Lecture Note 9

June 3, 2003

1 Isoperimetric inequalities

Let $G = (V, E)$. Let $S \subseteq V$. Define the vertex boundary, $\delta(S) := v \notin S : v \sim u \in S$ and the edge boundary, $\partial(x, y) \in E : x \in S, y \notin S$.

Problem 1

Given m, find S with $m \leq volS \leq vol\overline{S}$ st $|\partial(S)|$ is minimum.

Problem 2

Given m, find S with $m \leq volS \leq volS^c$ st $|\delta(S)|$ is minimum.

Let $S \subseteq V$

Definition

$$h_G(S) = \frac{|dS|}{vol(S)}$$

$$h_G = \underset{\substack{S \\ volS \leq vol\overline{S}}}{min} h_G(S) = \underset{S}{min} \frac{|\partial S|}{min(volS, vol\overline{S})}$$

$$g_G = \begin{matrix} min \\ S \end{matrix} \frac{vol\delta S}{min(volS, vol\overline{S})}$$

$$\overline{g}_G = \begin{matrix} min \\ S \end{matrix} \frac{|\delta S|}{min(|S|, |\overline{S}|)}$$

$$\overline{S} = \begin{matrix} min \\ S \end{matrix} \frac{|\partial S|}{min(volS, vol\overline{S})}$$

h_G is known as Cheeger's constant.

Cheeger's inequality

For G connected,

$$2h \geq \lambda_1 \geq \frac{h^2}{2}$$

where $h = h_G$ and λ_1 is the smallest nonzero eigenvalue of the Laplacian of a graph.

Proof:

Suppose S achieves $h_G = h_G(S) = \frac{|\partial S|}{volS}$.

$$\lambda_1 = \begin{matrix} inf \\ f \\ \sum f(x)d_x = 0 \end{matrix} \frac{\sum_{x \sim y}(f(x) - f(y))^2}{\sum f^2(x) d_x}$$

Define

$$g(x) = \begin{cases} \frac{1}{volS} & : \quad x \in S \\ -\frac{1}{vol\overline{S}} & : \quad x \notin S \end{cases}$$

Therefore,

$$\begin{aligned} \lambda_1 &= \leq \frac{\sum_{x \sim y}(g(x) - g(y))^2}{\sum g^2(x) d_x} \\ &= \frac{|\partial S|(\frac{1}{volS} + \frac{1}{vol\overline{S}})^2}{volS \frac{1}{volS^2} + vol\overline{S} \frac{1}{(vol\overline{S})^2}} \\ &= \frac{|\partial S| volG}{volS vol\overline{S}} \end{aligned}$$

$$
\begin{aligned}
&= h\frac{volG}{vol\overline{S}} \\
&\leq 2h
\end{aligned}
$$

For every x,

$$\sum_{\substack{y \\ y\sim x}} (f(x) - f(y)) = \lambda_1 f(x) d_x$$

f achieves λ_1.

Arrange vertices $v_0, v_1, \ldots, v_{n-1}$ such that

$$f(v_1) \leq f(v_2) \leq \ldots$$

and

$$\sum_{f(v)\geq o} d_v \geq \frac{volG}{2}.$$

$$c_i = \{\{v_k, v_l\} \in E : k \leq i, l \geq i\}$$

$$\alpha = \begin{matrix} min \\ i \end{matrix} \frac{|c_i|}{min(\sum_{j\leq i} d_j, \sum_{j\geq i} d_j)} \geq h_G$$

Define

$$g(x) = \begin{cases} f(x) & : \quad f(x) \geq 0 \\ 0 & : \quad otherwise \end{cases}$$

$$
\begin{aligned}
\lambda_1 &= \frac{\sum_{v\in V^+} f(v) \sum_{u\sim v} f(v) - f(u)}{\sum_{v\in V^+ f^2(v) d_v}}, V^+ = \{v : f(v) \geq 0\} \\
&\geq \frac{\sum_{\{u,v\}\in E^+} (g(u) - g(v))^2}{\sum_{v\in V^+} g^2(v) d_v}, E^+ = \{edges\ incident\ to\ V^+\} \\
&= \frac{\sum_{\{u,v\}\in E} (g(u) - g(v))^2 \sum_{\{u,v\}\in E^+} (g(u) + g(v))^2}{\sum_{v\in V^+} g^2(v) d_v \sum_{\{u,v\}\in E^+} (g(u) + g(v))^2} \\
&\geq \frac{(\sum \{u,v\} \in E^+ |g^2(u) - g^2(v)|)^2}{\sum v \in V^+ g^2(v) d_v (2 \sum_{v\in V^+} g^2(v) d_v)} \\
&\geq \frac{(\sum_i c_i |g^2(v_i) - g^2(v_{i-1}|)^2}{2 \sum_{v\in V^+} g^2(v) d_v}
\end{aligned}
$$

$$
\begin{aligned}
& (c_i \geq \alpha \sum_{j \leq i} d_j) \\
\geq & \frac{(\sum_i \alpha \sum_{j\leq i} d_j (g^2(v_i) - g^2(v_{i-1})))^2}{2\sum_{v\in V^+} g^2(v) d_v} \\
\geq & \frac{\alpha^2}{2}(\frac{\sum g^2(v_i)d_i}{\sum g^2(v_i)d_i})^2 \\
\geq & \frac{h^2}{2}
\end{aligned}
$$

2

Q_n, 2^n vertices $= \{a_1 \cdots a_n : a_i \in \{0,1\}\}$

Number of edges $= n2^{n-1}$, where $\{a_1 \cdots a_n\} \sim \{b_1 \cdots b_n\} \iff$ the Hamming distance is 1.

$h_{Q_n} = \frac{\partial(S)}{vol(S)}, |S| \leq 2^{n-1}$

Boolean function $f : \{0,1\}^n \to \{0,1\}$

binary strings of length n

$Q^n, S \subseteq V(Q^n)$

$S = \{x : f(x) = 1\}$

x is critical for f in i if $f(x) \neq f(x^{(i)})$

Define $c(f,x) = \{i : f(x) \neq f(x^{(i)})\}$

Critical complexity, $c(f)$ is defined as

$$c(f) = \underset{x}{max}\, c(f,x)$$

Now we can view Q^n as having 2 identical copies of Q^{n-1}, Q_1 and Q_2 and an edge joining a vertex from Q_1 to its copy in Q_2. Then there are 2^{n-1} edges joining Q_1 to Q_2. Hence

$h_{Q_n} \leq \frac{1}{n}$ since $vol(Q_1) = n2^{n-1}$.

Claim: $h_{Q_n} \geq \frac{1}{n}$

To prove the claim we need a lemma.

Let $S \subseteq V(Q^n)$. $G_S : induced\ subgraph\ in\ S.$

Lemma

$$|S| \geq 2^d \ i.e.\ log|S| \geq \frac{\sum_{v \in S} d_S(v)}{|S|}$$

$$d : average\ degree\ in\ G_s$$

Proof:

By induction on $|S|$.

True for $|S| = 1$.

Suppose it is true for $|S| \leq |S|$. Split Q^n into 2 copies of Q^{n-1}, called $Q^{(1)}$ and $Q^{(2)}$.

Let $S_i = S \cap V(Q^{(i)})$, $G_1 = G_{S_1}$.

Let $Z =$ no. of edges between S_1 and S_2, $|S_1| \leq |S_2|$. By induction hypothesis,

$$\begin{aligned} |S_1|log|S_1| &\geq \sum_{v \in S_1} deg_{S_1}(v) \\ &= \sum_{v \in S_1} deg_S(v) - Z \end{aligned}$$

Similarly $|S_2|log|S_2| \geq \sum_{v \in S_2} deg_S(v) - Z$.

Hence $|S_1|log|S_1| + |S_2|log|S_2| \geq \sum_{v \in S} deg_S(v) - 2Z$.

$$\begin{aligned} |S|log|S| &\geq |S_1|log(2log|S|) + |S_2|log(|S_2| + |S_1|) \\ &\geq |S_1|log|S_1 + |S_2|log|S_2| + 2|S_1| \\ &\geq \sum_{v \in S} deg_S(v)\ (Because\ |S_1| \geq Z) \end{aligned}$$

With this Lemma, we can prove the claim.