AT&T Bell Laboratories

subject: **A Brief Survey of Combinatorics**
Work Project No. 311401-1004
File Case 20878

date: **November 13, 1990**

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Combinatorics
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TECHNICAL MEMORANDUM

Introduction

It is not easy to give a precise description of the scope of combinatorics. Indeed, it may not even be possible. Combinatorics essentially deals with finite or discrete mathematical structures, and various relations between the elements of the structures. Almost every branch of mathematics (and many other subjects, as well) has combinatorial aspects to parts of it, and in fact, instances of combinatorial reasoning are routinely used in everyday life (e.g., one-to-one correspondence, or the box principle). It is not an accident that the words combine and combination have the same root as the word combinatorics.

One basic problem in combinatorics is to determine the number of possible configurations (e.g., arrays, graphs, designs) subject to certain restrictions. Even when the rules specifying the configurations are relatively simple, the

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determination, or even the estimation, of their number can be extremely challenging. In fact, it is often a major problem to decide if any such configurations can exist at all. For example, can there exist a map (of some hypothetical world) which requires five colors for coloring its regions, if adjacent regions must have different colors? (The answer is no, but this problem, known as the Four Color Conjecture, resisted the efforts of mathematicians for more than 125 years before it was finally settled in 1976.)

A major goal of combinatorics is to provide fundamental understanding for various classes of discrete structures, and for their properties and interrelations, and to identify and connect these principles to the broader spectrum of mathematics and its applications.

History

Early developments. Certain types of combinatorial problems have fascinated mathematicians since early times. Magic squares, for example, which are square arrays of numbers (usually consecutive) with the property that the rows, columns and diagonals all add up to the same number (see Figure 1),

4	9	2
3	5	7
8	1	6

Figure 1. The oldest known magic square.

occur in the I Ching, an ancient Chinese book dating back to 2200 B.C. The integer coefficients in the expansion of $(x+y)^n$, known as binomial coefficients, were studied by the 12^{th}-century Indian mathematician Bhaskara, who in his Līlāvati ("The Graceful"), dedicated to a beautiful woman, gave rules for calculating them together with illustrative examples. "Pascal's triangle", an arrangement of the binomial coefficients into a triangular array, had been taught by the 13^{th}-century Persian philosopher Kasir ad-Din al-Tusi (see Figure 2).

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

Figure 2. Pascal's triangle.

In the West, combinatorics is generally considered to have been begun in the 17^{th} century with Blaise Pascal and Pierre de Fermat, both of France, who discovered many classical combinatorical results in connection with their research into games of chance and the theory of probability, although early hints of the subject were evident in the 12^{th}-century writings of Leonardo Pisano (Fibonacci). The term combinatorial was first used in the modern mathematical sense by the German philosopher and mathematician Gottfried Wilhelm Leibniz in his Dissertatio de Arte Combinatoria ("Dissertation Concerning the Combinatorial Arts") in 1666. He foresaw the applications of this embryonic discipline to a whole range of sciences. The Swiss mathematician Leonhard Euler, inspired in part by Leibniz, was finally responsible for the establishment of a school of authentic combinatorial mathematics beginning with the 18^{th}-century. In particular, he became the founder of graph theory with his work on the "Königsberg bridge problem", and his famous conjecture on latin squares was not resolved until 1959. In England, Arthur Cayley, near the end of the 19th-century, made important contributions to enumerative graph theory, and James Joseph Syvester discovered many combinatorial results. The British mathematician George Boole at about the same time used combinatorial methods in connection with the development of symbolic logic. Many combinatorial problems were posed during the 19th-century as purely recreational problems and are identified by such names as "the problem of the eight queens" (in how many different ways can 8 mutually non-attacking queens be placed on a chessboard), the problem of the knight's tour (can a chess knight make a sequence of legal moves on a n by n chessboard and land on each square exactly once) and the Kirkman schoolgirl problem (see the main text). Indeed, the study of triple systems (collections of 3-element sets in which every pair of the elements appears exactly once) begun by Reverend Thomas P. Kirkman in 1847 and pursued by Jakob Steiner, a Swiss-born German mathematician, in the 1850's, arose from the "Kirkman schoolgirl problem" and was the beginning of the whole theory of designs. Among the earliest books devoted exclusively to combinatorics are the German mathematician Eugen Netto's "Lehrbuch der Kombinatorik (1901; "Textbook of Combinatorics") and the British mathematician Percy Alexander Mac Mahon's Combinatory Analysis (1915-16), which provide a view of combinatorics as it existed before 1920.

The modern era of combinatorics. Many factors have contributed to the accelerating pace of development of combinatorics during the past 25 years. Dominant among these was the emergence of the computer as a major force in science and technology. Because the mathematics appropriate for the design and analysis of algorithms used by computers is primarily discrete, as opposed to the more traditional continuous mathematics (e.g., the calculus), increasing attention has been focused on this rapidly developing area, which includes combinatorics at its core. Additional impetus for this shift of emphasis has also been supplied by numerous advances in other disciplines, such as the development of the statistical theory of experimental design, the use and maturation of coding theory in the quest for reliable communications over imperfect channels, and the recognition that graphs are very well suited for modelling a wide variety of problems occurring over the whole spectrum of the physical, natural and social sciences.

As a result of this activity, mathematicians have begun making significant progress in understanding the fundamental principles which underlie the basic mathematical discipline of combinatorics.

Enumeration

Permutations and combinations. Given a standard deck of 52 playing cards, in how many different orders can they be arranged? Since any one of the 52 cards can be chosen to be the bottom card, then any one of the 51 remaining cards can be chosen to be next, and so on, we see that the total number of arrangements is just the product $52 \cdot 51 \cdot 50 \cdot \ldots \cdot 2 \cdot 1$ of the integers from 52 through 1. More generally, if we wish to arrange n objects into a row, the total number of distinct possibilities is the product of the integers from n through 1. This product is denoted by $n!$ and is read "n factorial" (see (1)). Such an arrangement of objects is called a permutation. Mathematical interest in permutations arises primarily from group theory (permutation groups). In this case, the objects to be arranged are simply the numbers $1, 2, \ldots, n$, and a permutation is just a function ϕ such that the values $\phi(1), \phi(2), \ldots, \phi(n)$ are again the numbers $1, 2, \ldots, n$, although perhaps in some other order. Given two permutations ϕ and ρ, one can compose them to form another permutation $\phi\rho$ by defining $\phi\rho(i) = \phi(\rho(i))$. The $n!$ permutations endowed with this operation (composition) form the symmetric group which plays an important role in mathematics, and has been a key catalyst in the development of enumeration. Another simple question concerning our deck of 52 cards asks for the number of different hands of 5 cards a poker player could receive. As before, there are 52 choices for the first card, 51 choices for the second card, ..., and 48 choices for the fifth (and final) card. However, in this case the particular order in which cards were dealt is irrelevant; only the values of the cards matter. Since there are $5! = 120$ different orders in which any 5-card hand might have been dealt, the answer we desire is $\frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{120} = 2598960$. In general, a combination is an unordered choice of k objects from a set of n objects. The number of such combinations is denoted by $\binom{n}{k}$. It can be computed (as above) by the formula given in (2).

If we wish to select an unspecified number of objects from a set of n objects than we have two possibilities with respect to each object — choose it or do not choose it. Thus, the total number of ways to make selection is $2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$.

This leads to (3).

Another early source of interest in permutations and combinations come from analyzing games of chance. This explains in part the strong connection in many peoples' minds between combinatorics and probability theory.

The numbers $\binom{n}{k}$ are usually called binomial coefficients since they occur in the expansion of $(x+y)^n$ (see (4)). One can obtain (3) from (4) by setting $x=y=1$. Binomial coefficients satisfy a large number of identities, many of which were already known to medieval Chinese mathematicians. Perhaps the simplest is (5), which can be proved considering a choice of k objects from a set of n objects, and distinguishing two cases according to which the last object is or is not among the chosen ones.

The identity (6), known as Vandermonde's convolution, is equally simple. Choose k objects from a set of consisting of m red and n blue objects, and distinguish cases according to the number of red objects chosen.

One-to-one correspondence. This is a method in which one establishes a pairing between classes C in C', so that to each element c in C there is a unique element c' in C', and conversely. In this case the number of elements in C must be the same as the number of elements in C'. Let us give a simple example of this useful enumeration technique. Suppose in a lottery six numbers are chosen from the numbers $1, 2, 3, \ldots, 50$. How many choices are there in which no two consecutive numbers are chosen?

Let $a_0, a_1, a_2, \ldots, a_5$ be the numbers listed in increasing order, and form the new set $a_0, a_1 - 1, a_2 - 2, \ldots, a_5 - 5$. Thus, the new set consists of six *distinct* numbers from $1, 2, \ldots, 45$. In the other direction, given six distinct numbers between 1 and 45, say, $1 \le b_0 < b_1 < \cdots < b_5 \le 45$, we can define $a_0 = b_0$, $a_1 = b_1 + 1$, $a_2 = b_2 + 2, \ldots, a_5 = b_5 + 5$, and obtain six numbers between 1 and 50 with no two consecutive. Therefore, we have established a one-to-one correspondence between these two classes, and so the answer to the original question is $\binom{45}{6}$. The general case is given in (7). Similar considerations give (in (8)) the number of choices of k not necessarily distinct integers from $1, 2, \ldots, n$.

Recurrence relations. What is the number of ways of choosing some number (possibly zero) of distinct integers out of $1, 2, \ldots, n$ with no two consecutive? Denote the answer by f_n. If we distinguish two cases according to whether or not the integer 1 was chosen, we find that $f_n = f_{n-2} + f_{n-1}$ for $n \ge 3$. Since $f_1 = 2$ and $f_2 = 3$, we can use this equation to find further values $f_3 = 5$, $f_4 = 8$, $f_5 = 13, \ldots$. The numbers $F_n = f_{n-2}$, known as Fibonacci numbers (after the 12^{th} century mathematician Fibonacci = Leonardo Pisano), occur frequently in combinatorial enumeration. They satisfy the relation $F_n = F_{n-2} + F_{n-1}$ with $F_0 = 0$, $F_1 = 1$. Relations like this in which terms in a sequence are determined by preceding terms in the sequence are called recurrence relations. The most common recurrences are linear recurrences which have the form shown in (9). In this case an explicit formula can be given for the n^{th} term of the sequence determined by the recurrence and the k starting values $a_0, a_1, \ldots, a_{k-1}$. For example, an explicit expression for the n^{th} Fibonacci number F_n is given in (10).

Partitions. A partition of an integer n is a representation of n as a sum of positive integers, $n = x_1 + \cdots + x_k$, $x_i \geq 1$ for $i = 1, \ldots, k$. Partitions are usually written so that $x_1 \geq x_2 \geq \cdots \geq x_k$ since partitions which have the same set of x_i's but in different orders are considered to be the same. In (11) we list the seven partitions of 5. A related concept is that of an *ordered* partition, in which case the order of the x_i's matters.

One can think of generating ordered partitions of n by cutting apart a string of n beads. To obtain k parts, $k-1$ cuts are needed and these cuts can be chosen arbitrarily from among the $n-1$ places between the beads. Thus, the number of ordered partitions of n into k parts is $\binom{n-1}{k-1}$.

The number $P_k(n)$ of (unordered) partitions of n into k parts satisfies the recurrence relation (12). It follows from this that for fixed k, $P_k(n)$ has the asymptotic behavior given in (13).

A useful tool in investigating partitions is the Ferrers diagram. It is obtained from the partition $n = x_1 + x_2 + \cdots + x_k$ by forming a left-justified array of dots in which the i^{th} row has x_i dots. Figure 3 shows the Ferrers diagram for the partition $15 = 5 + 4 + 4 + 2$. By reading the columns of a Ferrers diagram from left to right instead of reading its rows from top to bottom we obtain the conjugate partition $15 = 4 + 4 + 3 + 3 + 1$

```
o o o o o
o o o o
o o o o
o o
```

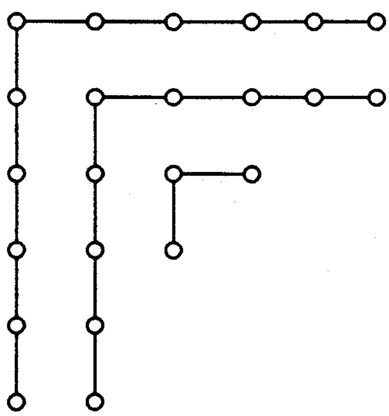
Ferrers diagram for
15=5+4+4+2

```
o o o o
o o o o
o o o
o o o
o
```

The conjugate partition
15=4+4+3+3+1

Figure 3

The number of parts in a partition is equal to the size of the largest part in the conjugate partition, (and vice versa, since the conjugate of the conjugate of a partition is just the original partition). Consequently, the number of partitions of n into k parts is the same as the number of partitions of n in which the largest part is equal to k. A partition is called self-conjugate if it is equal to its conjugate (i.e., if its Ferrers diagram is symmetric). Figure 4 shows how to associate uniquely with any self-conjugate partition, another one in which all parts are odd, thereby proving for any n, the numbers of these two types of partitions are the same.



⟷ 23=11+9+3

Self-conjugate partition of 23=6+6+4+3+2+2

Partition of 23 into distinct odd parts

Figure 4

The number $P(n)$ of (unrestricted) partitions of n is a much studied function in analytic number theory which has many remarkable properties. For example, it was shown by the Indian mathematician S. Ramanujan that: $P(5m+4) \equiv 0$ $(mod\ 5)$, $P(7m+5) \equiv 0$ $(mod\ 7)$, $P(11m+6) \equiv 0$ $(mod\ 11)$. The asymptotic behavior of $P(n)$ was determined by the English mathematician G. H. Hardy and Ramanujan and is shown in (14).

Generating functions. Given a sequence $A = (a_0, a_1, \ldots, a_n, \ldots)$ one can associate with it the algebraic expression $A(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots$, called the (ordinary) generating function for A. Generating functions provide a very compact way for representing sequences and provide one of the most powerful tools available in enumeration. Ignoring questions of convergence, we can write the generating function for the sequence $(0, 1, 2, 3, \ldots)$ as

$$x + 2x^2 + 3x^3 + \cdots = \frac{x}{(1-x)^2}$$

Setting $P(0)=1$, the generating function for the partition $P(n)$ is given in (15). Similarly, the generating function for partitions of n into unequal parts is $(1+x)(1+x^2) \cdots (1+x^k) \cdots$, and the generating function for partitions of n into odd parts is $(1-x)^{-1}(1-x^3)^{-1} \cdots (1-x^{2k-1})^{-1} \cdots$. In view of (16) it follows that these two types of partitions of any integer n are always equal in number.

For the Fibonacci numbers F_n mentioned above the generating function can be easily computed using (10). It is given in (17).

Another commonly occurring sequence in combinatorics is the sequence of Catalan numbers $c_0, c_1, c_2, \ldots, c_n, \ldots$. They can be defined by (18), or by the generating function (19). An example of the occurrence of the c_n is the following. Consider each possible sequence consisting of n A's and $n-1$ B's as a sequence of votes for two candidates A and B in which A wins by a single vote. Then c_n is just the number of such voting sequences in which B never leads.

Sieve methods. In a class of 50 students, suppose 32 like music, 41 like sports and 23 like both. How many students like neither music nor sports? The answer is $50-32-41+23=10$. This formula can be generalized to N students and n subjects and is called the formula of inclusion-exclusion. To describe it, we label the subjects $1, 2, \ldots, n$, and denote by $N(i_1, \ldots, i_s)$ the number of students who like all the subjects $i_1, \ldots, i_s$. The general formula for $\bar{N}$, the number of students liking *none* of the subjects is given by (20). We can express this in words as follows. To compute $\bar{N}$ we must first remove all students disliking each particular subject. However, this will count students disliking two (or more) subjects more than once, so these have to be added back in, and so on. This approach in enumeration is known as sieving and goes back to the Greek mathematician Eratosthenes, who counted the number of primes up to n by sieving out all the numbers divisible by the various primes not exceeding $\sqrt{n}$.

A simple application of (20) occurs in the so-called problem of arrangements. In this problem, the hats checked by n people are returned to them at random. In how many ways can this happen so that no one receives his own hat. The answer D_n is given by (21). Another application of (20) is in determining the number of integers not exceeding n which share no common prime factor with n. This number, called the Euler ϕ function, and denoted by $\phi(n)$, has important applications in number theory and combinatorics. An explicit formula for $\phi(n)$ is given in (22).

Polya's method. Let $c(n, k)$ denote the total number of different necklaces one can make out of n beads having k possible colors. Thus, $c(n, 1) = 1$. The answer, shown in (23), was found by the Hungarian-born U.S. mathematician George Polya and independently by the mathematician J. H. Redfield. Polya's techniques have evolved into one of the important techniques in enumeration.

Permanents. Given an n by n matrix M with general entry m_{ij}, its permanent is defined by (24). Thus, the permanent is analogous to the more familiar determinant, but is usually more difficult to compute. A matrix is called doubly stochastic if all its entries are nonnegative and each row and columns sums to 1. Doubly stochastic matrices often arise in probability theory, the simplest being $\frac{1}{n} \cdot J$, an n by n matrix with all entries equal to $1/n$. In 1926 the German mathematician B. L. van der Waerden conjectured that the inequality (25) always holds. This was proved in 1979 by the Russian mathematician D. P. Falikman. In the following year another Russian mathematician G. P. Egorychev showed that equality holds in (25) only if $M = \frac{1}{n} \cdot J$. His proof used the Alexandrov-Fenchel inequalities for the mixed volumes of convex bodies.

BOX

(1) $n! = n(n-1)\cdot \ldots \cdot 2 \cdot 1;\ \ 1! = 1, 2! = 2, 3! = 6, 4! = 24, 5! = 120$
$52! = 8.0658\ldots 10^{67},\quad 0! = 1$ (by convention).

(2)
$$\binom{n}{k} = \frac{n(n-1)\ \cdots\ (n-k+1)}{k!} = \frac{n!}{k!(n-k!)} = \binom{n}{n-k}$$

(3)
$$2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k}$$

(4)
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

(5)
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

(6)
$$\binom{n+m}{k} = \sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i}$$

(7) $\binom{n-k+1}{k}$ = number of ways of choosing k integers out of $1, 2, \ldots, n$ with no two consecutive

(8) $\binom{n+k-1}{k}$ = number of ways of choosing k not necessarily distinct integers out of $1, 2, \ldots, n$

(9) $a_{n+k} = \alpha_0 a_n + \alpha_1 a_{n+1} + \cdots + \alpha_{k-1} a_{k-1}$, $\alpha_0, \alpha_1, \ldots, \alpha_{k-1}$ are constants.

(10) $F_{n+2} = F_n + F_{n+1}\,,\ F_0 = 0,\ F_1 = 1$ implies

$$F_n = \frac{1}{\sqrt{5}}(\phi^n - \hat{\phi}^n) \text{ where } \phi = \frac{1+\sqrt{5}}{2},\ \hat{\phi} = \frac{1-\sqrt{5}}{2}$$

(11) $5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1$

(12)
$$P_k(n) = P_k(n-k) + P_{k-1}(n-1)$$

(13)
$$\lim_{n\to\infty} P_k(n)/n^{k-1} = 1/k!(k-1)!$$

(14)
$$\lim_{n\to\infty} P(n) \Big/ \left(\frac{1}{4n\sqrt{3}}\, e^{\pi\sqrt{2n/3}}\right) = 1$$

(15)
$$P(0) + P(1)x + \cdots + P(n)x^n + \cdots$$
$$= (1-x)^{-1}(1-x^2)^{-1} \cdot \ldots \cdot (1-x^n)^{-1} \ldots$$

(16)
$$(1+x^a)(1+x^{2a}) \cdot \ldots \cdot (1+x^{2^k a}) \cdot \ldots = (1-x^a)^{-1}$$

(17)
$$F_0 + F_1 x + F_2 x^2 + \cdots = \frac{x}{1-x-x^2}$$

(18)
$$c_n = \frac{1}{2n-1}\binom{2n-1}{n-1},\quad n \ge 1;\quad c_0 = 0$$

(19)
$$\sum_{n\ge 0} c_n x^n = \frac{1}{2}\left(1 - \sqrt{1-4x}\right)$$

(20)
$$\bar{N} = N - \sum N(i_1) + \sum N(i_1, i_2) - \cdots + (-1)^n \sum N(i_1, \ldots, i_n)$$

(21) $D_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)! = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$ = integer nearest to $n!/e$

where $e = \sum_{i=0}^{\infty} \frac{1}{i!} = 2.718 \cdots$ is the base for natural logarithms

(22) $\phi(n) = n \prod_{d} (1 - 1/d)$ where the product is taken over all divisors d of n

(23) $c(n, k) = \frac{1}{n} \sum_{d} \phi(d) k^{n/d}$ taken over all divisors d of n

(24) $perm(M) = \sum_{\pi} \prod_{i=1}^{n} m_{i\pi(i)}$ where the sum is taken over all $n!$ permutations π of $1, 2, \ldots, n$

(25) $perm(M) \geq \frac{n!}{n^n}$ holds for any n by n doubly stochastic matrix M

Graph Theory

Definitions. A graph consists of a set of vertices together with a set of arcs joining some pairs of vertices (see Figure 5). No vertex can be joined to itself and any pair of vertices can be joined by at most one arc. The shape of an arc is

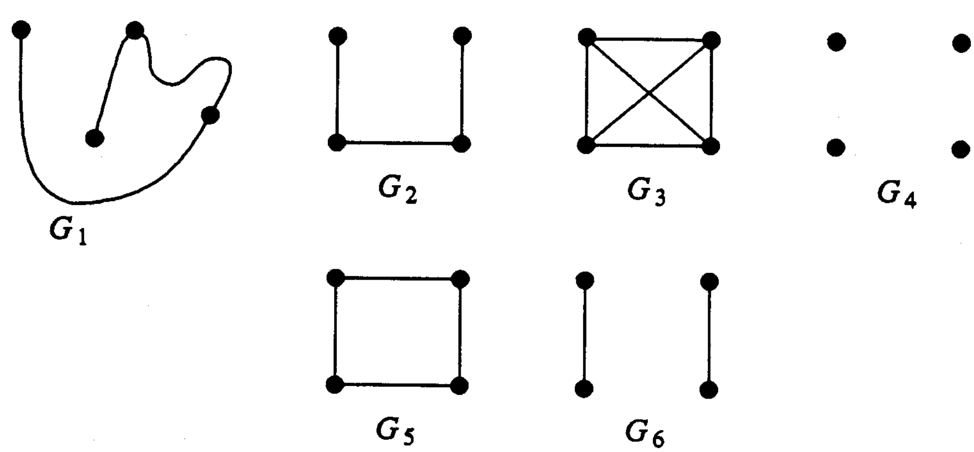


Figure 5. Some simple graphs.

immaterial, and two arcs may intersect. The only consideration in defining a graph is knowing which pairs of vertices are joined by arcs. Such vertices are said to be adjacent to each other. Arcs in a graph are usually called edges. For readers familiar with algebraic terminology, a graph is a non-empty set together with a non-reflexive, symmetric binary relation. If x and y are vertices than the edge joining them is denoted by xy. The sequence of edges x_0x_1, x_1x_2, $x_2x_3, \ldots, x_{n-1}x_n$ is called a path of length n if all the x_i are distinct. Thus, in Figure 3 both G_1 and G_2 represent paths of length 3. For $n \geq 2$, by adding the edge x_nx_1 to the path $x_1x_2, \ldots, x_{n-1}x_n$ forms what is called a cycle of length n, e.g., G_5 is a cycle of length 4. A graph in which all pairs of vertices are joined by edges is called a complete graph. Thus, G_3 is a complete graph on four vertices. Similarly, a graph without any edges is called an empty graph. In Figure 5, G_4 is an empty graph on four vertices.

A subgraph of a graph G is a graph which has a vertex set and edge set which are subsets of the vertex set and edge set, respectively, of G.

If the vertices of a graph can be partitioned into two classes so that all edges join two vertices which belong to different classes then the graph is said to be bipartite. For example, G_2, G_4 and G_5 are bipartite but G_3 is not. It is easy to show that a graph is bipartite if and only if it does not contain any cycle of odd length as a subgraph.

If we omit only edges but no vertices from a graph G, we obtain what is called a spanning subgraph of G. On the other hand, if we omit vertices together with all edges containing them, but no other edges, we obtain what is called an induced subgraph of G.

Much of graph theory is concerned with finding necessary and/or sufficient conditions for the existence of specific subgraphs (spanning or induced) in a graph.

For a graph G we denote by $V(G)$ its set of vertices, and $E(G)$ its set of edges. Also, we let $v(G) := |V(G)|$, $e(G) := |E(G)|$, the numbers of vertices and edges of G, respectively.

The degree of a vertex x is the number of edges ~~containing~~ which contain it. It is denoted by $deg_G(x)$. Formula (1) exhibits a simple relation between the degrees in a graph and its number of edges.

Connectivity and trees. A graph G is called connected if for any two of its vertices x and y, there exists a path $x_0x_1, x_1x_2, x_2x_3, \ldots, x_{n-1}x_n$ in G with $x_0 = x$, $x_n = y$. For example, in Figure 3, G_2 and G_3 are connected but G_4 and G_6 are not. For most graph properties it is useful to consider graphs which have a particular property but which fail to have it as soon as any edge (or vertex) is removed. Such graphs are called edge-critical (or vertex-critical) with respect to the property in question.

If a graph is critically connected, i.e., if the deletion of an arbitrary edge results in a non-connected graph, then it is easy to see that it contains no cycles. Critically connected graphs, that is, connected graph without cycles are called trees (see Figure 6).

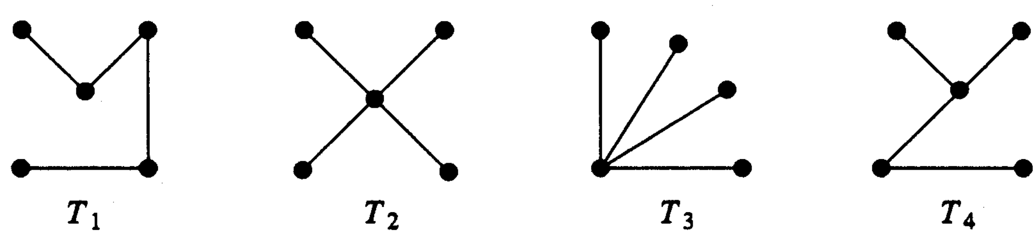


Figure 6. Some simple trees.

Since trees contain no cycles it follows that they must contain vertices of degree 1. Deleting a vertex of degree 1 (and the associated edge) does not destroy connectedness, so by induction we obtain (2). By the definition of criticality every connected graph contains a spanning subgraph which is a tree. Such a subgraph is called a spanning tree. In a tree any two vertices are connected by exactly one path. How many distinct trees are there on the vertex set $\{1, 2, \ldots, n\}$? This

[Name of Otter]

enumeration problem was solved by the British mathematician A. Cayley in 1889 (see (3)). If we consider unlabelled trees, that is, we do not distinguish between T_2 and T_3 then the problem becomes more difficult. Using Polya's method the American mathematician Otter in 1948 determined the generating function for the number of trees on n vertices. In general, two graphs are said to be isomorphic if there is a one-to-one correspondence between their vertex sets which preserves their edges. Thus, in Figure 6, T_2 and T_3 are isomorphic.

A graph G is said to be k-connected if no matter how we delete $k-1$ vertices from G, the remaining graph is connected. Thus, 1-connected just means connected. An important result of the German mathematician K. Menger asserts that in a k-connected graph, any two vertices x and y can be joined by at least k vertex-disjoint paths (i.e., the only vertices common to two of the paths are x and y). Menger's result can be strengthened as follows. For two vertices x and y of G, let us say that their connectivity is k if k is the maximum number of vertex-disjoint paths connecting x and y in G. The strong form of Menger's theorem states that one can always remove k vertices from G so that x and y cannot be connected by a path in the remaining graph (such a set is called a cutset for x and y). It is easy to see that in a k-connected graph, $k-1$ vertices would not be enough for this purpose. Thus, in this sense Menger's theorem is best possible. It also provides an excellent example of what is called a good characterization or min-max theorem, since it characterizes the connectivity through the size of cutsets: the maximum number of disjoint paths is the same as the minimum size of a cutset. This result was used by the American mathematicians L. Ford and D. R.

Fulkerson to build their theory of flows in networks, which is one of the important applications of graph theory. Another useful property of k-connected graphs for $k \geq 2$, proved by the mathematician G. Dirac, is that for any k vertices there is always a cycle in G containing all of them. Actually, for $k=2$ the above property characterizes 2-connected graphs. That is, if in a graph any two vertices are on some cycle then the graph is 2-connected.

Let G be a k-connected graph on $\{1, 2, \ldots, n\}$ and suppose $n = n_1 + n_2 + \cdots + n_k$. Then, as it was shown by the Hungarian mathematicians E. Györy and L. Lovàsz, $\{1, 2, \ldots, n\}$ can be partitioned into k sets of respective sizes $n_1, n_2, \ldots, n_k$ such that each set spans a connected subgraph of G.

We have defined trees as critically 1-connected graphs and we have seen that every tree has a vertex (in fact, at least two) of degree 1. The German mathematician W. Mader proved this holds more generally for arbitrary k (see (4)).

An edge joining two otherwise non-consecutive vertices of a cycle is called a chord. A cycle without a chord is called chordless. The American mathematician M. Plummer characterized critically 2-edge-connected graphs by showing that a 2-connected graph is critical if and only if all its cycles are chordless.

Matchings. If G is a graph then $\nu(G)$ denotes the matching number of G, that is, the maximum number of pairwise disjoint edges in G. It is obvious that $2\nu(G) \leq v(G)$ must hold. A collection of pairwise disjoint edges is called a matching. If a matching contains all the vertices of G it is called a perfect matching. When does a graph contain a perfect matching? This question was first answered for bipartite graphs by the Hungarian mathematician D. König, and later for all graphs by the British born Canadian mathematician W. T. Tutte. For a graph G define $\tau(G)$ as the minimum size of a subset Z of the vertices whose deletion results in an empty graph. Thus, every edge of G meets Z.

The bound $\nu(G) \leq \tau(G)$ is easily seen to hold for all graphs G. For a cycle C of length $2l+1$, one has $\nu(C) = l < l+1 = \tau(C)$. König's theorem asserts that for bipartite graphs one always has equality (see (5)). Given a collection $A_1, A_2, \ldots A_n$ of not necessarily distinct sets with union $A_1 \cup \cdots \cup A_n = \{x_1, \ldots, x_n\}$, one can make a bipartite graph B with vertex set $\{A_1, \ldots, A_n\} \cup \{x_1, \ldots, x_n\}$ by joining x_i with A_j if and only if x_i is an element of A_j. Clearly $\nu(B) \leq n$, and if one has equality then the collection of sets is said to possess a system of distinct representatives (or SDR, for short). Indeed, let $A_1 x_{i(1)}, \ldots, A_n x_{i(m)}$ be a matching of size n. Then $x_{i(j)} \in A_j$ and $x_{i(1)}, \ldots, x_{i(n)}$ are all distinct.

If a collection $A_1, \ldots, A_n$ has a SDR then clearly for all $1 \leq j_1 < j_2 < \cdots < j_r \leq n$, $|A_{j_1} \cup \cdots \cup A_{j_r}| \geq r$ holds. On the other hand, the British mathematician P. Hall proved that this condition is sufficient to guarantee the existence of an SDR. One can deduce this from (5) as well.

Let us call a graph d-regular if each of its vertices has degree d. A 1-regular graph is simply a perfect matching, and a 2-regular graph is the disjoint union of cycles.

/Petersen's criticals

The above results imply that a d-regular bipartite graph contains a perfect matching. Removing it from the graph, we are left with a $(d-1)$-regular bipartite graph which (for $d \geq 2$) again contains a perfect matching. Continuing this process we obtain (6).

For non-bipartite graphs the situation is more complex. Let $\Delta(G)$ denote the maximum of all degrees of vertices of G. Then in place of (6) one has the important result (7) proved by the Russian mathematician V. G. Vizing.

If G is a graph then one can define a relation $\sim$ on the vertices of G by setting $x \sim y$ if and only if $x = y$ or there is a path in G starting with x and ending with y. Thus, $x \sim y$ implies $y \sim x$, $x \sim x$, and $x \sim y \sim z$ implies $x \sim z$. Such a relation is called an equivalence relation. It decomposes the vertex set of G into non-empty subsets, each inducing a connected subgraph of G. These are called the connected components of G (some of which might consist of a single vertex). Let odd (G) denote the number of connected components of G consisting of an odd number of vertices.

For a subset X of $V(G)$ let $G \backslash X$ be the induced subgraph of G on $V(G) - X$. If G has a perfect matching then odd $(G \backslash X) \leq |X|$ must hold for all subsets X of $V(G)$. For example, if X is empty then the condition means that if G has a perfect matching then $\nu(G)$ is even. Tutte proved that this condition is in fact sufficient to ensure the existence of a perfect matching in G. The French mathematician C. Berge extended this result to show (8), which is also another example of a good characterization. The result (9) of the Danish mathematician Petersen is probably the oldest in matching theory. It can be deduced from Tutte's theorem.

A graph is called factor-critical if it has no perfect matching but after the deletion of any vertex it contains a perfect matching. The Hungarian mathematician T. Gallai has obtained important results on factor-critical graphs, extending Tutte's theorem.

Independent sets. A subset of the vertices of a graph is called independent if no two elements of it are connected by an edge, i.e., if these vertices span an empty graph. The independence number $\alpha(G)$ of a graph G is the maximum size of an independent subset of vertices of G. No effective method for determining $\alpha(G)$ is known. However, there many interesting results involving $\alpha(G)$.

A graph is called α-critical if $\alpha(G)$ increases after the deletion of any edge of G.

One can show that the only α-critical bipartite graphs are matchings. On the other hand every graph is an induced subgraph of some α-critical graph.

A vertex of a graph is called isolated if it has degree 0. An α-critical graph without isolated vertices must satisfy $v(G) \geq 2\alpha(G)$.

For a graph G let $\bar{G}$ denote its complement, that is $V(G) = V(\bar{G})$ and two vertices are adjacent in $\bar{G}$ if and only if they are not adjacent in G. Thus, $e(G) + e(\bar{G}) = \binom{v(G)}{2}$ holds.

The clique number $w(G)$ is the maximum size of a set of vertices spanning a complete subgraph of G. Clearly $w(G) = \alpha(\bar{G})$ always holds.

For a graph G, let $\bar{d}(G)$ be its average degree, that is, $\bar{d}(G) = (\sum_x d_G(x)/v(G)$.

Obviously, $\bar{d}(G) \leq \Delta(G)$ with equality if and only if G is $\Delta(G)$-regular. By induction one can easily prove that $\nu(G) \leq \alpha(G)(1+\Delta(G))$ holds. Turán proved the stronger inequality (10). Here equality holds if and only if G is the vertex disjoint union of $\alpha(G)$ complete graphs of size $1 + \Delta(G)$.

A graph which contains no K_3 as a subgraph is called triangle-free. For triangle-free graphs the American mathematician J. Shearer – using some ideas of the Hungarian mathematicians M. Ajtai, J. Komlòs and E. Szemerédi – proved the stronger inequality (11).

Colorings. A coloring of a graph G is an assignment of colors to its vertices in such a way that adjacent vertices get distinct colors. The minimum number of colors needed is called the chromatic number of G and is denoted by $\chi(G)$. Note that this is just the minimum number of independent sets whose union is $V(G)$.

In 1852 the English geographer Francis Guthrie noted that one can color with 4 colors the counties on a map of England so that adjacent counties have distinct colors. He wondered whether this was true for all maps. At first sight this is not a graph coloring problem. However, by associating with each county a vertex (say, its capital) and joining two vertices if the two counties are adjacent, one obtains a graph. Such a graph is called a planar graph because one can show that the edges can be drawn on a plane in such a way that they do not intersect except at the vertices. The above problem became known as the Four Color Problem.

Research on this problem was responsible for much of the early development of graph theory. In 1879 A. B. Kempe thought he had found a proof for (12) and his proof was generally accepted until in 1890 Heawood showed that it was incomplete. However, it was good enough to show that 5 colors are sufficient. Finally, in 1976, K. Appel and W. Haken found a complete proof of (12). Their proof relies not only on some of the ideas of Kempe but also on heavy use of a large computer in showing the reducibility of some 2000 configurations arising in the coloring process.

Various attempts to prove (12) have lead to new areas in chromatic graph theory. In 1912 G. D. Birkhoff introduced the chromatic polynomial $p_G(x)$ of a graph. It is a polynomial of degree $v(G)$ and its value for $x = k$, a positive integer, is the number of colorings of G by k colors. Tutte succeeded in deducing various properties of graphs from the chromatic polynomial and its generalizations.

A homomorphism of a graph G into another graph H is a map $\phi: V(G) \to V(H)$ such that whenever x and y are adjacent then so are $\phi(x)$ and $\phi(y)$. Thus, a graph G has a homomorphism into a complete graph on k vertices, which is denoted by K_k, if and only if $\chi(G) \leq k$ holds. The most well known unsolved problem in chromatic graph theory is a conjecture (13) of the German mathematician Hadwiger. Actually, (13) would imply (12).

A simple inequality for the chromatic number is expressed by the inequality (14). Equality holds, for example when $G = K_l$ or G is a cycle of odd length. The following strengthening of (14) was proved by Hajnal and Szemerédi. If $v(G) = st$, $\Delta(G) < t$ then G can be colored by t colors so that each color is used s times.

(14) was sharpened by the mathematician R. L. Brooks (see (15)). Another simple but useful inequality is expressed by (16).

A graph is G called k-critical if $\chi(G) = k$ but its chromatic number decreases after the deletion of an arbitrary edge. The only 2-critical graph is K_2. The 3-critical graphs are the cycles of odd length. There is a much larger variety of k-critical graphs for $k \geq 4$. The simplest 4-critical graphs are the odd wheels W_{2l+1} (see Figure 7).

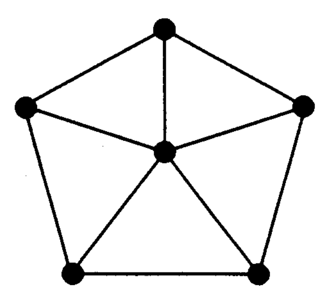


Figure 7. The wheel W_5

The k-critical graphs were first studied by G. Dirac in 1952, who showed (17).

An interesting inequality involving the independence number for vertex k-critical graphs (i.e., the deletion of any vertex decreases $\chi(G)$) was shown by Lovász (see (18)).

The determination of the chromatic number of a given graph is usually a very difficult problem. Let us define the graph $K(n, k, t)$ as a graph whose vertices are the k-element subsets of $[n]$ with two being joined if they overlap in fewer than t elements ($n \geq k \geq t \geq 1$). For example, $K(n, k, k)$ is simply $K_{\binom{n}{k}}$ while $K(2k, k, 1)$ is a perfect matching on $\binom{2k}{k}$ vertices. In 1955 the German mathematician M.

Kneser noted that $\chi(K(n, k, 1)) \le 1+\chi(K(n-1, k, 1))$ holds for all $n > 2k$ and he conjectured that equality (19) holds. This was proved 20 years later by Lovász using a deep algebraic topological argument. I. Bárány subsequently gave a simpler proof. The chromatic number of $K(n, k, t)$ for $t \ge 2$ was asymptotically determined by the Hungarian-born mathematician P. Frankl. What makes the chromatic number large? The inequality $\chi(G) \ge w(G)$ is obvious. However, it is not too hard to construct triangle-free graphs with arbitrarily high chromatic numbers. As a matter of fact, the Czech mathematician V. Rödl proved the much stronger statement (20).

The girth $g(G)$ of a graph G is the minimum integer l such that C_l, the cycle of length l, is a subgraph of G. (21) is a celebrated result of Paul Erdös. As much as Euler can be considered the father of combinatorics, the Hungarian mathematician Erdös is very much responsible for the rapid development of the field since the 1950's. He has written over 1000 research papers and his problems — which are not less numerous — have attracted many young mathematicians to combinatorics.

In a sense complementing (21) is (22), a result — conjectured by Erdös — and proved by H. A. Kierstead, Szemerédi and W. T. Trotter, shows that if G has large chromatic number and contains no short odd cycles, then it has many vertices.

Since bipartite graphs have chromatic number 2 it is interesting to know how far k-chromatic graphs must be from bipartite. Erdös has shown that from an arbitrary graph one can create a bipartite graph by omitting fewer than half the edges. In the other direction, Rödl and Z. Tuza showed that if G is k-critical and $v(G)$ is sufficiently large, then one has to omit at least $\binom{k-1}{2}$ edges to make G bipartite, and there are graphs for which this bound is attained.

Perfect graphs. As mentioned earlier, $\chi(G) \geq w(G)$ holds for any graph G. If equality holds for a graph and *all* its induced subgraphs then the graph is said to be perfect. Bipartite graphs and unions of complete graphs are perfect. On the other hand odd cycles of length 5 or more and their complements are not. An old conjecture of Berge (see (22)) which motivated most of the research on perfect graphs says that these are the only graphs which account for non-perfectness. Clearly (23) imposes the same condition on G as on $\bar{G}$. Indeed, Lovász proved (24). Lovász also proved another characterization of perfect graphs (see (25)).

There are numerous classes of graphs which are known to be perfect. These include chordal graphs (every cycle of length four or more has a chord), and comparability graphs (these graphs arise from partially ordered sets and we shall define them in a later section).

Euler tours and Hamiltonian cycles. The first acknowledged theorem in graph theory is credited to Euler in 1736. As the story goes, he was living in the small town of Konigsberg (now called Kaliningrad) in which there were seven bridges connecting two islands and the banks of the river (see Figure 8). Euler showed that as the townspeople had always suspected, it was not possible to find a walk which crossed each bridge exactly once. One can model the situation using a graph, or more precisely, a multigraph, since several pairs of vertices are joined by more than one edge (see Figure 8). The four vertices will be the two islands and two river banks. The seven edges will correspond to the seven bridges. In the terminology of graphs, what we are looking for in general is a list of all the edges $x_1y_1, x_2y_2, \ldots, x_{l(G)}y_{l(G)}$ of a graph G so that $y_i = x_{i+1}$ for $1 \leq i < l(G)$. Such a list is called an Euler trail. If the trial is closed, i.e., if $y_{l(G)} = x$, then it is called an Euler tour. It is not hard to see that if G has an Euler tour then it must be connected and all its degrees must be even. Euler proved that these conditions are also sufficient (see (20)).

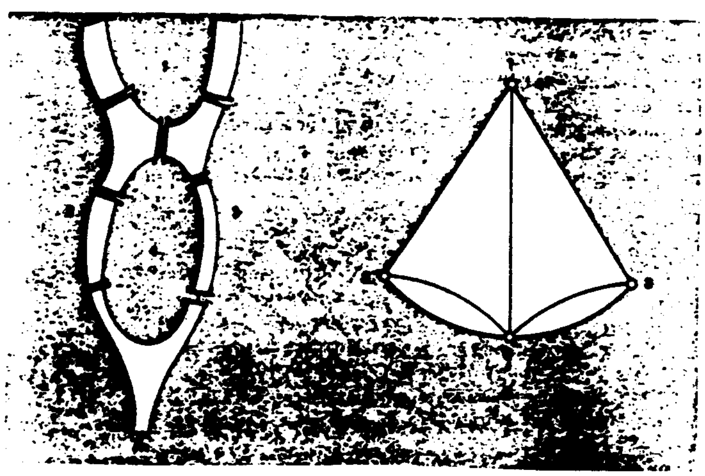


Figure 8. The bridges of Königsberg.

Suppose instead of traversing all the edges of G we wish to find a cycle which passes through each vertex of G exactly once. Such cycles are called Hamiltonian cycles, named after the well known British physicist W. R. Hamilton, who was particularly fond of this problem for the vertices and edges of a regular dodecahedron. There is no generally effective method known for finding

Hamiltonian cycles in a graph, or even deciding if one exists. and it is generally believed that there is no good characterization for graphs admitting Hamiltonian cycles. A useful necessary condition is given by (27). In the other direction, if $\delta(G)$ denotes the minimum degree of any vertex of G, then Dirac gave the sufficient condition (28) for quaranteeing the existence of a Hamiltonian cycle.

Reconstruction. Suppose we are given a list of all the (unlabelled) induced subgraphs of some graph. Is it always possible to decide in principle what the original graph was which generated the list? This problem, proposed by the Polish born American mathematician S. M. Ulam, is still unsolved, although it is generally believed that the answer is affirmative.

For the related version in which edges are deleted from G to form the list (rather than vertices), more is known. Call a graph edge-reconstructible if it is uniquely determined by the list (with multiplicities) of its edge-deleted subgraphs. The Czech mathematician V. Müller proved (29) that if G has relatively many edges than it is edge-reconstructible. L. Pyber showed that if G has a Hamiltonian cycle then it is edge-reconstructible.

Embeddings. The Four Color Conjecture (see (12)) was the genesis of a whole area of research in graph theory, namely, the embeddability of graphs into surfaces. Given a graph G and a surface S, we say that G can be embedded into S if we can represent the vertices of G by points in S and the edges of G by non-intersecting curves in S joining these points. When S is the plane then we obtain exactly planar graphs, as defined earlier.

If we have an embedding of a graph G into a surface S we can form a new graph G' (also embedded into S) by arbitrarily replacing edges of G by various paths between vertices of G (see Figure 9). In this case, G' is said to be a subdivision of G, and is called a topological G. If a graph G contains a subgraph which is a subdivision of a graph G', then G' is said to be a minor of G. Let $K_{m,n}$ denote the bipartite graph with m and n vertices in the two classes and all mn edges joining pairs of vertices in the two classes. In 1930, the Polish mathematician K. Kurátowski gave a characterization of planar graphs in terms of excluded minors (see (30)). A celebrated result of ~~the American mathematicians~~ N. Robertson and P. Seymour from 1985 asserts that for every graph property P which is minor-closed, i.e., if G has property P then so does every minor of G, there always exist a finite list $L(P)$ of graphs so that an arbitrary graph has property P if and only if it contains no minor in $L(P)$. However, for most properties the number of excluded minors is very large. As a consequence, this implies that for any surface S, graphs embeddable in S are characterized by a finite (though large) set of excluded minors.

Figure 9. K_4 and a topological K_4

Another attack on the Four Color Conjecture was made by the Hungarian mathematician L. Hajós who in 1961 conjectured that if $\chi(G) \geq k$ then G contains K_k as a minor. This was disapproved, however, by P. Catlin in 1979 for general k, although it is true for $k = 2, 3$ and 4.

It is known from topology that an oriented surface S is characterized by its genus $\gamma(S)$, which is a non-negative integer. For example, the genus of plane is 0, and the genus of the surface of a torus (doughnut) is 1. In 1890, Heawood raised the following question: What is the maximum chromatic number $\chi(S)$ of all graphs which can be embedded into S. He claimed a proof of (31) although it turned out that only his proof of the upper bound was correct. To complete the proof it was necessary to show that one can embed $K_{H(\gamma(S))}$ into S. This was finally achieved by G. Ringel and J. W. T. Youngs in 1968 for all but three values of γ. The remaining cases were settled by Mayer in the following year.

We remark that the proof of the upper bound in (31) is rather straightforward — one shows that if G is embeddable into S then $\Delta(G) < H(\gamma(S))$ must hold and uses (14). For example, for the torus T one finds from (31) that $\gamma(T) = 7$. Figure 10 illustrates an embedding of K_7 into T.

which join distinct X_i's.

We shall say that $ex(n, \boldsymbol{H})$ is of the order n^α if there exist positive constants a and b satisfying $an^\alpha \le ex(n, \boldsymbol{H}) \le bn^\alpha$. Also, we shall say that a function $f(n)$ is asymptotic to a function $g(n)$ if $\lim_{n \to \infty} f(n)/g(n) = 1$ holds.

Let k be the minimum of the chromatic numbers $\chi(H)$ for $H \in \boldsymbol{H}$.

In 1946 Erdös and A. Stone proved (33), thereby determining asymptotically $ex(n, \boldsymbol{H})$ when $\boldsymbol{H}$ contains no bipartite graph.

In 1954, T. Kövári, V. Sós and P. Turán gave an upper bound for $ex(n, K_{s,t})$ which is asymptotic to $\frac{1}{2}(t-1)^{1/2} n^{2-1/s}$ for all $2 \le s \le t$. It is generally believed that $ex(n, K_{s,t})$ is of the order $n^{2-1/s}$. However, this has only been established for $s=2$ and 3 by W. Brown. The best available construction for $s=t=2$ is due to Brown, Erdös and A. Rényi. This was shown by Z. Füredi to be optimal for infinitely many values of n (see (34)). Note that $K_{2,2} = C_4$.

Mörs has proved that (35) holds.

One of the well known open questions in extremal graph theory is (36). We discussed the $l=2$ case above. The cases $l=3$ and 5 were settled by C. T. Benson and R. R. Singleton. One has an upper bound of the corresponding order, but we lack good constructions for $l \ne 2, 3, 5$.

Erdös and M. Simonovits have a tantalizing conjecture, that every rational number α, $1 < \alpha < 2$, occurs as the exponent for some extremal graph problem (see (37)).

Another interesting question is this: if $e(G) > ex(n, H)$ then how many copies of graphs from H must G contain. B. Bollobás, Lovász and Simonovits have several strong results in this direction for the case $H = \{K_l\}$. For a graph G let $N_l(G)$ denote the number of K_l's in G. A weaker form of their results is given by (38). The result (39) — due to Erdös — is similar in flavor.

Another type of extremal problem related to complete graphs arises if we ask how many graphs are needed to cover all edges of a graph if we can only use complete subgraphs on l vertices and edges. In the case of $T(n, l)$, clearly $e(T(n, l))$ graphs are needed. Erdös, A. W. Goodman and L. Pòsa showed that this is always sufficient for $l = 3$ and Bollobás proved the general case (40).

In 1984, Pyber showed (41), settling a long-standing open problem of Gallai. The number $n-1$ is best possible, because trees have $n-1$ edges and no cycles. (42) gives a related best possible result of Lovász.

Random graphs. One of the many valuable contributions of Erdös to combinatorics is the introduction of ~~the~~ probabilistic method to graph theory. Often we encounter problems of the following type: does there exist a graph with some given property. Very often, efforts in constructing such graphs prove unsuccessful. Checking all $2^{\binom{n}{2}}$ graphs on n vertices is an ~~impossible~~ infeasible task when n becomes large. Erdös' idea was to endow the set of all graphs with a probability distribution and prove that graphs with the desired property form a subset with positive probability. This clearly proves their existence, but might not give any hint as just how to construct them — even though in some cases the proportion of "good" graphs might be very close to one. That is, almost all graphs (might) have the desired property.

As an example, we have (43), a result proved by Erdös in 1947. By a simple computation, Erdös shows that if all labelled graphs on n vertices have the same probability $2^{-\binom{n}{2}}$, and $n < 1+2^{k/2}$, then the proportion of graphs containing K_k is less than 1/2. The same holds by symmetry for the complements. Therefore, there must exist graphs containing no K_k whose complements also contain no K_k. Despite serious efforts, however, the largest explicitly constructed graphs with the above properties have only $k^{\log k/\log\log k}$ vertices — the construction being due to Frankl and R. M. Wilson.

Erdös also used the probabilistic method to prove the result (21) concerning the existence of graphs with large chromatic number and large girth. It was again used by Erdös and S. Fajtlowicz to show that Hajós' conjecture relating the chromatic number and complete minors is false in a very strong sense. For most graphs G on n vertices, $\chi(G) > n/2\log n$ holds but G has no K_l as a minor when $l > 2\sqrt{n}$.

Various improvements on the basic probabilitic method were given by Erdös, Lovász, J. Beck and Rödl, making the probabilistic method one of the most effective tools in graph theory and combinatorics.

Diameter and girth. A connected graph can be regarded as a finite metric space where the distance between two vertices is the minimum number of edges in any path joining x and y. The diameter $d(G)$ of a graph G is the maximum of all

[X Singleton

distances $d(x, y)$ between its vertices. For example, $d(G) = 1$ if and only if G is complete. It is immediate that $d(G) \geq \lfloor g(G)/2 \rfloor$, where $g(G)$ denotes the girth of the graph G. Two natural functions involving these quantities are given in (44) and (45). Tutte observed the lower bounds (46) on $n(g, \delta)$ and showed they are best possible if $\delta = 3$, $g = 3, 4, 5, 6$ or 8, and $\delta \geq 3$, $g = 4$.

It is also easy to show that (47) holds. In the case of equality in (47), the graph must have girth $2d+1$ and be regular of degree Δ, so that it also achieves equality in the first part of (46) (and conversely). Such graphs are known as Moore graphs. It was shown by the American mathematicians A. J. Hoffman and Singleton that if a Moore graph of diameter 2 and $\delta \geq 3$ exists then we must have $\Delta = \delta = 3$, 7 or 57. For $\delta = 3$, the corresponding Moore graph is known as the Petersen graph (see Figure 11); for $\delta = 7$, the Moore graph is called the Hoffman-Singleton graph. It is not known if the Moore graph corresponding to $\delta = 57$ exists. It has been shown by R. M. Damerell and, independently by E. Bannai and T. Ito, that for $d \geq 3$ and $\delta \geq 3$, no Moore graphs exists.

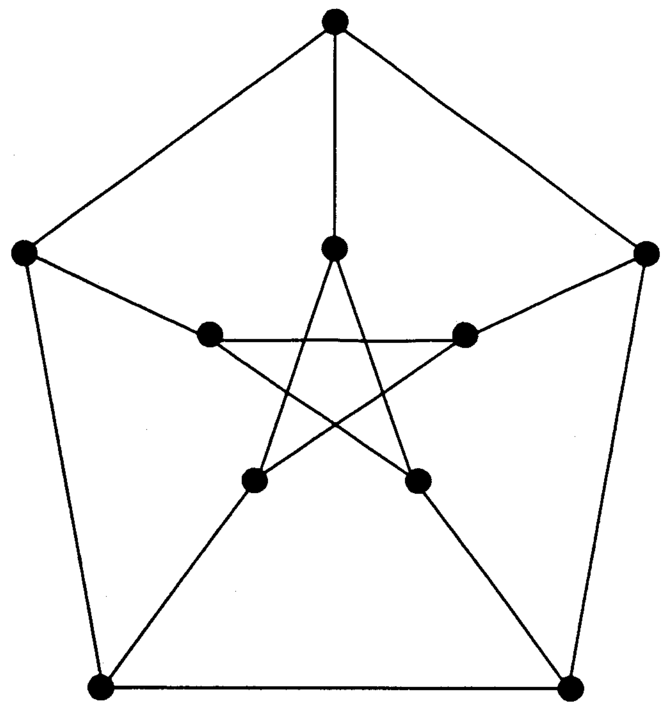


Figure 11. The Petersen graph.

Concerning Moore graphs with even girth $2a$ (i.e., satisfying equality in (46)), W. Feit and G. Higman showed that such graphs could not exist unless $a = 3, 4$ or 6. This was also shown independently by Singleton, who also gave a construction for these graphs when $a = 3$ and 4, and $\delta = q + 1$ with q a prime-power. An equivalent construction was given by Benson (which was mentioned in the extremal graph theory section as graphs having the maximum number of edges which do not contain C_6 as a subgraph). The best known upper bounds on $n(g, \delta)$ were obtained by an ingenious construction of the Russian mathematician G. Margulis (and modifications thereof).

Concerning $m(d, \Delta)$, by considering random Δ-regular graphs, Bollobás and

W. de la Vega proved (48). Graphs with small diameter and maximum degree, termed (d, Δ)-graphs, have important applications in the design of telecommunication networks. For these uses, it is necessary to have explicit constructions for large graphs of this type. The best general constructions currently known come from so-called de Bruijn graphs, which have as vertices all integer sequences $(x_1, \ldots, x_d)$ with $1 \le x_i \le \Delta/2$ and as edges all pairs $(x_1, \ldots, x_d)$, $(x_2, \ldots, x_d, x_{d+1})$.

Bollobás and F. R. K. Chung have shown that one can obtain relatively large (and, in particular, better than the de Bruijn graphs) (d, Δ)-graphs by a semi-random construction: add a random matching to specific Δ-regular graphs.

A related question asks by how much the diameter of a graph can decrease when a specified number of edges are added. Chung and M. R. Garey have established best possible bounds for this problem in (49).

BOX

(1) $\sum_{x \in V(G)} deg_G(x) = 2e(G)$.

(2) Every tree on n vertices has exactly $n-1$ edges.

(3) The number of trees on n labelled vertices is n^{n-2}.

(4) Every critically k-edge-connected graph has a vertex of degree k.

(5) $\nu(G) = \tau(G)$ holds for all bipartite graphs G.

(6) Every d-regular bipartite graph is the union of d perfect matchings.

(7) Every graph G is the union of $\Delta(G)$ or $\Delta(G) + 1$ matchings.

(8) $2\nu(G) = \min\{v(G) + |X| - \text{odd}(G \setminus X) : X \text{ a subset of } V(G)\}$.

(9) Every 2-connected 3-regular graph has a perfect matching.

(10) $v(G) \leq \alpha(G)(1 + \bar{d}(G))$

(11) Let G be triangle-free and $\bar{d} = \bar{d}(G) > 1$, then $\alpha(G) \geq v(G)(\bar{d}\log\bar{d} - \bar{d} + 1)/(\bar{d}-1)^2$.

(12) The Four Color Theorem: the chromatic number of every planar graph is at most four.

(13) Hadwiger's Conjecture: $X(G) \geq k$ implies that G has a homomorphism into K_k.

(14) $\chi(G) \leq \Delta(G) + 1$.

(15) $\chi(G) \leq \Delta(G)$ holds if $G \neq K_{\Delta(G)+1}$ and G is 3-connected.

(16) $\chi(G) \leq \chi(G_1)\chi(G_2)$ holds if $V(G) = V(G_1) = V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

(17) Every $(k+1)$-critical graph is k-edge-connected.

(18) If G is vertex k-critical then $\alpha(G) \leq \binom{n-\alpha(G)-1}{k-2}$.

(19) $\chi(K(n, 2k, 1)) = n - 2k + 2$ for $n \geq 2k$.

(20) For every integer k there exists an $n = n(k)$ such that every graph with chromatic number at least n contains a triangle-free subgraph of chromatic number k.

(21) For all integers $k, l \geq 3$ there exists a graph G with $\chi(G) = k$, $g(G) = l$.

(22) Every graph G with $\chi(G) = k+1 \geq 3$, $v(G) = n$, contains an odd cycle of length at most $4kn^{1/k}$.

(23) G is perfect if and only if neither G nor $\overline{G}$ contains an induced odd cycle of length 5 or more.

(24) G is perfect if and only if $\overline{G}$ is perfect.

(25) A graph G is perfect if and only if $\alpha(G')\omega(G') \geq v(G')$ holds for all induced subgraphs G' of G.

(26) A connected graph G has an Euler tour if and only if all its vertices have even degree.

(27) If G has a Hamiltonian cycle then $c(G \backslash X) \leq |X|$ for all sets X of vertices.

(28) If $2\delta(G) \geq v(G)$ then G has a Hamiltonian cycle.

(29) If $e(G) \geq v(G) \log_2 v(G)$ then G is edge-reconstructible.

(30) G is planar if and only if neither K_5 nor $K_{3,3}$ are minors of it.

(31) For all S with $\gamma(S) \geq 1$ one has $\chi(S) = \left\lfloor (7 + \sqrt{1+48\gamma(S)})/2 \right\rfloor \stackrel{\text{def}}{=} H(\gamma(S))$ ($\lfloor x \rfloor$ is largest integer not exceeding x).

(32) If G contains no K_3 then $e(G) \leq \left\lfloor v(G)^2/4 \right\rfloor$.

(33) $ex(n, \mathbf{H})$ is asymptotic to $\dfrac{k-2}{2(k-1)} n^2$ if $k = \min\{\chi(H) : H \in \mathbf{H}\} \geq 3$.

(34) $ex(q^2+q+1, C_4) = (q+1)^2 q/2$ for q a prime power.

(35) $ex(n, K_{2,t})$ is asymptotic to $\frac{1}{2}\sqrt{(t-1)n^3}$ for all $t \geq 2$.

(36) Is $ex(n, C_{2l})$ of the order $n^{1+1/l}$?

(37) For every rational α, $1 < \alpha < 2$ there exists $\mathbf{H}$ with $ex(n, \mathbf{H})$ of the order n^{α}.

(38) Define the real number y by $e(G) = \left(1 - \dfrac{1}{y}\right)\dfrac{n^2}{2}$. If $y \geq l-1$ then $N_l(G) \geq \binom{y}{l}\left(\dfrac{n}{y}\right)^l$ holds.

(39) Let $2 \leq l < k$ be integers. If G contains no K_k then $N_l(G) \leq N_l(T(n, k))$.

(40) The edges of every graph on n vertices can be covered by at most $e(T(n, l))$ edges and edge-disjoint complete subgraphs on l vertices.

(41) The edges of every graph on n vertices can be covered by $n-1$ edges and cycles.

(42) The edges of every graph can be covered by $\lfloor n/2 \rfloor$ edge-disjoint paths and cycles.

(43) For every $k \geq 3$ there exists a graph G on at least $2^{k/2}$ vertices such that neither G nor $\overline{G}$ contains K_k as a subgraph.

(44) $n(g, \delta)$ = the minimum number of vertices in a graph with girth g and minimum degree δ.

(45) $m(d, \Delta)$ = the maximum number of vertices in a graph with diameter d and maximum degree Δ.

(46) $n(2a+1, \delta) \geq 1 + \delta \frac{(\delta-1)^a - 1}{\delta-2}$, $n(2a, \delta) \geq 2 \frac{(\delta-1)^a - 1}{\delta-2}$, $(\delta \geq 3)$.

(47) $m(d, \Delta) \leq 1 + \Delta \frac{(\Delta-1)^d - 1}{\Delta-2}$.

(48) $m(d, \Delta) \geq (\Delta-1)^{d-1}/2\Delta \log(\Delta-1)$.

(49) Adding t edges to a graph G, the new graph G' satisfies $d(G') \geq (d(G) - t)/(t+1)$; equality is achieved for paths.

Design theory

Existence theorems. A collection $D = \{D_1, \ldots, D_m\}$ of k-element subsets of $[n] := \{1, 2, \ldots, n\}$ a t-design, or, more exactly, a $\lambda-(n, k, t)$-design if all t-subsets of $[n]$ are contained in exactly λ sets in the collection. To avoid trivialities one assumes $n > k > t \geq 1$. If no k-set occurs more than once in D then the design is called simple. The members of D are called blocks.

Designs are conveniently represented by their incidence matrices $M(D)$ which is an m by n 0-1 matrix, with general entry $m_{ij} = 0$ or 1 according to whether D_i contains j or not. A design is simple if and only if $M(D)$ has no repeated rows.

The simplest example of a non-trivial design is provided by the 1-(7, 3, 2)-design, called the Fano-plane, in Figure 12. The sets are formed by the sets of colinear points on the six lines together with the three points on the circle.

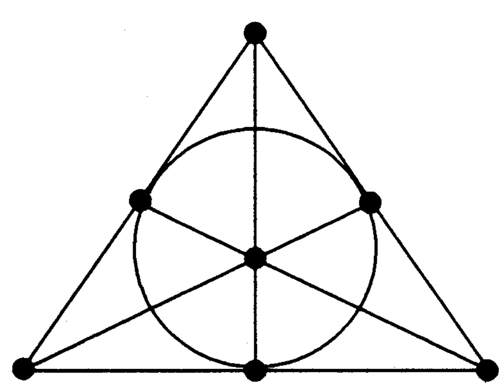


Figure 12. The Fano plane.

The main problem of design theory is to decide for which values of the parameters (n, k, t, λ) t-designs exist.

In general, by counting in two different ways the number of pairs (T, D) where T is a t-element set, D is a block and $T \subseteq D$ holds, one obtains (1).

For $t=1$ (1) gives the necessary condition that $m = \lambda n/k$ must be an integer. This condition is easily seen to be sufficient for the existence of 1-designs. To see this define $D_i = \{(i-1k+1, (i-1)k+2, \ldots, ik\}$ for $i=1, \ldots, m$; where computation is done modulo n.

For $t \geq 2$ the situation is much more complex. For an element $i \in [n]$ define the derived and residual designs $D(i)$ and $D(\bar{i})$ by (2). Then $D(i)$ is a $\lambda-(n-1, k-1, t-1)$-design and $D(\bar{i})$ a $\mu-(n-1, k, t-1)$-design with $\mu = \lambda(n-k)/(k-t+1)$.

Taking derived designs $t-1$ times and using (1) we obtain the necessary conditions (3) for the existence of a t-design.

The simplest non-trivial choice for k, t and λ is $k=3$, $t=2$, $\lambda=1$. In this case (3) is equivalent to $n \equiv 1$ or 3 (*mod* 6). The British ~~reverend~~ clergyman T. Kirkman proved in 1847 that (3) is indeed sufficient for teh existence of designs with these parameters (see (4)).

The $1-(n, k, t)$ designs are denoted by $S(n, k, t)$ and are called Steiner-systems, because the Swiss mathematician Steiner asked in 1853 – obviously unaware of Kirkman's prior results – for which n $S(n, 3, 2)$ can exist. The requirement that a collection D forms a Steiner system is a strong restriction. However, as Wilson showed, the number of distinct $S(n, 3, 2)$ for $n \equiv 1$ or 3 (*mod* 6) grows very rapidly as a function of n. L. Babai used this fact to show that almost all of them are asymmetric. An $S(n, 3, 2)$ D is called asymmetric if there is no non-trivial permutation $\pi(1), \ldots, \pi(n)$ such that $\{a, b, c\} \in D$ implies $\{\pi(a), \pi(b), \pi(c)\} \in D$. The Israeli mathematiçian H. Hanani proved more than 100 years after Kirkman introduced the problem that the necessary conditions for the existence of an $S(n, k, t)$ are also sufficient for $(k, t) = (4, 2)$, $(5, 2)$ and $(4, 3)$ (see (5), (6) and (7)). In 1961 he also proved also that $\lambda - (n, k, 2)$-designs with $k = 3$ or 4 exist for all values of n satisfying the necessary conditions (3).

For $t \geq 4$ there are only four Steiner-systems known. An $S(4, 5, 11)$, an $S(5, 6, 12)$, an $S(4, 7, 23)$ and an $S(5, 8, 24)$. These were constructed independently by Witt (1938) and Carmichael (1937) using the sporadic simple groups M_{11}, M_{12}, M_{23} and M_{24} (which were discovered by the French mathematician Mathieu in 1861 and 1873, respectively).

One of the most powerful constructions for 2-designs is based on the concept of a difference system. For a set $I = \{i_1, \ldots, i_k\}$ of integers, we denote by $K(I)$ the collection of all differences $i_a - i_b$, $1 \leq a \neq b \leq k$, where all numbers are reduced modulo n and, in $K(I)$ the same number may occur several times. A collection of sets $\boldsymbol{I} = \{I_1, \ldots, I_s\}$ is called a difference system (modulo n) with repetition number λ if all integers $1 \leq j < n$ occur altogether exactly λ times in the sets $K(I)$, $I \in \boldsymbol{I}$. It is routine to show that if $\boldsymbol{I}$ is a difference system then $s = \lambda(n-1)/k(k-1)$ and the collection D defined in (8) is a $\lambda - (n, k, 2)$ design.

In 1938 Peltesohn used difference sets to give an alternate, direct construction for $S(n, 3, 2)$ for all n with $n \equiv 1 \ (mod\ 6)$.

In Wilson's celebrated existence proof of $S(n, k, 2)$, see (9) difference sets are an important building block. Let us mention, however, that Wilson proves the existence of these difference sets by a probabilistic argument. That is, he shows their existence without actually producing them.

One must also mention here the beautiful results (10) of Teirlinck (1987) which show the existence of simple t-designs for all t.

Inequalities and symmetric designs. Apart from the divisibility conditions (3) there are several known inequalities which the parameters of a design must satisfy. The statistician R. Fisher proved in 1940 that the number of blocks is not smaller than the number of points in any 2-design. The inequality (11), which was proved by Wilson in 1983 generalizes Fisher's inequality and simultaneously extends earlier results of H. B. Mann and D. Ray-Chaudhuri/Wilson.

Simple $2s$-designs for which equality holds in (11) are said to be tight. Bannai proved that for every $s \geq 2$ there can be at most a finite number of tight designs. For $s = 2$ the only one is the $S(4, 7, 23)$ as it was proved by N. Ito.

For $s = 1$, tight designs D are called symmetric because the corresponding incidence matrix $M(D)$ has as many rows as columns. It follows from the definition that $M(D)$ satisfies the matrix equation (12) where I is the identity, J the all 1 matrix – both of order n. From (3) using $m = n$ we deduce $(n-1)\lambda = k(k-1)$. Since $M(D)$ is a square matrix, (12) implies (13). This result is due to Bruck, Chowla and Ryser who for n odd also succeeded in using the matrix equation (12) in obtaining the necessary condition (13) for the existence of a symmetric design.

The most studied symmetric designs are those with $\lambda=1$. In this case $n=k^2-k+1$ follows from (3) and $m=n$. It can be shown also that any two blocks intersect in exactly 2 points. Such a design ($S(k^2-k+1, k, 2)$) is called a projective plane of order $k-1$. If $k-1$ is the power of a prime then these can be easily constructed using a simple algebraic structure called a field. A field is a set together with 2 operations: addition and multiplication which satisfy the familiar rules of algebra (e.g., $a+b=b+a$, $a(b+c)=ab+ac$). If it has a finite number of elements then it is called a finite field. There is exactly one finite field with p^a elements for each prime power p^a. The simplest field has 2 elements, 0 and 1. For each prime p, the numbers $0, 1, \ldots, p-1$ form a field where addition and multiplication are performed modulo p.

There is no projective plane of order d *known* for $d \neq p^a$, although for given $d=p^a$, $a \geq 2$ there are several non-isomorphic planes of order d. From (14) the non-existence of projective planes follows for an infinity of values of d, and in particular, for $d=6$. The first undecided case is $d=12$.

It was shown by Singer that the projective planes obtained over finite fields are cyclic. That is, they can be obtained using an appropriate difference set (a difference system consisting of only one set). The same applies to projective spaces of higher dimensions as well (see (15)).

For given $\lambda \geq 2$ there are only a finite number of symmetric designs known and most people believe that this is not by chance. We also point out (16), the inequality of J. Tits for the parameters of an $S(n, k, t)$.

Resolvable designs. A $\lambda-(n, k, 2)$ design $\boldsymbol{D}$ is called resolvable if there is a partition of the blocks $\boldsymbol{D}=\boldsymbol{D}_1 \cup \cdots \cup \boldsymbol{D}_s$ such that for all $1 \leq i < s$, the blocks in $\boldsymbol{D}_i$ form a partition of $[n]$. Resolvability thus implies that n is a multiple of k. Since by (3) $(k-1)$ divides $(n-1)$, $n \equiv k$ $(mod\ k(k-1))$ must hold. In 1850 Kirkman raised the problem of whether a resolvable $S(n, 3, 2)$ exists for $n \equiv 3$ $(mod\ 6)$. He constructed such a design for $n=9$ and 15. He was motivated by the following problem, which has become known as Kirkman's schoolgirl problem. A teacher has a class of 15 girl students whom he takes for a walk on each day of the week. The girls walk in 5 rows of three. Can he arrange that every girl will be in the same row with every other girl on exactly one day of each week? Kirkman gave a positive answer for $n=15$ and in his honor a resolvable $S(n, 3, 2)$ is called a Kirkman system of order n. Their existence for all $n \equiv 3$ $(mod\ 6)$ was finally established in 1971 by the Indian mathematician Ray-Chaudhuri and Wilson (see (17)). In 1972, Hanani, Ray-Chaudhuri and Wilson settled the case $k=4$ as well (see (18)).

Another longstanding open problem was solved by the Hungarian mathematician Baranyai in 1974 who showed that the set of all k-subsets of $[n]$ is in fact a resolvable design (see (19)).

Orthogonal arrays and latin squares. Orthogonal arrays, like designs, first arose in statistics. Let $X_1, \ldots, X_k$ be disjoint n-element sets (we consider them as copies of $[n]$). An orthogonal array of strength t is a collection $\boldsymbol{F}$ of k-element sets such that (20) holds. It follows from (20) that for every choice of

$1 \leq i_1 < \cdots < i_t \leq k$ and elements $x_j \in X_j$, $1 \leq j \leq t$, there is a unique set in $\boldsymbol{F}$ containing $\{x_1, \ldots, x_t\}$. Orthogonal arrays were introduced by Rao, and, following Hanani, we call them transversal designs and denote them by $T(n, k, t)$. Transversal designs are an important tool for the recursive construction of designs. For example, if $\boldsymbol{D}_i$ is an $S(n, k, 2)$ on X_i and $\boldsymbol{F}$ is a $T(n, k, 2)$ then $\boldsymbol{D}_1 \cup \cdots \cup \boldsymbol{D}_k \cup \boldsymbol{F}$ is an $S(kn, k, 2)$. Transversal designs are often easier to construct (e.g., (21) holds).

A latin square is an n by n matrix for which every row and every column is a permutation of $[n]$. It is equivalent to a $T(n, 3, 2)$. Indeed if (i, j, k) is a set in the $T(n, 3, 2)$ then let the (i, j) entry of the latin square be k. A third way of looking at latin squares is the following. Consider the "square" $S = \{(i, j) : 1 \leq i, j \leq n\}$. Let $\boldsymbol{Y} = \{Y_1, \ldots, Y_n\}$ and $\boldsymbol{X} = \{X_1, \ldots, X_n\}$ be its standard partitions, that is, $Y_i = \{(i,j) : 1 \leq j \leq n\}$, $X_j = \{(i,j) : 1 \leq i \leq n\}$. Call two partitions $\boldsymbol{P} = \{P_1, \ldots, P_n\}$ and $\boldsymbol{Q} = \{Q_1, \ldots, Q_n\}$ of S into n-sets orthogonal if $|P_i \cap Q_j| = 1$ for all i, j. Then a latin square is simply a partition $\boldsymbol{P}$ which is orthogonal to both $\boldsymbol{Y}$ and $\boldsymbol{X}$. Indeed let the (i,j) entry of the matrix be k if and only if $(i,j) \in P_k$.

Two latin squares $\boldsymbol{P}$ and $\boldsymbol{Q}$ are called orthogonal if the two partitions are orthogonal. In terms of the entries in the latin squares, this means that in a superposition of the two squares all pairs (a,b), $1 \leq a,b \leq n$, occur exactly once. This definition goes back to Euler, who tried unsuccessfully to construct two orthogonal latin squares of order 6 and conjectured that no such pairs of squares exist of order n if $n \equiv 2 \ (mod\ 4)$. The non-existence of orthogonal latin squares of order 6 was proved in 1900 by Tarry. Euler's conjecture was disproved by the Indian mathematicians R. C. Bose, S. Shrikhande and the American mathematician E. Parker in 1960, who succeeded in constructing a pair of orthogonal latin squares of order $4k+2$ for all $k \geq 2$ (see (22)). Their discovery made front-page news in the New York Times. Two orthogonal latin squares of order 10 are given in Figure 13. From the third definition it is easy to see that there can be at most $n-1$ pairwise orthogonal latin squares of order n and that their existence is equivalent to a resolvable $S(n^2, n, 2)$, which is also called an affine plane of order n. It is true that an affine plane of order n exists if and only if a projective plane of order n exists.

We also point out that a $T(n, k+2, 2)$ is equivalent to a set of k pairwise orthogonal latin squares of order n.

01234	56789
34012	79865
43120	97658
12407	85396
20375	68941
57698	34120
89756	12034
65981	43207
98563	01472
76849	20513

01923	84657
67895	23104
93746	58210
38254	79061
14507	36982
25619	40873
40138	62795
56480	17329
82071	95436
79362	01548

Figure 13. Two orthogonal latin squares of order ten.

Hadamard matrices. The nineteenth century French mathematician J. Hadamard proved the inequality (23) for square matrices with ± 1 entries. A matrix achieving equality in (23) is called a Hadamard matrix. Simple considerations show that if a Hadamard matrix of order n exists then $n=2$ or n is a multiple of four. In 1933, Paley constructed an infinite series of Hadamard matrices (see (24)). However, it is still unknown whether Hadamard matrices of order $4n$ exist for all n.

By forming tensor products Hadamard matrices of order ab, can be constructed from those of orders a and b.

From (23) it follows that $M^T M = nI$, that is, not only the rows but also the columns of Hadamard matrices are pairwise orthogonal. Multiplying a row or a column by -1 will not change this property. Therefore, one may suppose that the first row is made up entirely of 1's. Each of the remaining rows of a Hadamard matrix of order $4n$ then defines a partition of $[4n]$ into two subsets of size $2n$. In 1933, Todd showed that these $8n-2$ sets form an $(n-1)-(4n, 2n, 3)$-design. The derived design $D(i)$, which is a symmetric $(n-1)-(4n-1, 2n-1, 2)$-design, is called a Hadamard design.

Association schemes and coding theory. Given a connected graph G, one can define a metric on $V(G)$ by defining the distance $d(x, y)$ of two vertices as the minimum number of edges (in a path) connecting x and y in G. If for all triples d, e, f there exists a constant $\alpha(d, e, f)$ such that (25) holds then G is called distance regular. The two most commonly occurring distance regular graphs are the Johnson scheme $J(n, k)$ and the Hamming scheme $H(n, a)$ defined in (26) and (27).

In the case $a=2$ the vertices of $H(n, 2)$ may be thought of as all subsets of $[n]$

with $d(A, B) = |A \Delta B|$ ($A \Delta B$ denotes the symmetric difference $(A-B) \cup (B-A)$). For a distance regular graph G, define the matrices $M_i = M_i(G)$ by (28). Then the products $M_i M_j$ can be expressed as linear combinations of the M_k (see (29)). That is these matrices define an algebra, called the Bose-Mesner algebra. This may also happen for other sets of (0, 1)-matrices $M_0 = I$, $M_1, \ldots, M_s$ satisfying $M_0 + M_1 + \cdots + M_s = J_m$, the all 1 matrix of order m. Then one has what was introduced by Bose and Mesner as an association scheme. The study of association schemes is a well-developed topic resembling that of finite groups. The importance of association schemes in combinatorics became evident after the seminal work of Delsarte in 1973. The techniques developed by Delsarte, in particular, his linear programming bound, is widely used in graph theory extremal set theory, coding theory and geometry.

An error-correcting code in a distance regular graph G is a subset C of $V(G)$ satisfying $d(x, y) \geq 2e+1$ for all distinct $x, y \in C$. In the case of $H(n, a)$, we can think of the elements of C as messages, which may get modified (in at most e components) during transmission. If $x \in C$ is transmitted and $\tilde{x}$ is the message actually received, then by the definition of C x is the unique closest element of C for $\tilde{x}$ (see (30)).

Error-correcting codes are widely used in communication theory. For a vertex x in G one defines the sphere $S_e(x)$ of radius e centered at x by (31). For an e-error-correcting code those spheres have to be disjoint, which gives (32), known as the sphere-packing bound. In case equality holds in (32), C is called a perfect code. It is known that almost no perfect codes exist. In particular, for $H(n, 2)$ it was proved by the Finnish mathematician Tietäväinen and independently by the Russian mathematicians Leontiev and Zinoviev in 1971 that (33) holds. Perfect codes for $e=1$ exist in $H(n, 2)$ if and only if $n+1$ is a power of 2. They are called Hamming codes, because the American mathematician R. W. Hamming was the first to construct them.

There are various bounds on the maximum size of e-error-correcting codes, which are sharp for various values of the parameters n, a and e. The most well known are those due to Elias, Griesmer, Plotkin and Singleton. Probably the deepest is due to McEliece, Rodemich, Rumsey and Welch and it is deduced by using Delsarte's linear programming technique. The best known lower bound (which is not difficult to prove) is the Gilbert-Varshamov bound (34). If q is a prime power then $H(n, q)$ can be viewed as a vector space of dimension n over the field of q elements. A code is called linear if it forms a vector subspace. Linear codes are typically easier to construct and analyze than general non-linear codes. Good linear codes were constructed by Bose, Ray-Chaudhuri and Hocquenheim in 1960. These are called BCH codes after the initials of these three.

Deep algebraic geometric results were applied by Goppa, Tsafsman, Vladuts and Zink in the 1980's to construct codes in $H(n, q)$ with q a large prime power, which are larger than the lower bound in (34).

Linear codes have their orthogonal complements called dual codes. The code and its dual are connected by the MacWilliams identities.

One should also mention the (asymptotically best possible) results of C.

Shannon for codes which can correct *nearly all* occurrences of e errors. They are not linear and are obtained by the probabilistic method.

BOX

(1) $m\binom{k}{t} = \lambda\binom{n}{t}$.

(2) $D(i) = \{D - \{i\} : i \in D \in \mathbf{D}\}$, $D(\bar{i}) = \{D : i \notin D \in \mathbf{D}\}$.

(3) $\lambda\binom{n-j}{t-j} / \binom{k-j}{t-j}$ is an integer for $j = 0, \ldots, t-1$.

(4) $1-(n, 3, 2)$ designs exist if and only if $n \equiv 1$ or 3 $(mod\ 6)$.

(5) $S(n, 4, 2)$ exists if and only if $n \equiv 1$ or 4 $(mod\ 12)$.

(6) $S(n, 5, 2)$ exists if and only if $n \equiv 1$ or 5 $(mod\ 20)$.

(7) $S(n, 4, 3)$ exists if and only if $n \equiv 2$ or 4 $(mod\ 6)$.

(8) $\mathbf{D} = \{I + j : I \in \mathbf{I},\ 0 \le j < n\}$ and $I + j = \{i + j : i \in I\}$.

(9) For k fixed and n large, $n > n_0(k)$, an $S(n, k, 2)$ exists if and only if both $\binom{n}{2} / \binom{k}{2}$ and $(n-1)/(k-1)$ are integers.

(10) If $\lambda = (t+1)!^{2t+1}$ divides $n-t$ then $\binom{[n]}{t+1}$ can be partitioned into $\lambda-(n, t+1, t)$-designs.

(11) If $\mathbf{D}$ is a $2s$-design in which some block occurs r times then $|\mathbf{D}| \ge r\binom{n}{s}$ holds.

(12) $M(\mathbf{D})^T M(\mathbf{D}) = (k-\lambda)I + \lambda J$.

(13) $\det M(D) = \sqrt{k^2(k-\lambda)^{n-1}}$ is an integer; thus, if n is odd then $k-\lambda$ is a perfect square.

(14) If $k > \lambda$ and n is odd then the diophantine equation $x^2 = (k-\lambda)y^2 + (-1)^{(n-1)/2}\lambda z^2$ has a non-trivial integer solution.

(15) A projective space of dimension d is a $\lambda-(n, k, 2)$-design with $n = \frac{q^{d+1}-1}{q-1}$, $k = \frac{q^d-1}{q-1}$ and $\lambda = \frac{q^{d-1}-1}{q-1}$, where q is a prime power, $d \geq 3$.

(16) If an $S(n, k, t)$ exists then $n \geq (k-t+1)(t+1)$.

(17) A Kirkman-system of order n exists if and only if $n \equiv 3 \ (mod\ 6)$.

(18) A resolvable $S(n, 4, 2)$ exists if and only if $n \equiv 4 \ (mod\ 12)$.

(19) $\binom{[n]}{k}$ can be partitioned into $\binom{n-1}{k-1}$ partitions of $[n]$ for all n which are multiples of k.

(20) $|F \cap X_i| = 1$ for all $F \in \mathbf{F}$, $1 \leq i \leq k$, $|\mathbf{F}| = n^t$ and $|F \cap F'| < t$ for all $F, F' \in \mathbf{F}$.

(21) $\{\{x_1, \ldots, x_k\} : x_1 + \cdots + x_k \equiv 0 (mod\, n),\ x_i \in X_i\}$ is a $T(n, k, k-1)$ for all n and k.

(22) There is a pair of orthogonal Latin squares of order n for all $n \neq 2, 6$.

(23) $\det M \leq n^{n/2}$ holds for all ± 1 matrices of order n with equality iff $MM^T = nI$ holds.

(24) If $n = 2^e(q+1)$ is divisible by 4 and q is a prime power then there exist Hadamard matrices of order n.

(25) For all $x, y \in V(G)$ with $d(x, y) = d$, the number of vertices $z \in V(G)$ satisfying $d(x, z) = e$, $d(y, z) = f$ is $\alpha(d, e, f)$.

(26) $J(n, k)$ has as vertices all k-subsets of $[n]$, and $d(A, B) = k - |A \cap B|$.

(27) The vertices of $H(n, a)$ are all sequences $\vec{x} = (x_1, \ldots, x_n)$ with $0 \leq x_i < a$, and $d(\vec{x}, \vec{y})$ is the number of i, $1 \leq i \leq n$ where $x_i \neq y_i$.

(28) $M_i(G)$ is a symmetric $(0-1)$-matrix with both rows and columns indexed by the vertices of G. The general entry $m_i(x, y) = 1$ or 0 according to whether $d(x, y)$ is i or not. In particular, $M_0(G)$ is the identity matrix.

(29) $M_i(G) M_j(G) = \sum_k \alpha(k, i, j) M_k(G)$.

(30) $d(x, \tilde{x}) \leq e < 2e + 1 - e \leq d(y, \tilde{x})$ for all $x \neq y \in C$.

(31) $S_e(x) = \{y \in V(G) : d(x, y) \leq e\}$, $s(e) = |S_e(x)| = |S_e(y)|$ for all $y \in C$.

(32) $|C| \leq |V(G)| / s(e)$.

(33) For $e \geq 2$, the only perfect code in $H(n, 2)$ is the 3-error-correcting Golay-code of length 23.

(34) $|C| \geq |V(G)| / s(2e)$ holds for every e-error-correcting code which cannot be extended to a larger one.

Extremal set theory

Intersection theorems. In this topic, one considers families $\boldsymbol{F}$ of distinct subsets of $[n]=\{1,2,\ldots,n\}$, and asks for the maximum (or minimum) size of such families subject to various restrictions on the sets in $\boldsymbol{F}$. Formally, $\boldsymbol{F}\subset 2^{[n]}$, the set of all subsets of $[n]$. A typical restriction is to require that $F\cap F'\neq\emptyset$ for all $F, F'\in\boldsymbol{F}$. Such families are called intersecting. In this case, (1) holds, as is easily seen by noting that $\boldsymbol{F}$ cannot contain both a set F and its complement $[n]-F$. Furthermore, the bound (1) is best possible by considering the family of all subsets of $[n]$ which contain the element 1.

Erdös, the Chinese mathematician Chao Ko and the German-born English mathematician R. Rado proved in 1938 that a similar result holds for $\boldsymbol{F}\subset\binom{[n]}{k}=\{X\subset[n]:\ |X|=k\}$ (see (2)).

A family $\boldsymbol{F}$ is called t-intersecting if $|F\cap F'|\geq t$ holds for all $F, F'\in\boldsymbol{F}$. Two examples of t-intersecting families are exhibited in (3). Settling a conjecture of Erdös, Ko and Rado, it was shown by the Hungarian mathematician G. O. H. Katona that $\boldsymbol{F}$(even) and $\boldsymbol{F}$(odd) are the unique largest t-intersecting families for $t\geq 2$.

The case $\boldsymbol{F}\subset\binom{[n]}{k}$ (such families are called uniform) is more complicated. For $0\leq i\leq k-t$, define families $\boldsymbol{A}_i$ by (4). These are t-intersecting. For example, $\boldsymbol{A}_0$ is simply the collection of all k-subsets containing a fixed t-set. Comparing $\boldsymbol{A}_0$ and $\boldsymbol{A}_1$ gives (5). One can use this to give an alternate proof of Tits' inequality (Designs (16)). On the other hand, Frankl and Wilson showed that (6) holds.

The general problem would be settled by (7) which was conjectured by Frankl in 1976.

Call $F \subset 2^{[n]}$ a union-family if $F \cup F' \neq [n]$ holds for all $F, F' \in F$. The bound (1) (with the same proof) remains valid for union-families. What happens for intersecting union-families? It was proved by D. Daykin and Lovász (and independently, by J. Marica and J. Schönhein) that 2^{n-2} is the correct answer (taking all sets containing 1 but missing n shows that it is a lower bound). Actually, this result can be deduced using a very important inequality due to Kleitman.

Call a family C a complex if along with each $C \in C$ all subsets of C belong to C. Kleitman's result is expressed by (8). Extensions of (8) proved by the three physicists Fortuin, Kastelyn and Ginibre proved very useful in statistical mechanics. Other important extensions (to distributive lattices) were proved by R. Ahlswede and Daykin, Kleitman actually proved (8) in order to show (9).

The isoperimetric inequality proved by Harper is another important tool in extremal set theory. Define a graph on $2^{[n]}$ by putting an edge between E and F if $E \subset F$, $|F - E| = 1$. This brings us back to the association scheme $H(n, 2)$ (cf. the design section). For a family $F \subset 2^{[n]}$, define its boundary ∂F as those sets $E \notin F$ which are joined by an edge to some set in F. A sphere of radius r is defined by (10). A set F with $S_r(F) \subsetneq F \subsetneq S_{r+1}(F)$ is called a generalized sphere.

Harper proved that among all families of given size, generalized spheres have the smallest boundary. Harper's result can be used to give an alternate proof of

Katona's theorem on t-intersecting families. Despite repeated efforts, the isoperimetric problem remains open for the Johnson scheme $J(n, k)$.

Given a family $F \subset \binom{[n]}{k}$, define its l-shadow $\sigma_l(F)$ by (11). Probably the single most important result in extremal set theory is the Kruskal-Katona theorem which for $|F|$ fixed gives best possible lower bounds on $|\sigma_l(F)|$. In (12) a more numerical consequence of it – deduced by Lovász – is described.

Given a sequence $(f_0, f_1, \ldots, f_n)$ of integers, the Kruskal-Katona theorem gives necessary and sufficient conditions for the existence of a complex $C \subset 2^{[n]}$ which contains exactly f_i sets of size i for $0 \leq i \leq n$. Katona, and independently, Daykin showed that (2) also follows from the Kruskal-Katona theorem.

A family $\{F_1, \ldots, F_s\}$ is called a sunflower of size s is for some set C, $F_i \cap F_j = C$ holds for all $1 \leq i < j \leq s$.

In 1960, Erdös and Rado proved (13).

Erdös conjectures that in (13), $k!$ can be replaced by $c(s)^k$ where $c(s)$ does not depend on k.

The lines of a projective plane of order $(k-1)$ given an example of $k^2 - k + 1$ sets intersecting pairwise in one element.

For a set of non-negative integers L, a family F is called an L-system if $|F \cap F'| \in L$ for all distinct F, F' in F. In 1974, M. Deza showed (14), thereby settling a conjecture of Erdös and Lovász. A general class of problems is given by (15). Ray-Chaudhuri and Wilson showed that $\binom{n}{|L|}$ is always an upper bound with equality corresponding to tight $2|L|$-designs. Another general upper bound proved by Deza, Erdös and Frankl is (16). Equality in (16) for $L=\{0, 1, \ldots, t-1\}$ corresponds to Steiner-systems $S(n, k, t)$. The dual case $L=\{t, t+1, \ldots, k-1\}$ corresponds to t-intersecting families and proves (6) for n sufficiently large.

There is a reduction lemma due to the Hungarian mathematician Z. Füredi, which permits rather good upper bounds to be obtained for the order of $m(n, k, L)$. It was used by Frankl to prove (16), which shows the richness of these problems.

For applications, the case when only one intersection size is forbidden, is the most useful. Frankl and Füredi proved (17) which sharpens (6). Solving a longstanding conjecture of Erdös, it was shown in 1985 by Frankl and Rödl that (18) holds.

Hypergraphs. A family $\boldsymbol{F}$ of non-empty sets is often called a hypergraph, and its members of $\boldsymbol{F}$ are called edges. This terminology is used when one wants to emphasize the close relationship to graph theory — a graph being a family with all edges of size 2. The most important parameters of a hypergraph $\boldsymbol{F}$ are the covering number $\tau(\boldsymbol{F})$ and the matching number $\nu(\boldsymbol{F})$ (see (19)). Practically all combinatorial problems can be formulated as the determination of $\tau(\boldsymbol{F})$ for some hypergraph $\boldsymbol{F}$. A family $\boldsymbol{F}=\{F_1, \ldots, F_m\}$ is called τ-critical if $\tau(\boldsymbol{F})$ decreases after deleting an arbitrary edge. This implies the existence of a family $\boldsymbol{G}=\{G_1, \ldots, G_m\}$ satisfying $|G_i|=\tau(\boldsymbol{F})-1$, $F_i \cap G_i=\varnothing$ and $F_i \cap G_j \neq \varnothing$ for all $i \neq j$. That is, G_i meets all the edges of $\boldsymbol{F}-\{F_i\}$.

In 1965, Bollobás proved a very useful result on such families (see (20)). If $F = \{F_1, \ldots, F_m\}$ is an antichain, that is, no edge contains another one, then one can define $G_i = [n] - F_i$ and obtain from (20) what is called the LYM-inequality — first proved by the Japanese mathematician K. Yamamoto in 1954. Since the binomial coefficient $\binom{n}{k}$ is largest for $k = \lfloor n/2 \rfloor$ one deduces Sperner's theorem (21), which is discussed at greater length in the section on partially ordered sets.

The rank $r(H)$ of a hypergraph is the maximum cardinality of an edge in H. Another consequence of (21) is (22). This inequality is best possible as can be seen by looking at $\binom{[r+\tau-1]}{r}$.

Let $A = \{A_1, \ldots, A_m\}$ and $B = \{B_1, \ldots, B_m\}$ be hypergraphs (23) and (24), due to Tuza and Füredi, are important variations of (20). Perles used (23) to show that there are at most 2^{d+1} disjoint simplices in $\mathbb{R}^d$ which intersect each other in a facet. N. Alon and G. Kalai use (24) to give a simple proof for the upper bound theorem on convex polytopes.

For G, a graph without isolated vertices, define the quantity $\sigma(G)$ by $\sigma(G) = 2\tau(G) - \nu(G)$. Gallai proved that $\sigma(G)$ is always non-negative (it is zero only if G is a matching). A. Hajnal proved that the only connected graphs with $\sigma(G) = 1$ were cycles of odd length. He also showed that $\Delta(G) \leq \sigma(G) + 1$ holds for the maximum degree.

Lovász (proving a conjecture of Gallai) showed that (25) is true. For τ-critical hypergraphs, inequality (26) was proved by A. Gyárfás, J. Lehel and Tuza.

It is often instructive to consider fractional covers of hypergraphs. A fractional cover is a function f defined on the vertices of F and satisfying (27). One defines $w(f) = \sum_x f(x)$ and its minimum over all fractional covers by $\tau^*(F)$. The inequalities (28) are easy to show. The upper bound (29) was proved by Füredi, J. Kahn and P. Seymour.

Using results on τ^*, Frankl and Füredi proved (30). Here, equality holds if F is a symmetric $t-(n, k, 2)$ design. A. R. Calderbank gave a short proof using Delsarte's linear programming bound.

For graphs, the value of $2\tau^*$ is always an integer. It was proved by Chung, Füredi, M. R. Garey and R. L. Graham that the denominator of τ^* can be arbitrarily large for hypergraphs and, in fact, (31) even holds for hypergraphs of rank 3.

Excluded configurations. Unless otherwise stated F here denotes a family of k-subsets of $[n]$. By analogy with graphs one defines $ex(n, H_1, \ldots, H_s)$ by (32). A simple but useful result of Katona, Nemetz and Simonovits is (33). A family F is called **k-partite** if there is a partition $[n] = X_1 \cup \cdots \cup X_k$ with $|X_i \cap F| = 1$ for all $F \in F$ and $i = 1, \ldots, k$. Erdös proved in 1964 that $\phi(H_1, \ldots, H_s) = 0$ if and only if all H_j are k-partite. The most well-known case is that of complete k-graphs, that is, $ex\left(n, \binom{[l]}{k}\right)$. Despite numerous attacks, this problem remains unsolved for all $l > k \geq 3$. It dates back to Turán in 1961 who made several plausible conjectures (see (34), (35)).

The value of $\phi(\boldsymbol{H}_1, \ldots, \boldsymbol{H}_s)$ is known in only a very few cases. Let $C(k) = \{\{1, \ldots, k\}, \{1, \ldots, k-1, k+1\}, \{k, k+1, \ldots, 2k-1\}\}$. Then, respective results of Mantel, Bollobás, Sidorenko and Frankl/Füredi determine $\phi(C(k))$ for $k = 2, 3, 4$ and 5-6 (see (35)).

Define $R = \{\{1, 2, 3, 5, 7\}, \{1, 2, 3, 6, 8\}, \{1, 2, 4, 5, 8\}\}$. Using results of I. Ruzsa and Szemerédi, it was shown by Frankl and Füredi that $ex(n, R)$ has no exponent, that is, (36) holds.

The chromatic number. There are two natural ways to define the chromatic number of a hypergraph $\boldsymbol{H}$. One is to define $\chi(\boldsymbol{H}) = \chi(\sigma_2(\boldsymbol{H}))$, that is, as the chromatic number of the graph obtained by replacing each edge by a complete graph on the same vertex set. This is called the strong chromatic number and (37) is a well known open problem due to Erdös, Lovász and Faber (1973) related to it.

The other way of defining the chromatic number $\kappa(\boldsymbol{H})$ for hypergraphs $\boldsymbol{H}$ in which every edge has size at least 2 is given by (38). Let $m(k)$ denote the minimum number of edges in a hypergraph $\boldsymbol{H}$ of k-element sets with $\kappa(\boldsymbol{H}) = 3$. The best known bounds for $m(k)$ were obtained using the probabilistic method. The upper bound is due to Erdös, the lower bound to Beck (see (39)).

BOX

(1) $|F| \le 2^{n-1}$ holds for all intersecting families $\subset 2^{[n]}$.

(2) $|F| \le \binom{n-1}{k-1}$ holds if $F \subset \binom{[n]}{k}$ is intersecting and $n \ge 2k$.

(3) (i) $n+t$ even, $F(\text{even}) = \{F \subset [n] : |F| \ge (n+t)/2\}$.

(ii) $n+t$ odd, $F(\text{odd}) = \{F \subset n] : |F \cap [n-1]| \ge (n-1+t)/2\}$.

(4) $A_i = \{A \in \binom{[n]}{k} : \{A \cap [t+2i] \ge t+i\}$.

(5) $|A_0| \lesseqqgtr |A_1|$ according to whether $n \lesseqqgtr (k-t+1)(t+1)$

(6) If $n \ge (k-t+1)(t+1)$ and $F \subset \binom{[n]}{k}$ is t-intersecting then $|F| \le \binom{n-t}{k-t} = |A_0|$.

(7) If $F \subset \binom{[n]}{k}$ is t-intersecting, $n \ge 2k-t$, then $|F| \le \max |A_i|$.

(8) If $C, D \subset 2^{[n]}$ are complexes then $|C \cap D|/2^n \ge (|C|/2^n)(|D|/2^n)$.

(9) If $F_1, \ldots, F_k$ are intersecting families then $|F_1 \cup \cdots \cup F_k| \le 2^n - 2^{n-k}$.

(10) $S_r(F) = \{G \subset [n] : |F \Delta G| \le r\}$ is the sphere centered at F.

(11) $\sigma_l(F) = \{H \in \binom{[n]}{l} : \exists F \in F, H \subset F\}$.

(12) Let $F \subset \binom{[n]}{k}$, $|F| = \binom{x}{k}$, $x \ge k$, x real. Then $|\sigma_l(F)| \ge \binom{x}{l}$ for all $0 \le l \le k$.

(13) If $F \subset \binom{[n]}{k}$, $|F| > (s-1)^k k!$ then F contains a sunflower of size s.

(14) If F is an $\{l\}$-system of k-sets with $|F| > k^2 - k + 1$ then F is a sunflower.

(15) Determine or estimate $m(n, k, L)$, the maximum size of an L-system $F \subset \binom{[n]}{k}$.

(16) For every rational number $\alpha \geq 1$ there exist k and L so that $m(n, k, L)$ is of the order n^α.

(17) Suppose $F \subset \binom{[n]}{k}$, $|F \cap F'| \neq t-1$, holds for all $F, F' \in F$. If $n > n_0(k, t)$, $k \geq 2t$, then $|F| \leq \binom{n-t}{k-t}$ with equality if and only if F consists of all k subsets containing a fixed t-subset.

(18) Suppose $F \subset 2^{[n]}$, $|F| > 1.99^n$. Then there are $F, F' \in F$ with $|F \cap F'| = \lfloor n/4 \rfloor$.

(19) $\tau(F) = \min\{|T| : T \cap F \neq \emptyset$ for all $F \in F\}$; $\nu(F) = \max\{s :$ there are s pairwise disjoint edges in $F\}$.

(20) Let (F_i, G_i), $i = 1, \ldots, m$ be pairs of disjoint sets satisfying $F_i \cap G_j \neq \emptyset$ for $i \neq j$. Then $\sum_{1 \leq i \leq m} 1 \Big/ \binom{|F_i| + |G_i|}{|F_i|} \leq 1$.

(21) If F is an antichain then $|F| \leq \binom{n}{\lfloor n/2 \rfloor}$.

(22) If H is τ-critical then $|H| \leq \binom{\tau(F) + r(F) - 1}{r(F)}$.

(23) Suppose that $A_i \cap B_i = \emptyset$ but for all $i \neq j$, either $A_i \cap B_j$ or $A_j \cap B_i$ is non-empty. Then $\sum_i p^{|A_i|} q^{|B_i|} \leq 1$ holds for all positive reals p, q with

$p+q=1.$

(24) Suppose that $|A_i \cap B_i| < t$ but $|A_i \cap B_j| \geq t$ for $1 \leq i < j \leq m$. Then

$$m \leq \binom{r(A)+r(B)-2t}{r(A)-t}.$$

(25) For every $\sigma \geq 0$ there are only a finite number of connected graphs with minimum degree 3 which satisfy $\sigma(G) = \sigma$.

(26) $\nu(H) \leq \tau \binom{\tau+r-2}{r-2} + \tau^{r-1}.$

(27) $\sum_{x \in F} f(x) \geq 1$ for all $F \in \boldsymbol{F}$, $f(x) \geq 0$ for all $x \in V(\boldsymbol{F})$.

(28) $\nu(\boldsymbol{F}) \leq \tau^*(\boldsymbol{F}) \leq \tau(\boldsymbol{F}) \leq r(\boldsymbol{F})\nu(\boldsymbol{F}).$

(29) There is a collection $F_1, \ldots, F_s$ of disjoint edges in every hypergraph $\boldsymbol{F}$ such that

$$\tau^*(\boldsymbol{F}) \leq \sum_{1 \leq i \leq s} |F_i| - 1 + \frac{1}{|F_i|} \leq \nu(\boldsymbol{F}) \left(r(\boldsymbol{F}) - 1 + \frac{1}{r(\boldsymbol{F})} \right).$$

(30) $\tau^*(\boldsymbol{F}) \leq (k^2-k+t)/t$ for every t-intersecting family $\boldsymbol{F}$ of rank k.

(31) For every rational $0 \leq \beta < 1$, there is a hypergraph $\boldsymbol{H}$ with $\tau^*(\boldsymbol{H}) - \beta$ being integer.

(32) $ex(n, \boldsymbol{H}_1, \ldots, \boldsymbol{H}_s) = \max\{|\boldsymbol{F}| : \boldsymbol{H}_i$ is not a subhypergraph of $\boldsymbol{F}$, $i = 1, \ldots, s\}$.

(33) $ex(n, \boldsymbol{H}_1, \ldots, \boldsymbol{H}_s) \,/\, \binom{n}{k}$ is monotone decreasing in n and so,

$$\phi(\boldsymbol{H}_1, \ldots, \boldsymbol{H}_s) = \lim_{n \to \infty} ex(n, \boldsymbol{H}_1, \ldots, \boldsymbol{H}_s) \,/\, \binom{n}{k} \text{ exists.}$$

(34) $\phi\left(\binom{[4]}{3}\right) = 5/9$, $\phi\left(\binom{[2k-1]}{k}\right) = 1 - 1/2^{k-1}$.

(35) $\phi(C(k)) = k!/k^k$ for $k = 2, 3, 4$; $\phi(C(5)) = 6/11^4$ and $\phi(C(6)) = 11/12^5$.

(36) $\lim_{n\to\infty} ex(n, R)/n^4 = 0$ and, for every $\beta < 4$, $\lim_{n\to\infty} ex(n, R)/n^\beta = \infty$.

(37) Let $F = \{F_1, \ldots, F_k\}$ be a collection of k-sets with $|F_1 \cap F_j| \leq 1$ for $1 \leq i < j \leq k$. Then $\chi(F) \leq k$.

(38) $\kappa(H) = \min\{s$: there is a partition $V(H) = X_1 \cup \cdots \cup X_s$ with no X_i containing an edge of $H\}$.

(39) There are constants c and d such that

$$ck^{1/3} < m(k)/2^k < dk^2 .$$

Partially ordered sets

A partially ordered set (or, poset, for short) is a pair $(P, \leq)$ where P is a set (for us always finite) and $\leq$ is a reflexive, anti-symmetric and transitive relation on P (see (1)).

The standard examples of posets are (i) $[n] = \{1, 2, \ldots, n\}$ with $\leq$ having the usual meaning, (ii) $2^{[n]}$ with $A \leq B$ if A is a subset of B, (iii) $[n]$ with $a \leq b$ if a divides b.

A special property of (i) is that any two elements in it are comparable. Such a partially ordered set is called a chain or linear order. A chain consisting of $l+1$ elements is said to have length l. The opposite concept of a chain is an antichain, a subset of a poset in which no two elements are comparable. For a finite poset P we denote by $s(P)$ its Sperner number, i.e., the maximum size of an antichain in P.

In 1950 Dilworth proved that (2) holds.

Posets are most conveniently represented by their Hasse diagrams. This is a directed graph in which one draws an edge from a to b if $a < b$, provided there is no third element c with $a < c < b$ (we then say that b covers a) (see Figure 14).

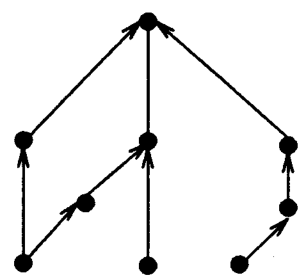


Figure 14. The Hasse diagram of a poset.

A poset P is called ranked if all maximal chains in P have the same length, or, equivalently, if P can be partitioned into $P = P_0 \cup P_1 \cup \cdots \cup P_r$ such that all edges of the Hasse diagram go from P_i to P_{i+1} for some $i < r$. In this case, $r(P) = r$ is called the rank of the poset, e.g., $r(2^{[n]}) = n$, if $x \in P_i$ then the rank $r(x)$ of x is defined to be i. The f-vector $f = (f_0, \ldots, f_r)$ of a ranked poset is given by $f_i = |P_i|$. Clearly, each P_i is an antichain. If $s(P) = f_i$ for some i, then P is said to have the Sperner property since Sperner proved in 1928 that $2^{[n]}$ has this property. A stronger property, called the LYM property, is defined by (3). In 1974 Kleitman showed that the LYM property is equivalent to the normalized matching property of Graham and L. Harper (see (4)) and to the existence of a regular covering by chains (see (5)).

The poset $2^{[n]}$ is a LYM poset, as is $L_n(q)$, the poset of all subspaces of an n-dimensional vector space over the finite field of q elements.

Taking the union of l ranks in a ranked poset, we obtain a set which contains no chain of length l or more. In LYM posets (6) holds. For $2^{[n]}$, (6) was proved by Erdös in connection with a problem of Littlewood and Offord. For the more general problem Kleitman proved (7). Kleitman's proof proceeds by partitioning the set of all 2^n signed sums into $\binom{n}{\lfloor n/2 \rfloor}$ groups in such a way that any two sums in the same group are at distance at least two, and therefore, could not be in the same ball. The idea of this proof is based on what is called a symmetric chain decomposition of a poset. A chain in a poset of rank n is called symmetric if for some $i \leq \frac{n}{2}$, it consists of $n-2i+1$ elements, one for each rank j, $i \leq j \leq n-i$. A poset is a symmetric chain poset if its elements can be partitioned into symmetric chains. An important property of symmetric chain posets is unimodulity and rank symmetry (see (8)). The posets $L_n(q)$ are symmetric chain posets. Define the posets $L(k, n)$ by (9). A deep result of Stanley, whose proof uses the hard Lefshetz theorem for complex varieties, asserts that $L(k, n)$ is a symmetric chain poset.

Another poset, which was investigated mostly in connection with the Littlewood-Offord problem, is $P(X_1, \ldots, X_k)$, where the X_i are disjoint sets, the elements are the subsets of $X_1 \cup \cdots \cup X_k$ with $A \leq B$ if $A \subset B$ and for all but one i, $A \cap X_i = B \cap X_i$ holds. Katona and Kleitman proved that for $k=2$ this poset has the Sperner-property and is a symmetric chain poset. This is no longer true for $k \geq 3$. Asymptotically best possible results on $s(P(X_1, \ldots, X_k))$ were obtained by Füredi, Griggs, Odlyzko and Shearer.

Möbius inversion and lattices. The theory of posets plays an important unifying role in enumerative combinatorics. In 1935 Weisner discovered that the principle of inclusion and exclusion can be extended to posets.

This idea was taken up by G.-C. Rota who in a seminal paper written in 1964 began the systematic study of posets arising in combinatorics.

One can define a function, called the Möbius function, $\mu(x, y)$ (see (10)) for any poset. Then the Möbius inversion theorem can be stated by (11). If the poset P has a unique minimal element, say $\hat{0}$, and a unique maximal element $\hat{1}$ then the value $\mu(\hat{0}, \hat{1})$ of the Möbius function is precisely the reduced Euler characteristic of a topological space, the order complex of P. This permits the use of the well-developed machinery of algebraic topology in conjunction with Möbius inversion. For example, Cohen-Macaulay posets can be defined in terms of homology properties of the subposets $\{z : x \leq z \leq y\}$ where $x, y \in P$. Then (12) expresses an important property of these posets. In 1975, Stanley used the Cohen-Macaulay property in proving the upper-bound-theorem for triangulated spheres. Lattices are specific posets P for which any two elements x, y, their join $x \vee y$ and their meet $x \wedge y$ satisfying (13) exist. There are general methods for computing the Möbius function of lattices. Every lattice has a $\hat{0}$ and a $\hat{1}$. Elements of rank 1 in a lattice are called atoms, and if every element of the lattice is the join of atoms then it is called atomic. Examples (i) and (ii) above are lattices; however (iii) is not. Also (ii) is atomic but (i) is not.

Matroids. For a ranked poset P and $x \in P$, let $r(x)$ denote the rank of x. A lattice is called semi-modular if (14) holds. An atomic semimodular lattice M is called a matroid. Often, a matroid M is defined via independent sets. Call the set of atoms $\{x_1, \ldots, x_k\}$ independent if $r(x_1 \vee \cdots \vee x_k) = k$. The family $I(M)$ of all independent sets of M satisfies (15). If one defines matroids via independent sets and (15), then it might happen that some 1- and 2-element sets are not independent. However, deleting non-independent (isolated) elements and contracting 2-element non-independent sets brings us back to matroids arising from lattices.

To every graph G one can associate a matroid $M(G)$ whose elements are the edges of G, and a set of edges being independent if they form a forest (contain no cycle).

Matroids were introduced by Whitney in 1935 in order to isolate the abstract properties of linear independence. Indeed, if B is an arbitrary finite subset of a vector space over some field (finite or infinite) then one can define a matroid by saying that $A \subset B$ is independent if the smallest affine subspace containing A has dimension $|A| - 1$. A matroid is said to be representable over a field $\mathbb{F}$ if it is isomorphic to a matroid obtainable in the above way for some vector space over $\mathbb{F}$.

Whitney characterized (in terms of excluded minors) matroids representable over $GF(2)$. In 1958 Tutte gave a similar characterization of matroids representable over all fields. Matroids representable over $GF(3)$ were characterized by R. Bixby and Seymour. However, for $GF(q)$, $q > 3$ the problem is still wide open. Tutte has also characterized graphic matroids, that is, matroids arising from graphs. Let us call a matrix M totally unimodular if all square submatrices of M have determinant 0 to ± 1. An elegant decomposition theorem for totally unimodular matrices was given by Seymour (which shows, in particular, how any such matrix can be built up in a precise way from a small collection of simple pieces), using deep applications of matroid theory.

The posets of matroids are Cohen-Macaulay in a strong sense and therefore the Möbius function satisfies (12) — this was proved by J. H. Folkman. Also, $\mu(x, y) \neq 0$ for $x \leq y$ in a matroid. A well known conjecture of Rota states that the f-vector of every matroid is unimodular (see (16)).

Note that in a lattice $f_0 = f_n = 1$ holds. A celebrated result of Dowling and Wilson from 1975 is (17). Even the special case $k=1$ of (17), that is $f_1 \leq f_{n-1}$ is fairly non-trivial and extends earlier results of deBruijn and Erdös (1948) and Motzkin (1951). Let us mention that (17) can be quite easily deduced from a 1969 result of Lindström involving the Möbius function of lattices. An even stronger conjecture for the f-vectors of geometric lattices is log-concavity (see (18)).

Linear extensions and dimensions. A linear extension of a poset P is an arrangement $x_1, \ldots, x_n$ of the elements of P such that $x_i \leq x_j$ implies $i \leq j$.

For a fixed $x \in P$ let g_i be the number of linear extensions with $x = x_i$. In response to a conjecture of Chung, Fishburn and Graham, it was proved by Stanley that the sequence $g_1, \ldots, g_n$ is log concave.

The most tantilizing open problem dealing with linear extensions is the so-called "$\frac{1}{3}, \frac{2}{3}$" conjecture (see (19)). The best bounds currently known are $\frac{3}{11}$ and $\frac{8}{11}$, and are due to J. Kahn and M. Saks. The proof of both of the mentioned results used the Alexandrov-Fenchel inequality.

The FKG-inequality was used by Shepp to prove several positive correlation results about linear extensions. A simple example is (20), where $p(x < y)$ is the proportion of linear extensions with x preceding y among all linear extensions.

The dimension of a poset P (also called Dushnik-Miller dimension) *dim* P is defined as the minimum integer m for which there exist linear extensions $L_1, \ldots, L_m$ of P with $x \leq y$ in P if and only if $x \leq y$ in all the L_i.

Clearly, *dim* $P = 1$ if and only if P is a chain. Also, *dim* $2^{[n]} = n$ is not hard to verify. The simplest example of a poset of dimension n is the crown C_n (see Figure 15) which is a poset of rank 1, with Hasse diagram equal to $K_{n,n}$ minus a perfect matching.

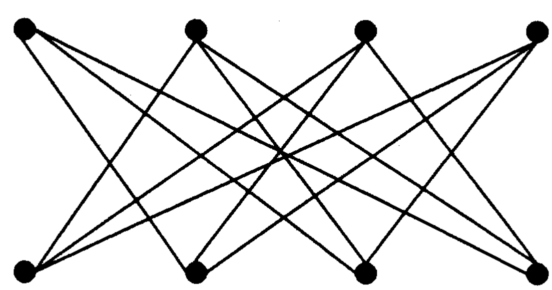


Figure 15. The poset C_4.

Füredi and Kahn proved that if every element of a poset P is comparable to at most k other elements, then *dim* $(P) < 50k \log^2 k$. The example of C_{k+1} shows that *dim* $P \leq k$ is not true in general.

BOX

(1) $a \leq a$; $a \leq b$ and $b \leq a$ imply $a = b$; $a \leq b$ and $b \leq c$ imply $a \leq c$.

(2) every finite poset P is the union of $s(P)$ chains.

(3) P is a LYM poset if $\sum_i |A \cap P_i| / f_i \leq 1$ holds for all antichains A.

(4) Any subset $B \subset P_i$ is connected (in the Hasse diagram) to at least $|B| f_{i+1} / f_i$ elements in P_{i+1}.

(5) There is a non-empty collection of maximal chains such that for every i every element of P_i occurs in the same number of chains.

(6) If $A \subset P$ contains no chain of length l then $|A|$ is at most the sum of the largest l among the numbers f_i.

(7) If $v_1, v_2, \ldots, v_n$ are vectors of at least unit length in $\mathbb{R}^d$ then out of the 2^n signed sums $\sum_{i=1}^{n} \epsilon_i v_i$, $\epsilon_i = \pm 1$ at most $\binom{n}{\lfloor n/2 \rfloor}$ can be in any particular open ball of radius 1.

(8) $f_0 \leq f_1 \leq \cdots \leq f_{\lfloor n/2 \rfloor} \geq f_{\lfloor n/2 \rfloor + 1} \geq \cdots \geq f_n$; $f_i = f_{n-i}$.

(9) $L(k, n) = \{(x_1, \ldots, x_k) : x_i$ integer, $0 \leq x_i \leq \cdots \leq x_k \leq n\}$ and $(x_1, \ldots, x_k) \leq (y_1, \ldots, y_k)$ if $x_i \leq y_i$ for all $1 \leq i \leq k$. The rank of an element is $x_1 + \cdots + x_k$.

(10) $\mu(x, y) = 0$ unless $x \leq y$. For fixed x and $x \leq y$ $\mu(x, y)$ is defined inductively by $\mu(x, x) = 1$, $\mu(x, y) = -\sum_{x \leq z < y} \mu(x, z)$.

(11) Let $f, g : P \to \mathbb{R}$ be functions. Then "$g(x) = \sum_{y \leq x} f(x)$ for all $x \in P$" is equivalent to "$f(x) = \sum_{y \leq x} \mu(y, x) g(x)$ for all $x \in P$."

(12) If $x \leq y \leq z$ and z covers y with x, y, z elements of a Cohen-Macaulay poset then $\mu(x, y)\mu(x, z) \leq 0$, i.e., $\mu(x, y)$ alternates in sign.

(13) If $x \leq z$, $y \leq z$ then $x \vee y \leq z$ and if $u \leq x$, $u \leq y$ then $u \leq x \wedge y$.

(14) $r(x \vee y) + r(x \wedge y) \leq r(x) + r(y)$ for all $x, y \in P$.

(15) (i) $I(M)$ is a non-empty complex

(ii) if $I, J, \in I(M)$ with $|I| < |J|$ then for some $x \in J$ one has $(I \cup \{x\}) \in I(M)$.

(16) $f = (f_0, \ldots, f_n)$ is called unimodular if for some $0 \leq j \leq u$

$$f_0 \leq \cdots \leq f_j \geq f_{j+1} \geq \cdots \geq f_n .$$

(17) $f_0 + f_1 + \cdots + f_k \leq f_{n-k} + \cdots + f_n$ holds for all $k < n/2$.

(18) $f_i^2 \geq f_{i-1} f_{i+1}$ for $1 \leq i < n$.

(19) In every poset P which is *not* a linear order, there exist two elements x, y with $x < y$ holding in at least 1/3 but at most 2/3 of all linear extensions of P.

(20) Let x, y, z be elements of a poset P then $p(x < y$ and $x < z) \geq p(x < y) p(x < z)$.

Ramsey theory

Ramsey theory is concerned with finding in large collections of objects a relatively large subset which is homogeneous in some sense.

The simplest example is the box principle: if we have $kn+1$ objects distributed into n boxes, then one of the boxes will contain at least $k+1$ of them.

A more difficult example is provided by a theorem of Erdös and Szekeres (see (1)).

Another example is (2), which is an easy corollary of Dilworth's theorem. Ramsey-type theorems often turn up in combinatorics, algebra and geometry; we shall treat them separately.

Ramsey theorem for graphs and hypergraphs. In 1930, the brilliant young English logician Ramsey proved (3). For example, for $k=1$, (3) is just a reformulation of the box principle. For $k=r=2$ it can be stated, as follows: for n sufficiently large every graph on n vertices or its complement contains a complete graph on l vertices. Ramsey was not concerned with bounds on $R(l, k, r)$.

Erdös and Szekeres discovered Ramsey's theorem independently in 1935 and they proved the bound (4) for graphs. Despite considerable efforts, (4) has not been substantially improved since then, and even (5), a conjecture of Erdös, remains unresolved. By (4), if the limit exists, its value is at most 4 and by a result of Erdös (cited earlier) it is at least $\sqrt{2}$. $R(3,2,2)=6$ is easy to see and $R(4,2,2)=17$ was established by A. Gleason and R. Greenwood. However, the determination of the exact value of $R(5,2,2)$ (not to mention, say, $R(10,2,2)$)

appears to be hopeless at our present stage of knowledge. For $k \geq 3$ the best upper bound is (6). Hajnal showed that for $r \geq 4$ there is a corresponding lower bound with $c(r)$ replaced by a suitable, small positive constant $b(r)$.

Homogeneous equations and Rado's theorem. Let $\mathbb{N}$ denote the set of all positive integers. A k-cube M is defined by (7). What can be said to be the first result in Ramsey theory was due to the celebrated German mathematician David Hilbert, who showed in 1892 that for every partition of $\mathbb{N}$ into r classes, one of the classes will contain infinitely many k-cubes.

In 1916, Schur proved that if one partitions the integers up to $\lfloor r!e \rfloor$ into r classes, then one of the classes must contain x, y and z with $x+y=z$.

Another classical result — due to B. L. van der Waerden in 1927 — is that there exists a minimum integer $w(k, r)$ such that if $[w(k, r)]$ is partitioned into r classes then one of the classes always contains an arithmetic progression of length k, i.e., $\{a, a+d, \ldots, a+(k-1)d\}$ for some $a, d \geq 1$.

Van der Waerden's result can be restated as saying that for all partitions of $[w(k, r)]$ into r classes, one of the classes contains a non-trivial solution to the system of homogeneous equations (8). The general case of systems of homogeneous equations (see (9)) was settled by Schur's student R. Rado in 1933.

Rado proved that a system has a non-trivial monochromatic solution for every partition of $\mathbb{N}$ into finitely many classes if and only if M satisfies the columns-condition (10). In the case of (8) one takes $t=1$ to see that (10) is satisfied. We point out that Hilbert's theorem corresponds to the special case when M is a (0-1)-

matrix with $M(1, \ldots, 1)^T = \vec{0}$. If a system (9) satisfies the columns-condition (10), then it is called regular.

Call a set $L \subset \mathbb{N}$ large if every regular system has a non-trivial solution in L. Proving a conjecture of Rado, it was shown in 1973 by W. Deuber that for every partition $L = L_1 \cup \cdots \cup L_r$, some L_i must be large. Rado's theorem can be used to establish (11), which has been also proved by J. Folkman and J. Sanders. (11) is known as the finite unions theorem. In 1974, N. Hindman proved the infinite version (11a) of the finite union theorem, thereby settling a conjecture of Graham and B. L. Rothschild.

Euclidean Ramsey theory. The reason behind Erdös and Szekeres' interest in the existence of $R(l, k, r)$ was their desire to show the existence of a function $c(l)$ with the property that for any set of $c(l)$ points chosen in the plane (with no three lying on a straight line), one could always find l of them forming the vertices of a convex polygon. Their proof showed that it sufficies to take $c(l) = R(l, 4, 2)$. The best bounds known, also due to Erdös and Szekeres, are given in (12). It is generally believed that the lower bound gives the exact answer (this is true for $l = 3, 4$ and 5). Since the problem originated with Esther Klein, who subsequently became Mrs. Szekeres, the result has sometimes been called the Happy End theorem.

In a series of papers in 1972-73 Erdös, Graham, P. Montgomery, Rothschild, Spencer and E. G. Straus introduced another class of geometric Ramsey problems. Given a finite set S of points in $\mathbb{E}^d$, d-dimensional Euclidean space, we say that S is Ramsey if for all r there exists $n = n(r, S)$ with the property that for every partition of $\mathbb{E}^n$ into r sets one of the sets contains a subset S congruent (isometric) to S.

This and other Ramsey-type problems can be formulated in the language of hypergraphs. Let $E(n, S)$ be the hypergraph with vertex set $\mathbb{E}^n$ and having as edges those subsets S which are congruent to S. Then S is Ramsey if and only if $\chi(E(n, S))$ tends to infinity with $n \to \infty$.

The above authors proved that if S is a subset of the vertex set of some brick (rectangular parallelepiped) then S is Ramsey and if S is Ramsey, then S is spherical (all $x \in S$ are at the same (Euclidean) distance from some point of $\mathbb{E}^d$).

Frankl and Rödl showed that if the points of S are affinely independent, then it is Ramsey. They actually showed that $\chi(E(n, S))$ tends to infinitely exponentially rapidly both in this case, and if S is a brick. At present no Ramsey set $S \subset \mathbb{E}^d$ with $|S| > 2^d$ is known. (13) and (14) are the two most outstanding open problems of this subfield.

Parameter sets and spaces. The family of k-dimensional subspaces of an n-dimensional vector space V over the field of q elements shares many common partition properties with $\binom{[n]}{k}$. We denote it by $\begin{bmatrix} V \\ k \end{bmatrix}$. It was conjectured by Rota and proved by Graham, K. Leeb and Rothschild that $\begin{bmatrix} V \\ k \end{bmatrix}$ has the Ramsey property, that is, (15) holds. They proved the corresponding statement for affine and projective spaces as well. These results are non-trivial even for $k=0$. Namely, van der Waerden's theorem can be deduced from this case of the affine Ramsey theorem.

Consider all functions $f : [n] \to [a]$, or equivalently, all integer sequences $(x_1, \ldots, x_n)$ with $1 \le x_i \le a$. One can define specific a-element subsets, called combinatorial lines by (16). This structure is called the Hales-Jewett cube and is denoted by $HJ(n, a)$.

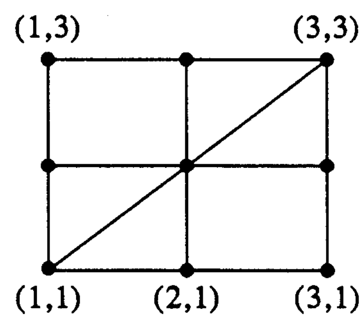


Figure 16. The lines in $HJ(2,3)$.

A result of Hales and Jewett from 1963 asserts that $HJ(n, a)$ has the Ramsey property for $n \ge h(a, r)$.

One can define combinatorial subcubes of dimension k, $1 \le k \le n$, in $HJ(n, a)$ (also called k-parameter sets) in a way analogous to lines (M is replaced by $M_1 \cup \cdots \cup M_k$ and $x_i = x_j$ if $i, j \in M_t$ for some t. A k-dimensional subcube is a special a^k-element subset. A basic theorem of Graham and Rothschild (1971) shows that (17) holds.

In 1987, the Israeli logician S. Shelah proved that the function $h(a, l, k, q, r)$ is primitive recursive as a function of a, although the upper bound he provides has the growth rate of $f_6(a)$ (see (18)).

Induced and restricted Ramsey theorems. Most of the results here can be stated and proved for hypergraphs and systems of spaces (vector, affine, projective or combinatorial). However, to make them easier to understand, we shall restrict ourselves to graphs. Let us start with the definition (19) of induced Ramsey graphs. It was proved independently by three sets of authors, Deuber, Erdös-Hajnal-Pósa and Rödl in 1973 that for every r and every G there is an induced Ramsey graph for G with r colors. None of the proofs worked for hypergraphs, however. A few years later Nesetril and Rödl invented a powerful construction, called (partite) amalgamation which enables one to obtain strong induced Ramsey statements for structures for which a Ramsey theorem is known (hypergraphs, spaces).

Nesetril and Rödl used their methods to show the existence of induced Ramsey graphs H for arbitrary G and r with the additional restriction $\omega(H)=\omega(G)$ and $g(H)=g(G)$. Such a result is called a restricted Ramsey theorem. The first such result is the restricted version of van der Waerden's theorem (20). This is an easy consequence of the Hales-Jewett theorem for $a=l$. Namely, with $(x_1, \ldots, x_n)$ one associates the integer $\sum_{i=1}^{n} x_i a^{2i}$.

Density theorems. For an infinite sequence A of integers $a_1 < a_2 < \cdots < a_n < \cdots$, one defines the upper density $\bar{d}(A)$ by (21).

It was conjectured by Erdös and Turán in 1936, and proved by Szemerédi almost 40 years later, that if A has positive upper density then it contains arithmetic progressions of arbitrary finite length. In 1977 Fürstenberg (using tools from ergodic theory) gave a new proof of Szemerédi's theorem.

Define (k, l)-boxes by (22) as special subsets of the lattice points in $\mathbb{R}^k$. For $k=1$, a (k, l)-box is just an arithmetic progression of length l. Using the powerful methods from Fürstenberg's approach, Fürstenberg and Katznelson proved a powerful density theorem (see (23)) for boxes.

The corresponding Ramsey theorem was due to Gallai and W. Witt.

The density theorem for affine spaces was proved by Fürstenberg, Katznelson and B. Weiss (see (24)).

It is still a challenging open problem as to whether the analogue of (24) holds for the Hales-Jewett cube $HJ(n, a)$.

For regular sets of equations Szemerédi's theorem can be used to show that a density statement holds if and only if $M\vec{1}=\vec{0}$.

(25) is an interesting open problem due to P. Cameron, which would be a density version of Schur's theorem.

We close this section by mentioning the following conjecture of Erdös (for which he has a standing offer of \$3000 for its solution, which is unsolved even for the case $l=3$).

Let $A=\{a_1 < \cdots < a_n < \cdots\}$ be an infinite sequence of integers with $\sum_{i\geq 1} 1/a_i = \infty$. Then A contains arbitrarily long arithmetic progressions.

Note that if true, this implies the existence of arbitrarily long arithmetic progressions among the primes.

Canonical colorings. In Ramsey's theorem, the number of partition classes (or colors) is fixed. What can be true if an unlimited number of colors is used. Call a coloring of the set of unlimited ordered k-tuples of $S \subset [n]$ canonical if (26) holds.

In 1950, Erdös and Rado proved the canonical Ramsey theorem (27). In this connection let us mention one of the rare exact bounds in Ramsey theory, (28), given by Chung and Graham.

Erdös and Graham obtained the canonical version of van der Waerden's theorem, stating that for all colorings of N either there is a monochromatic arithmetic progression of length l, or one with all members having distinct colors.

The canonical version of the full Rado theorem was proved by Lefmann in 1985.

BOX

(1) Every sequence of $kn+1$ integers contains either an increasing subsequence of $k+1$ terms or a decreasing subsequence of $n+1$ terms.

(2) If P is a poset, $|P| \geq kn+1$, then either P contains a chain of $n+1$ elements or an antichain of $k+1$ elements.

(3) For any positive integers l, k, r; $l \geq k$ there exists an $n = R(l, k, r)$ such that for every partition of $\binom{[n]}{k}$ into r classes, one of the classes contains $\binom{|Y|}{k}$ for some $Y \in \binom{[n]}{l}$.

(4) $R(l+1, 2, r) \leq (rl)! \,/\, l!^r < r^{rl}$.

(5) Does $\lim_{l \to \infty} R(l, 2, 2)^{1/l}$ exist?

(6) $R(l, k, r) < 2^{2^{\cdot^{\cdot^{\cdot^{c(r)l}}}}}$ (with $k-2$ 2's and where $c(r)$ is a constant depending on r).

(7) Let $a, d_1, \ldots, d_k \in \mathbb{N}$ and define $M = \{a + \sum_{i=1}^{k} \epsilon_i d_i : \epsilon_i = 0 \text{ or } 1\}$.

(8) $x_i - 2x_{i+1} + x_{i+2} = 0$, $i = 1, \ldots, k-2$.

(9) $M\vec{x}^T = \vec{0}$ where $\vec{x} = (x_1, x_2, \ldots, x_k)$ and M is an s by k integer matrix of rank s.

(10) After possibly rearranging the columns of M, there exist integers $1 \leq n_1 < n_2 < \cdots < n_t = k$ such that for the column vectors $\vec{c}_1, \ldots, \vec{c}_k$, the following is true: $\vec{c}_{n_i+1} + \cdots + \vec{c}_{n_{i+1}}$ is a rational linear combination of $\vec{c}_1, \ldots, \vec{c}_{n_i}$; $i = 0, \ldots, t-1$ (in particular, $\vec{c}_1 + \cdots + \vec{c}_{n_1} = \vec{0}$.

(11) For $n > n_0(k, r)$, if $2^{[n]}$ is partitioned into r families $F_1, \ldots, F_r$ then for some F_i, there are k pairwise disjoint sets $A_1, \ldots, A_k$ such that all $2^k - 1$ unions $\bigcup_{j \in I} A_j$, $\varnothing \neq I \subset [k]$, are in F_i.

(11a) If all the finite subsets of the integers are partitioned into finitely many classes than one of the classes contains infinitely many pairwise disjoint sets along with all their (finite) unions.

(12) $1 + 2^{k-2} \leq c(k) \leq \binom{2k-2}{k-1} + 1.$

(13) Are all spherical sets Ramsey?

(14) Does $\chi(E(2, S)) \geq 3$ hold for all non-collinear sets $S = \{x, y, z\}$ for x, y, z not forming an equilateral triangle?

(15) For $n > n_0(l, k, q, r)$ if $\begin{bmatrix} V \\ k \end{bmatrix}$ is partitioned into r classes, one of the classes must contain $\begin{bmatrix} W \\ k \end{bmatrix}$ for some l-dimensional subspace W of V.

(16) Let $[n] = C \cup M$ be a partition with $M \neq \varnothing$ and $b_c \in [a]$ for all $c \in C$ be given. Define

$$L := \{(x_1, \ldots, x_n) : x_c = b_c \text{ for } c \in C \text{ and } x_i = x_j \text{ for } i, j \in M\}.$$

(17) For given a, l, k, q and r, there exists $h = h(a, l, k, q, r)$ such that for all partitions of the k-dimensional subcubes of $HJ(n, a)$ into r classes some l-dimensional subcube has all its k-dimensional subcubes in the same class.

(18) Set $f_1(x) = 2x$ and define $f_{i+1}(x) = f_i^{(x)}(x)$ where $f^{(x)}(y)$ denotes the xth iterate of $f(y)$, e.g., $f_2(x) = 2^x$, $f_3(x) = 2^{2^{\cdot^{\cdot^{2}}}}$ with the number of 2's equal

to x.

(19) The graph H is called an induced Ramsey graph for G with r colors if for all partitions of the edges of H into r classes, one can find a subset $V \subset V(H)$ such that V spans in H a subgraph isomorphic to G, and all the edges of this subgraph are in the same class.

(20) For every r and l there exists a finite subset S of the integers which contains no arithmetic progression of length $l+1$ but for any partition into r classes, one of the classes contains one of length l.

(21) $\bar{d}(A) := \limsup_{n \to \infty} |A \cap [n]|/n.$

(22) Let $a_1, \ldots, a_k$ and d be positive integers. Then the set

$$\{(a_1 + x_1 d, \ldots, a_k + x_k d) : 0 \leq x_i < l\}$$

is called a (k, l)-box.

(23) For every real $\epsilon > 0$ and $k, l \geq 2$ there exists an $n = n_0(\epsilon, k, l)$ such that if A is a set of lattice points $(x_i, \ldots, x_k)$ with $1 \leq x_i \leq n$ and $|A| > \epsilon n^k$ then A contains a (k, l)-box.

(24) For every $\epsilon > 0$ and q there is an $n = n_0(\epsilon, q)$ such that any subset of size at least ϵq^n of an n-dimensional vector space over the field of q elements contains an affine line.

(25) Does there exist an infinite sequence $a_1 < a_2 < a_3 \ldots$ of integers such that every subsequence B of positive relative upper density contains three elements x, y, z with $x + y = z$?

(26) Let T be a (possibly empty) subset of $[k]$. Then S is said to be canonically colored if for all $A = \{a_1, \ldots, a_k\}$, $B = \{b_1, \ldots, b_k\} \subset S$ with $a_1 < \cdots < a_k$, $b_1 < \cdots < b_k$, A and B have the same color if and only if $a_i = b_i$ holds for all $i \in T$.

(27) For $n > n_0(l, k)$ and for all colorings of $\binom{[n]}{k}$, some l-subset S is colored canonically.

(28) Suppose that $\binom{[n]}{2}$ is colored by $2r$ colors so that all triangles have (exactly) 2 colors. Then $n \leq 5^r$ and this is best possible.

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Atts.
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