Integer Sets Containing no Solution to x + y = 3z

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Summary. We prove that a maximum subset of $\{1, 2, \ldots, n\}$ containing no solutions to $x + y = 3z$ has $\lceil \frac{n}{2} \rceil$ elements if $n \neq 4$, thus settling a conjecture of Erdős. For $n \geq 23$ the set of all odd integers less than or equal to n is the unique maximum such subset.

1. Introduction

Many classical problems in computational number theory focus upon subsets S of positive integers with the property that for all x, y, z in S, we have $x+y \neq kz$, for a fixed positive integer k. The history can be dated from 1916, when Schur [5], in work related to Fermat's Last Theorem, proved that the set of positive integers cannot be partitioned into finitely many sum-free sets, i.e., sets having no solution to $x+y = z$. This result is a Ramsey-type theorem which predates Ramsey's Theorem. In 1927, van der Waerden [9] considered subsets having no solution to $x + y = 2z$, or in other words, subsets containing no three-term arithmetic progression. His celebrated theorem states that the positive integers cannot be partitioned into finitely many subsets each of which contains no k-term arithmetic progressions. In 1952, Roth proved that a set of positive upper density contains three-term progressions [4]. This was improved by Szemerédi to four-term progressions [7] and later to the general k-term progressions [8].

A problem which appears in several undergraduate combinatorics texts is to show that a maximum subset of $\{1, \ldots, n\}$ containing no solutions to $x + y = z$ (x, y, z not necessarily distinct) has size $\lceil \frac{n}{2} \rceil$. In section 2 of this paper we find all such subsets and prove the following theorem:

Theorem 1.1. *A maximum subset of $\{1, \ldots, n\}$ containing no solution to $x + y = z$ (x, y, z not necessarily distinct) has size $\lceil \frac{n}{2} \rceil$. If $n \geq 3$ is odd there are precisely two maximum subsets: the odd integers less than or equal to n and $\left\{x \in Z \mid \frac{n+1}{2} \leq x \leq n\right\}$. If $n \geq 4$ is even there are at least three maximum subsets: $\left\{x \in Z \mid \frac{n+1}{2} \leq x \leq n\right\}$ and the two maximum subsets for the odd number $n - 1$. For even $n \geq 10$ these three are the only ones. For smaller even numbers, $\{1, 4\}$, $\{2, 5, 6\}$, $\{1, 4, 6\}$, and $\{2, 3, 7, 8\}$ are the only additional ones.*

Let $f^*(n, 2)$ denote the maximum size subset of $\{1, \ldots, n\}$ containing no three-term arithmetic progressions (such subsets contain no solutions to $x + y = 2z$, but now, for the problem to make sense, x, y, z are distinct). Roth first showed [4] that

$$f^*(n, 2) = O\left(\frac{n}{\log\ \log\ n}\right).$$

The current best bounds are, for appropriate absolute constant c_i,

$$ne^{-c_1\sqrt{\log\ n}} < f^*(n,2) < \frac{c_2 n}{(\log\ n)^{c_3}}$$

where the lower bound was proved by Salem and Spencer [6] (see also Behrend [1]), and the upper bound was proved by Heath-Brown and Szemerédi [3].

Erdős conjectured that a maximum subset of $\{1,\ldots,n\}$ having no solutions to $x+y=3z$ (x,y,z not necessarily distinct) has size no more than a small constant more than $\lceil\frac{n}{2}\rceil$. In section 3 we verify this conjecture by proving the following theorem:

Theorem 1.2. *Let T_n be a subset of $\{1,\ldots,n\}$ of maximum size such that $x+y=3z$ has no solutions with $x,y,z\in T_n$ (x,y,z not necessarily distinct). If $n\neq 4$ then $|T_n|=\lceil\frac{n}{2}\rceil$.*

In section 4 we show that for sufficiently large n there is a unique maximum such subset:

Theorem 1.3. *If $n\geq 23$ and T_n is a subset of maximum size of $\{1,\ldots,n\}$ having no solutions to $x+y=3z$ then T_n is the set of all odd integers less than or equal to n.*

We use the standard notation $\lceil\ \rceil$ and $\lfloor\ \rfloor$ for least integer not less than and greatest integer not greater than, respectively. For a and b nonnegative integers we let $[a,b]$ denote the set of all integers x such that $a\leq x\leq b$.

2. Maximum sum-free sets of $\{1,\ldots,n\}$

Proof (of Theorem 1.1). First we show the maximum size is always $\lceil\frac{n}{2}\rceil$. Let U_n be a maximum sum-free subset of $\{1,\ldots,n\}$ and let p be the largest integer in U_n. Then at most one integer in each of the pairs $(i,\ p-i)$, $i=1,2,\ldots,\lceil\frac{p-2}{2}\rceil$ is in U_n, so $|U_n|\leq\lceil\frac{p}{2}\rceil\leq\lceil\frac{n}{2}\rceil$. Clearly, there are subsets which attain this bound, so $|U_n|=\lceil\frac{n}{2}\rceil$.

To characterize the maximum subsets we consider two cases depending on the parity of n.

Case 2.1 (n odd). Let $n\geq 5$ be the smallest odd integer such that there exists a maximum sum-free subset U_n of $\{1,\ldots,n\}$ which is not the odd integers less than or equal to n or $\left[\frac{n+1}{2},\ n\right]$. Clearly $n\in U_n$ (or else $|U_n|\leq\lceil\frac{n-1}{2}\rceil<\lceil\frac{n}{2}\rceil$), so if $n-1\notin U_n$ then, by the minimality of n, either U_n is the set of all odd integers less than or equal to n (which is impossible by assumption) or $U_n=\left[\frac{n-1}{2},\ n-2\right]\cup\{n\}$ which is impossible since $\frac{n-1}{2}+\frac{n+1}{2}=n$. So we can assume $n-1$ and n are in U_n.

Let G be the graph with vertex set

$V=\left\{v_i\mid i\in\left[2,\frac{n-3}{2}\right]\cup\left[\frac{n+1}{2},\ n-2\right]\right\}$ of size $n-4$ where $\{v_i,v_j\}$ is an edge of G if and only if $i+j=n$ or $i+j=n-1$. Then G is the path $v_{n-2},v_2,v_{n-3},v_3,\ldots,v_{\frac{n-3}{2}},v_{\frac{n+1}{2}}$ (with an odd number of vertices). Since $n-1$ and n are in U_n, 1 and $\frac{n-1}{2}$ are not, so the other $\frac{n-3}{2}$ integers in U_n must be the indices of an independent set of vertices in G (i.e., no two of them adjacent). The only sufficiently large independent set in G is the maximum independent set which has indices $\left[\frac{n+1}{2},\ n-2\right]$, so $U_n=\left[\frac{n+1}{2},\ n\right]$.

Case 2.2 (n even). If $n \geq 4$ is even and $n \notin U_n$, then certainly U_n must be one of the two maximum subsets for the odd integer $n-1$. It is easy to check that the statement in the theorem about when n is 4, 6, or 8 is correct. Let $n \geq 10$ be the smallest even integer such that there exists a maximum sum-free subset U_n of $\{1, \ldots, n\}$ which contains n but is not $\left[\frac{n}{2}+1,\ n\right]$. If $n-1 \notin U_n$, we let $U_{n-2} = U_n \cap [1,\ n-2]$, so that $\mid U_{n-2} \mid = \frac{n-2}{2}$. U_{n-2} cannot be the odd integers less than or equal to $n-3$ because $3+(n-3)=n$. And U_{n-2} cannot be $\left[\frac{n-2}{2},\ n-3\right]$ or $\left[\frac{n}{2},\ n-2\right]$ because $\frac{n}{2} \notin U_n$. So U_{n-2} cannot be any of the three kinds of maximum subsets for even n described in the theorem. So by the minimality of n we would have to have $n=10$ and $U_{n-2} = \{2,3,7,8\}$. This cannot be because $2+8=10$. Hence, as with the odd case, $n-1$ and n are both in U_n.

Now let H be the graph with vertex set $V = \left\{v_i \mid i \in \left[2,\ \frac{n-2}{2}\right] \cup \left[\frac{n+2}{2},\ n-2\right]\right\}$ of size $n-4$ where $\{v_i, v_j\}$ is an edge of H if and only if $i+j=n$ or $i+j=n-1$ or $\{i,j\} = \left\{n-2,\ \frac{n-2}{2}\right\}$. Then H is the cycle $v_{n-2}, v_2, v_{n-3}, v_3, \ldots, v_{\frac{n+2}{2}}, v_{\frac{n-2}{2}}$. Since $n-1$ and n are in U_n, 1 and $\frac{n}{2}$ are not, so the other $\frac{n-4}{2}$ integers in U_n must be the indices of an independent set of vertices in H. There are two possibilities: $\left[2,\ \frac{n-2}{2}\right]$ and $\left[\frac{n+2}{2},\ n-2\right]$. If $n \geq 10$ the first of these cannot occur because it contains 2 and 4 (If $n=6$ the first possibility gives us $\{2,5,6\}$ while if $n=8$ it gives $\{2,3,7,8\}$.). So we have $U_n = \left[\frac{n}{2}+1,\ n\right]$ which completes the proof.

3. The Size of a set with no solution to $x+y=3z$

We observe that if T_n is a set containing no solutions to $x+y=3z$ and if $w \in T_n$, then $\frac{1}{2}w$, $\frac{2}{3}w$, $\frac{3}{2}w$, and $2w$ cannot be in T_n (because x, y, z need not be distinct).

Proof (of Theorem 1.2). The set of all odd integers less than or equal to n has no solutions to $x+y=3z$ so $\mid T_n \mid \geq \lceil \frac{n}{2} \rceil$. It is easy to check that $\mid T_n \mid = \lceil \frac{n}{2} \rceil$ for $n = 1,2,3,5$ (There are three ways to choose T_5: $\{1,3,4\}, \{1,3,5\}$, or $\{1,4,5\}$. The first of these shows that $\mid T_4 \mid = 3$.) So it remains to show $\mid T_n \mid \leq \lceil \frac{n}{2} \rceil$ for $n \geq 6$. Let n be the smallest integer greater than or equal to 6 such that $\mid T_n \mid > \lceil \frac{n}{2} \rceil$. We can assume n is even and $n \in T_n$ (otherwise $n-1$ is a smaller counter-example).

Case 3.1. T_n has no integer x such that $\frac{n}{3} < x \leq \frac{2n}{3}$. By the minimality of n at most $\left\lceil \frac{\lfloor \frac{n}{3} \rfloor}{2} \right\rceil$ of the integers in $\left[1,\ \lfloor \frac{n}{3} \rfloor\right]$ are in T_n provided $\lfloor \frac{n}{3} \rfloor \neq 4$. So

$$
\begin{aligned}
\mid T_n \mid &\leq \left\lceil \frac{\lfloor \frac{n}{3} \rfloor}{2} \right\rceil + \mid \left[\lfloor \frac{2n}{3} \rfloor + 1, n \right] \mid \\
&= \begin{cases} \frac{n}{2} & \text{if } n = 0 \text{ or } 2 \pmod 6 \\ \frac{n}{2} + 1 & \text{if } n = 4 \pmod 6\ . \end{cases}
\end{aligned}
$$

So we are done if $n \neq 4 \pmod 6$ and $\lfloor \frac{n}{3} \rfloor \neq 4$. If $\lfloor \frac{n}{3} \rfloor = 4$ then $n = 12$ or $n = 14$ and the respective candidates for a set of size greater than $\frac{n}{2}$ are $\{1,3,4,9,10,11,12\}$ and $\{1,3,4,10,11,12,13,14\}$. However, neither is acceptable because $1+11 = 3 \cdot 4$.

It remains only to consider $n = 6k+4$ ($k = 1,2,3,\ldots$), in which case the candidate for a counter-example is to choose $k+1$ of the integers in $[1, 2k+1]$ and all the integers in $[4k+3,\ 6k+4]$. If $2k+1 \notin T_n$ then more than half of the first $2k$ integers are in T_n, so $2k=4$, $n=16$, and the candidate is $\{1,3,4,11,12,13,14,15,16\}$

which fails again because it contains $1, 4$, and 11. So $2k+1 \in T_n$ and $6k+3$ is a forbidden sum. Since $[4k+3,\ 6k+2] \subseteq T_n$ it follows that $T_n \cap [1, 2k] = \emptyset$. This is impossible since $k+1$ of the first $2k+1$ integers are in T_n.

Case 3.2. T_n has an integer x such that $\frac{n}{3} < x \le \frac{2n}{3}$.

In fact $x \ne \frac{2n}{3}$ since $n \in T_n$. Assume x is the largest integer in T_n such $\frac{n}{3} < x < \frac{2n}{3}$. Then the integers in $W = [3x-n, n]$ can be arranged in pairs as follows:

$$(3x-n+j,\ n-j) \qquad j = 0, 1, 2, \ldots n - \left\lceil \frac{3x}{2} \right\rceil$$

Since the sum of the integers in each pair is $3x$, at most one integer from each pair can be in T_n.

If x is even then $|W|$ is odd and one of the pairs is $\left(\frac{3x}{2}, \frac{3x}{2}\right)$. In this case T_n contains at most $\frac{1}{2}$ $(|W|-1)$ integers from W and, by the minimality of n, at most $\frac{n-|W|+1}{2}$ integers from $[1, 3x-n-1]$. So $|T_n| \le \frac{n}{2}$.

If x is odd, then $|W|$ is even and at most $\frac{1}{2}|W|$ integers from W can be in T_n. So at most $\frac{1}{2}(n - |W|)$ integers from $[1,\ 3x-n-1]$ can be in T_n provided $3x-n-1 \ne 4$. So we are done except for the possibility that $3x-n=5$. In this case one of the pairs of integers in W is $\left(\frac{3x-1}{2}, \frac{3x+1}{2}\right) = \left(\frac{n+4}{2}, \frac{n+6}{2}\right)$. Since x is the largest integer in T_n which is less than $\frac{2n}{3}$ and since $x = \frac{n+5}{3} < \frac{n+4}{2} < \frac{n+6}{2}$, we must have $\frac{n+6}{2} > \frac{2n}{3}$ (so that one integer in this pair can be in T_n). Solving this gives $n < 18$. Since $n \ge 6$, the only possibilities are $n = 10,\ x = 5$ and $n = 16,\ x = 7$. The first of these is impossible because $n \in T_n$ but T_n cannot contain both 5 and 10. For the second possibility, since $7 \in T_n$ certainly $14 \notin T_n$ so the only candidate is $\{1, 3, 4, 7, 11, 12, 13, 15, 16\}$. But T_n cannot contain 1, 4, and 11 so the proof is complete.

4. Maximum sets with no solutions to $x + y = 3z$

Choosing lots of smaller integers to go into a set T_n which has no solutions to $x+y=3z$ clearly eliminates some of the larger integers from inclusion. If the smaller included integers follow a simple pattern it may be possible to get a simple description of the eliminated larger integers.

Lemma 4.1. *Let w be an odd integer greater than or equal to 3. If T is a set which contains no solutions to $x+y=3z$ and if T contains all odd positive integers less than or equal to w, then T contains no even integer less than $3w$.*

Proof. The result is easy to verify for $w = 3$. If $w \ge 5$ and v is any even number less than $3w$, then (precisely) one of the integers $v+1$, $v+3$, $v+5$ is equal to $3t$ where t is an odd number less than or equal to w.

In proving Theorem 1.3 we will make frequent use of the maximum size subsets of $\{1, \ldots, n\}$ with no solutions to $x+y=3z$ for $n \le 22$. We have calculated them all and display them for even n between 6 and 22 inclusive. To get all such maximum subsets for $n = 2p-1$ for $p = 3,\ 4, \ldots, 11$, just choose the ones for $n = 2p$ which do not include the integer $2p$.

n	T_n	
6	1 3 4	1 5 6
	1 4 5	2 5 6
8	1 3 4 7	1 5 6 8
	1 5 6 7	2 5 6 8
	2 5 6 7	1 6 7 8
	1 4 5 8	2 6 7 8
10	1 3 4 7 10	1 7 8 9 10
	1 3 7 9 10	2 7 8 9 10
	1 4 7 9 10	3 7 8 9 10
12	1 3 4 7 10 12	1 3 9 10 11 12
14	1 3 9 10 11 12 13	1 3 10 11 12 13 14
	1 3 9 10 11 12 14	3 4 10 11 12 13 14
	1 3 4 10 12 13 14	
16	1 3 4 7 10 12 13 16	1 3 4 12 13 14 15 16
	3 4 7 11 12 13 15 16	1 3 11 12 13 14 15 16
	1 3 4 7 12 13 15 16	3 4 11 12 13 14 15 16
	1 3 7 11 12 13 15 16	
18	1 3 4 12 13 14 15 16 17	1 3 4 13 14 15 16 17 18
20	1 3 4 13 14 15 16 17 18 19	1 3 4 14 15 16 17 18 19 20
	1 3 4 13 14 15 16 17 18 20	
22	1 3 4 7 10 12 13 16 19 21 22	1 3 5 15 16 17 18 19 20 21 22
	1 3 4 15 16 17 18 19 20 21 22	1 4 5 15 16 17 18 19 20 21 22

Table 4.1. Maximum subsets of $1, \ldots, n$ with no solutions to $x + y = 3z$ for $6 \le n \le 22$ (except the set of all odd integers less than or equal to n).

Proof (of Theorem 1.3). Suppose the theorem is false and let $n \ge 23$ be the smallest counter-example with T_n a subset of $\{1, \ldots, n\}$ of size $\lceil \frac{n}{2} \rceil$ which contains an even integer. It is easy to see that 23 cannot be added to any of the maximum subsets of $\{1, \ldots, 22\}$ listed in the table without producing a solution to $x + y = 3z$, so $n \ge 24$. By the minimality of the counter-example we can assume n is even and $n \in T_n$. We divide the proof into cases (and subcases) along the lines of the proof of Theorem 1.2.

Case 4.1. T_n has no integer x such that $\frac{n}{3} < x \le \frac{2n}{3}$. Let y be the largest integer in T_n such that $y \le \frac{n}{3}$. Then $3y$ is a forbidden sum and at most one member of each pair $(i, 3y - i)$ $i = 1, 2, \ldots, y$ can be in T_n. Since T_n has no integers strictly between y and $2y$

$$\frac{n}{2} = |T_n| \le y + |[3y, n]| = n - 2y + 1.$$

So, $y \le \lfloor \frac{1}{4}(n+2) \rfloor$ and, since $T_n \cap \left[y + 1, \lfloor \frac{2n}{3} \rfloor\right] = \emptyset$,

$$\begin{aligned} \frac{n}{2} &\le |T_n \cap [1, y]| + \left\lceil \frac{n}{3} \right\rceil \\ &\le \left\lceil \frac{y}{2} \right\rceil + \left\lceil \frac{n}{3} \right\rceil \qquad (if \;\; y \ne 4) \\ &\le \left\lfloor \frac{n+6}{8} \right\rfloor + \left\lceil \frac{n}{3} \right\rceil . \end{aligned}$$

The only even solutions greater than 22 for this inequality are $n = 26,\ 28,\ 34$ with corresponding values $y = 7,\ 7,\ 9$ respectively. If $n = 34$ and $y = 9$ then T_n must contain five of the first nine integers, which (see Table) must be $\{1, 3, 5, 7, 9\}$, and everything in $[23, 34]$. Since $24 + 3 = 3 \cdot 9$ this cannot happen. If $y = 7$ and $n = 26$ or $n = 28$, then T_n contains everything in $[18, 26]$ or $[19, 28]$ respectively. But $7 \in T_n$, so 21 is a forbidden sum, and, since 19 and 20 are in T_n (in both cases), neither 1 nor 2 can be in T_n. Since four of the first seven positive integers must be in T_n this is a contradiction (see Table). If $y = 4$, then $|T_n \cap [1, y]|$ could be as much as 3 in the above inequalities, but $\frac{n}{2} \leq 3 + \lceil \frac{n}{3} \rceil$ has no solutions for $n \geq 24$. So Case 1 cannot occur.

Case 4.2. T_n has an integer y such that

$$\frac{n}{3} < y \leq \frac{2n}{3} \tag{4.1}$$

Let x be the largest integer in T_n satisfying (4.1). Then $3x$ is a forbidden sum and the integers in $W = [3x - n,\ n]$ can be arranged in pairs $\left(\lfloor \frac{3x}{2} \rfloor - i, \lceil \frac{3x}{2} \rceil + i\right)$ $i = 0,\ 1, \ldots, n - \lceil \frac{3x}{2} \rceil$ such that the sum of the integers in each pair is $3x$. If x is even then $\frac{3x}{2}$ is paired with itself. All other pairs (for x even or odd) have distinct integers.

If x is even then, because of the above pairing,

$$|T_n \cap W| \leq \frac{|W| - 1}{2} \tag{4.2}$$

But by Theorem 1.2,

$$T_n \cap [1,\ 3x - n - 1] \leq \left\lceil \frac{3x - n - 1}{2} \right\rceil = \frac{n - |W| + 1}{2} \tag{4.3}$$

Since $\frac{n-|W|+1}{2} + \frac{|W|-1}{2} = \frac{n}{2}$, equality must hold in (4.2) and (4.3). So T_n contains precisely one integer out of each pair of distinct integers above. And $T_n \cap [1,\ 3x - n - 1]$ must be a maximum subset of $[1,\ 3x - n - 1]$ containing no solutions to $x + y = 3z$.

If x is odd then $|W|$ is even, so T_n must contain precisely one integer out of each pair of W and $T_n \cap [1,\ 3x - n - 1] = \frac{3x-n-1}{2}$, unless $3x - n - 1 = 4$. If $3x - n - 1 = 4$ then it would be possible to have $T_n \cap [1, 4] = \{1, 3, 4\}$ and T_n contain precisely one integer from all but one of the pairs and no integer from that one pair.

Subcase 4.21. $\lceil \frac{3x+1}{2} \rceil \leq \frac{2}{3}n$.

If x is even then both integers of the pair $\left(\frac{3x}{2} - 1, \frac{3x}{2} + 1\right)$ are less than or equal to $\frac{2}{3}n$. Since x is the largest integer in T_n which is less than or equal to $\frac{2}{3}n$, neither integer in this pair is in T_n which is a contradiction.

If x is odd, neither integer in the pair $\left(\lfloor \frac{3x}{2} \rfloor, \lceil \frac{3x}{2} \rceil\right)$ can be in T_n, which again is a contradiction unless $3x - n - 1 = 4$. But since $n \geq 24$

$$\frac{3x + 3}{2} = \frac{n + 8}{2} \leq \frac{2}{3}n$$

so neither integer in the pair $\left(\lfloor \frac{3x}{2} \rfloor - 1, \lceil \frac{3x}{2} \rceil + 1\right)$ can be in T_n either, so subcase 2a cannot occur.

Subcase 4.22. Assume the two following inequalities:

$$\left\lceil \frac{3x+1}{2} \right\rceil > \frac{2}{3}n \tag{4.4}$$

and

$$3x - n > 23 \tag{4.5}$$

Since $T_n \cap [1,\ 3x-n-1]$ is a maximum subset of $[1,\ 3x-n-1]$ containing no solutions to $x+y=3z$ and since $23 \le 3x-n-1 < n$, by the minimality of n as a counter-example, $T_n \cap [1,\ 3x-n-1]$ must be the set of all odd integers less than or equal to $3x-n-1$. By Lemma 1, T_n contains no even integer less than or equal to 68, so $n \ge 70$.

If x is even, then inequality (4.4) becomes

$$x > \frac{4n-6}{9}. \tag{4.6}$$

And $3x-n-1$ is odd so, by Lemma 1, T_n contains no even integers less than or equal to $3(3x-n-1)-1$, which by (4.6) is greater than $n-10$. If x is odd then (4.4) becomes

$$x > \frac{4n-3}{9} \tag{4.7}$$

and T_n contains no even integers less than or equal to $3(3x-n-2)-1$ which by (4.7) is also greater than $n-10$. So T_n contains at most five even integers, and hence there are at most five odd integers less than n which are not in T_n. Let $m_i = 2\lceil \frac{n}{6} \rceil + 2i - 1$ and $p_i = 3m_i - n, i = 1, 2, \ldots, 11$. It is easy to check that $\{m_i\}$ and $\{p_i\}$ are each sets of 11 distinct odd integers less than or equal to n and clearly not both m_i and p_i can be in T_n for any $i = 1, \ldots, 11$. Hence there are at least six odd integers less than n which are not in T_n, which shows subcase 2b cannot occur.

Subcase 4.23. Inequality (4.4) holds but (4.5) does not.

Hence

$$\frac{2}{3}n < \left\lceil \frac{3x+1}{2} \right\rceil \le \frac{n}{2} + 12 \tag{4.8}$$

from which it follows that $n \le 70$. So only a finite number of possibilities remain to be checked, and this could be done one by one (by hand or computer). We prefer to avoid this by considering the following possibilities.

4.23(i). Assume (4.8) holds and also assume

$$\left\lceil \frac{n}{3} \right\rceil + 3 \le x \le \frac{n}{2} \tag{4.9}$$

If x is even, inequality (4.8) simplifies to

$$\frac{4n-6}{9} < x \le \frac{n+22}{3} \tag{4.10}$$

And if x is even then, as we showed before, $|T_n \cap [1,\ 3x-n-1]| = \frac{3x-n}{2}$. By inequality (4.9), $3x-n-1 \ge 9 > 5$, so $3x-n-1 \in T_n$. Since $T_n \cap \left[x+1, \lfloor \frac{2n}{3} \rfloor\right] = \emptyset$, we must have $\left[3x - \lfloor \frac{2n}{3} \rfloor + 1, 2x-1\right] \subseteq T_n$, since these integers are each paired with an excluded integer in the pairing of $[3x-n,\ n]$ we discussed before ($3x - \lfloor \frac{2n}{3} \rfloor$ might not be in T_n because it might be equal to $\frac{3x}{2}$). But $3(3x-n-1)$ is a forbidden sum so $T_n \cap \left[7x - 3n - 2,\ 6x - 3n + \lfloor\frac{2n}{3}\rfloor - 4\right] = \emptyset$. Let $a = 7x - 3n - 2, b = 6x - 3n + \lfloor\frac{2n}{3}\rfloor - 4$, and $c = 3x - n - 1$. With x satisfying (4.9) and (4.10) it is easy to check that $a \geq 0$ and $c \geq 9$. Since $T_n \cap [a, b] = \emptyset$ and $c \in T_n$ we certainly have a contradiction if $a \leq c \leq b$. We also have a contradiction if $a \leq b < c$ and $b - a + 1 \geq c - b + 2$ because then

$$\begin{aligned} \frac{c+1}{2} = |T_n \cap [1, c]| &\leq |T_n \cap [1, a-1]| + (c - b) \\ &\leq \frac{a}{2} + c - b \\ &\leq \frac{c-1}{2}. \end{aligned}$$

Hence we have a contradiction if the conditions $a \leq c$ and $2b \geq a + c + 1$ are both satisfied, i.e., if

$$n - \left\lfloor \frac{2n}{3} \right\rfloor + 3 \leq x \leq \frac{2n+1}{4}.$$

But that is precisely our assumption in (4.9).

If x is odd then inequality (4.8) simplifies to

$$\frac{4n-3}{9} < \frac{n+23}{3} \tag{4.11}$$

and the argument is similar. In this case it turns out that $\left[3x - \lfloor\frac{2n}{3}\rfloor, 2x - 1\right] \subseteq T_n$ and that $9x - 3n - 3$ or $9x - 3n - 6$ is a forbidden sum, but we still get a contradiction with inequality (4.9).

4.23(ii). Assume (4.8) holds and $x > \frac{n}{2}$.

If x is even there are eight ordered pairs of values for $n \geq 24$ and x which satisfy (4.8) when $x > \frac{n}{2}$. We list them as triples $(n, x, 3x - n - 1)$:

(24, 14, 17)	(30, 16, 17)
(26, 14, 15)	(32, 18, 21)
(26, 16, 21)	(34, 18, 19)
(28, 16, 19)	(38, 20, 21)

Since $|T_n \cap [1,\ 3x - n - 1]| = \frac{3x-n}{2}$ we see (by the Table) that if $3x - n - 1$ is equal to 21 or 15 then T_n must contain all the odd integers less than or equal to 15 and (by Lemma 1) no even integers at all. That eliminates four triples. Since $3x - n - 1 \in T_n$ and x is the largest integer in T_n less than $\frac{2n}{3}$, we cannot have $x < 3x - n - 1 \leq \frac{2n}{3}$. That eliminates three more triples, leaving only $(24, 14, 17)$. If $3x - n - 1 = 17$, then $15 \in T_n$ (by the Table). But $x = 14 < 15 \leq \frac{2}{3} \cdot 24$, so we get a contradiction here as well.

If x is odd there are ten such triples $(n, x, 3x - n - 1)$:

(24, 13, 14)	(30, 17, 20)
(24, 15, 20)	(32, 17, 18)
(26, 15, 18)	(34, 19, 22)
(28, 15, 16)	(36, 19, 20)
(28, 17, 22)	(40, 21, 22)

We will show just the argument for $(32, 17, 18)$ here. Since $\lfloor\frac{2n}{3}\rfloor = 21$, we must have $T_n \cap [18, 21] = \emptyset$. Since $|T_n \cap [1, 18]| = 9$, by the Table (or by Theorem 2) 17 must be in T_n. Also, $|T_n \cap [22, 32]|$ must be 7. Since 51 is a forbidden sum, at most one integer in each of the pairs (22, 29), (23, 28), (24, 27), (25, 26) is in T_n. By the Table $15 \in T_n$, so $30 \in T_n$, which means $|T_n \cap [22, 32]| \leq 6$ a contradiction. The other nine triples can be disposed of with similar (and mostly simpler) arguments.

4.23(iii). Assume (4.8) holds and $x < \lceil \frac{n}{3} \rceil + 3$.

The only possibility here is $n = 28$ and $x = 12$. Then $3x - n = 8$, so $|T_n \cap [1, 7]| = 4$ and T_n contains precisely one member of each pair $(18 - i, 18 + i)$ $i = 1, 2, \ldots, 10$. Since $\lfloor \frac{2n}{3} \rfloor = 18, T_n \cap [13, 18] = \emptyset$, so $[19, 23] \subseteq T_n$. But 21 is a forbidden sum, so neither 1 nor 2 can be in T_n. This is a contradiction since (by the Table) if $|T_n \cap [1, 7]| = 4$, either 1 or 2 must be in T_n.

5. Related problems and remarks

For k a positive integer not equal to 2, let $f(n, k)$ be the maximum size of a subset S of $\{1, \ldots, n\}$ such that there are no solutions to $x + y = kz$ with x, y, z (not necessarily distinct) integers in S (the problem does not make sense for $k = 2$). In this paper we determined $f(n, 1)$ and $f(n, 3)$ for all n and found all maximum such subsets. The determination of $f(n, k)$ when $k \geq 4$ has a very different flavor, and we have some results in this direction [2].

Let $g(t)$ be the maximum "size" (appropriately defined) of a subset S of the closed interval $[0, 1]$ having no solutions to $x + y = tx$ where t is a fixed positive number. Finding $g(k)$ is a continuous analog of finding $f(n, k)$. It turns out there is a strong connection between these problems if $k \geq 4$, but not for $k = 3$. For $k = 1$ we remark that the maximum set $\left[\lfloor \frac{n+1}{2} \rfloor, n\right]$ of Theorem 1 does have a continuous analog, while the set of all odd integers less than or equal to n does not.

In [2] we show that if $k \geq 3$ then the positive integers can be partitioned into finitely many subsets each of which has no solutions to $x + y = kz$.

For k any positive integer, let $f^*(n, k)$ be the maximum size of a subset of $\{1, \ldots, n\}$ having no solutions to $x + y = kz$ where x, y, z must be distinct. Clearly, $f(n, k) \leq f^*(n, k)$. It is easy to show that $f^*(n, 1) = \lceil \frac{n+1}{2} \rceil$. There are many values of n for which $f(n, 3)$ is smaller than $f^*(n, 3)$. For example, the set $[1, 4] \cup [12, 18]$ shows that $f^*(n, 3) \geq 11$. However, we join Paul Erdős in conjecturing that $f^*(n, 3) = f(n, 3) = \lceil \frac{n}{2} \rceil$ for sufficiently large n. We also suspect there is a unique maximum set for sufficiently large n. One could do some computer work to obtain some information as to the likelihood of this conjecture being correct. To get a proof one could follow the lines of our proofs in this paper. Unfortunately, the minor inconvenience of $f(4, 3)$ being greater than 2 would become the major headache of $f^*(n, 3)$ being greater than $\lceil \frac{n}{2} \rceil$ for many small values of n.

Of course the problem of narrowing the bounds for $f^*(n, 2)$ is one of the most intriguing problems in combinatorics.

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