Math 262A Lecture Notes

Lemma 1: Recall this lemma from the last lecture. Given a forest F on n vertices and a positive real number k with $n \geq k+1$, we can remove a single vertex v from F such that some subforest F' of $F \setminus v$ has size $k \leq |F'| < 2k$.

In particular, we can remove one vertex from a tree on n vertices to obtain a subforest F of size $n/3 \leq F < 2n/3$. Really we can't do any better than this by removing one vertex, as shown by a tree whose root has three subtrees each of size $(n-1)/3$.

Lemma 2: Given a forest F on n vertices, a positive integer w and a positive real number K with $|F| \geq k + w$, there exists a set of w vertices $v_1 \dots v_w$ in F such that some subforest F' of $F \setminus \{v_1 \dots v_w\}$ has size $||F'| - K| \leq K/3^w$.

Proof: If $w = 1$ then let k=(2/3)K. By lemma 1, we can remove a single vertex v such that $F \setminus v$ contains a subforest F' with $\frac{2}{3}K \leq |F'| < \frac{4}{3}K$, which implies $||F'| - k| \leq K/3$.

We proceed by induction. Assume we can remove i vertices from any forest F and find a subforest F_i with $(1 - 1/3^i)K \leq |F_i| < (1 + 1/3^i)$. There are two cases to consider. We may assume that either

$$(1 - 1/3^{i+1})K \leq |F_i| < (1 + 1/3^i) \quad \text{or}$$

$$(1 - 1/3^i)K \leq |F_i| < (1 + 1/3^{i+1})$$

We only consider the first of these cases for now. Case 1: If $(1 - 1/3^{i+1})K \leq |F_i| < (1 + 1/3^i)$, use lemma 1 with $F = F_i$ and $k = 2/3(F_i - K)$. We can remove an additional vertex $v_i + 1$ to obtain a subforest F_i' of F_i such that $2/3(|F_i| - K) \leq |F_i'| < 4/3(|F_i| - K)$. We let $F_{i+1} = F_i - F_i'$ and we have $(1 - 1/3^{i+1})K \leq |F_i| < (1 + 1/3^{i+1})$.

Corollary: You can separate a tree into two equal-sized parts by removing $\lfloor log(n)/log(3) \rfloor + 1$ vertices.

Universal Graphs: Let $\mathcal{F}$ be a family of graphs. A graph H is a universal graph for $\mathcal{F}$ if it contains every graph in $\mathcal{F}$ as a subgraph. In particular we will consider $\mathcal{F}_n = \{T : \text{T a tree on } n \text{ vxs}\}$. We wish to find a graph H with a minimal number of edges which is universal for $\mathcal{F}$.

Here is a recursive way to build a universal graph U_n for $\mathcal{F}_n$. Let U_n consist of a vertex v, a universal graph $U_{\lfloor n/2 \rfloor}$ for $\mathcal{F}_{\lfloor n/2 \rfloor}$, and a universal graph $U_{\lceil 2n/3 \rceil}$ for $\mathcal{F}_{\lceil 2n/3 \rceil}$, with an edge going from v to each of the vertices in $U_{\lfloor n/2 \rfloor}$ and $U_{\lceil 2n/3 \rceil}$. It is clear from lemma 1 that this is a universal graph for $\mathcal{F}_n$.

Excercise: Let $f(n)$ and $g(n)$ be the number of vertices and edges in U_n. Solve the recurrence

$$f(n) \leq 1 + f(n/2) + f(2n/3)$$

$$g(n) \leq f(n/2) + f(2n/3) + g(n/2) + g(2n/3)$$

to get an upper bound for the size of a universal graph for trees on n vertices.

That construction just used Lemma 1. We can do better by using Lemma 2. We can build U_n from w vertices $v_1 \dots v_w$ with all $\binom{w}{2}$ possible edges between them, two universal graphs $U_{\lfloor n/2 \rfloor}$ and $U_{\lfloor \frac{n}{2} + \frac{n}{2*3^w} \rfloor}$, and with all possible edges connecting $v_1 \dots v_w$ to the two smaller universal subgraphs. You should figure out which w gives you the best recursion.

Lemma 3: Let $T_{3,2t}$ be the complete ternary tree with 2t levels. $|V(T_{3,2t}| = (3^{2t} - 1)/2$. How many vertices should we remove to partition the tree into two equal sized parts? I'll let you think about it.

Hint: You don't have to remove more than $log(n)/log(3) - 2log(n)/loglog(n)$ vertices.

I wrote a chapter of a book on this stuff, but it never turned into a book.