibly moved forward, to maintain the non-decreasing order in v_{i+1}. Using the above strategy, we can ensure that no more than $(k-2)$ comparisons have been used in this round unless a_{i1} or a_{ik} gets incremented, in which case, their positions in the list do not change. Therefore, by the inductive hypothesis, we have

$$t_{i+1} \leq (k-1)a_{(i+1)1} + (k-2)\sum_{j=2}^{k-1} a_{(i+1)j} + (k-1)a_{(i+1)k} - 2$$

This proves the claim.

At state v_i, let r_i be the number of the remaining balls that have not been involved in any queries. Phase I ends when one of the following happens:

(A) $a_{i1} = a_{i2} = ... = a_{ik}$
(B) $r_i = a_{i1} - a_{i2} - 1$
(C) $r_i = a_{i1} - a_{i2}$

(Note that one of (A), (B), (C) will eventually occur.) To prove the theorem, we use induction where the cases for $n \leq 3$ are easy to verify. More comparisons may be needed in Phase II depending on in which case Phase I ends. If Phase I ends in case (A), we use the induction hypothesis; in case (B), no more comparisons are needed because A_{i1} is a Plurality ball; in case (C), we need no more than r_i more comparisons to identify A_{i1} or A_{i2} as a Plurality ball. In all cases, we can show (using the claim) with arguments similar to those in [2] that

$$PA_k(n) \leq (k-1)n - n/k - 2 = (k - \frac{1}{k} - 1)n - 2$$

.

This proves the theorem. □

5 Conclusion

In this paper, we established upper and lower bounds for oblivious and adaptive strategies used to solve the *Majority* and *Plurality problems*. These problems originally arose from applications in fault tolerant system design. However, the interactive nature of these problems also places them in the general area of interactive computing. It is therefore desirable to develop more techniques to solve this type of problems efficiently as well as to understand the limits of our ability in doing so.

References

1. M. Aigner, “Variants of the Majority Problem”, *Applied Discrete Mathematics,* **137***, (2004), 3-25.*
2. M. Aigner, G. De Marco, M. Montangero, “The Plurality Problem with Three Colors”, *STACS 2004, 513-521.*
3. M. Aigner, G. De Marco, M. Montangero, “The Plurality Problem with Three Colors and More”, *Theoretical Computer Science, to appear, (2005).*
4. N. Alon, “Eigenvalues and Expanders”, *Combinatorica* **6** *(1986), 86-96.*
5. L. Alonso, E. Reingold, R. Schott, “Average-case Complexity of Determining the Majority”, *SIAM J. Computing* **26***, (1997), 1-14.*
6. P. M. Blecher, “On a Logical Problem”, *Discrete Mathematics* **43***, (1983), 107-110*
7. B. Bollobás, “Random graphs”, 2nd ed., *Cambridge Studies in Advanced Mathematics, 73. Cambridge University Press, Cambridge, 2001*
8. F. R. K. Chung, “Spectral Graph Theory”, *CBMS Lecture Notes, AMS Publications, 1997.*
9. F. R. K. Chung, R. L. Graham, J. Mao, and A. C. Yao, “Finding Favorites”, *Electronic Colloquium on Computational Complexity(ECCC) (078): 2003*
10. M. J. Fischer and S. L. Salzberg, “Finding a Majority among n Votes”, *J. Algorithms 3: 375-379 (1982).*
11. D. E. Knuth, *personal communication.*
12. D. E. Knuth, “The Art of Computer Programming, Volume 3. Sorting and Searching”, *Addison-Wesley Publishing Co., Reading, Mass., 1973.*
13. A. Lubotzky, R. Phillips and P. Sarnak, “Ramanujan Graphs”, *Combinatorica* **8** *(1988), 261-277.*
14. J. Moore, “Proposed problem 81-5”, *J. Algorithms 2: 208-210 (1981).*
15. M. Saks and M. Werman, “On Computing Majority by Comparisons”, *Combinatorica* **11***(4) (1991), 383-387.*
16. A. Taylor and W. Zwicker, *personal communication.*
17. G. Wiener, “Search for a Majority Element”, *J. Statistical Planning and Inference 100 (2002), 313-318.*