Ramsey Theory in the Work of Paul Erdős

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Summary. Ramsey's theorem was not discovered by P. Erdős. But perhaps one could say that Ramsey theory was created largely by him. This paper will attempt to demonstrate this claim.

1. Introduction

Ramsey's theorem was not discovered by Paul Erdős. This was barely technically possible: Ramsey proved his theorem in 1928 (or 1930, depending on the quoted source) and this is well before the earliest Erdős publication in 1932. He was then 19. At such an early age four years makes a big difference. And also at this time Erdős was not even predominantly active in combinatorics. The absolute majority of the earliest publications of Erdős is devoted to number theory, as can be seen from the following table:

	1932	1933	1934	1935	1936	1937	1938	1939
all papers	2	0	5	10	11	10	13	13
number theory	2	0	5	9	10	10	12	13

The three combinatorial exceptions among his first 8^2 papers published in 8 years are 2 papers on infinite Eulerian graphs and the paper [1] by Erdős and G. Szekeres. Thus, the very young P. Erdős was not a driving force of the development of Ramsey theory or Ramsey-type theorems in the thirties. That position should be reserved for Issac Schur who not only proved his sum theorem [2] in 1916 but, as it appears now [3], also conjectured van der Waerden's theorem [4], proved an important extension, and thus put it into a context which inspired his student R. Rado to completely settle (in 1933) the question of monochromatic solutions of linear equations [5]. This result stands apart even after 60 years.

Yet, in retrospect, it is fair to say that P. Erdős was responsible for the continuously growing popularity of the field. Ever since his pioneering work in the thirties he proved, conjectured and asked seminal questions which together, some 40 to 50 years later, formed Ramsey theory. And for Erdős, Ramsey theory was a constant source of problems which motivated some of the key pieces of his combinatorial research.

It is the purpose of this note to partially justify these claims, using a few examples of Erdős' activity in Ramsey theory which we will discuss from a contemporary point of view.

In the first section we cover paper [1] and later development in a great detail. In Section 2, we consider the development based on Erdős' work related to bounds on various Ramsey functions. Finally, in Section 3 we consider his work related to structural extensions of Ramsey's theorem.

No mention will be made of his work on infinite extensions of Ramsey's theorem. This is covered in these volumes by the comprehensive paper of A. Hajnal.

2. The Erdős-Szekeres Theorem

F. P. Ramsey discovered his theorem [6] in a sound mathematical context (of the decision problem for a class of first-order formulas). But since the time of Dirichlet the "Schubfach principle" and its extensions and variations played a distinguished role in mathematics. The same holds for the other early contributions of Hilbert [19], Schur [2] and van der Waerden [4].

Perhaps because of this context Ramsey's theorem was never regarded as a puzzle and/or a combinatorial curiosity only. Thanks to Erdős and Szekeres [1] the theorem found an early application in a quite different context, namely, plane geometry:

Theorem 2.1 **([1]).** *Let n be a positive integer. Then there exists a least integer $N(n)$ with the following property: If X is a set of $N(n)$ points in the plane in general position (i.e. no three of which are collinear) then X contains an n-tuple which forms the vertices of a convex n-gon.*

One should note that (like in Ramsey's original application in logic) this statement does not involve any coloring (or partition) and thus, by itself, fails to be of "Ramsey type". Rather it fits to a more philosophical description of Ramsey type statements as formulated by Mirsky:

> "There are numerous theorems in mathematics which assert, crudely speaking, that every system of a certain class possesses a large subsystem with a higher degree of organization than the original system."

It is perhaps noteworthy to list the main features of the paper. What a wealth of ideas it contains!

I. It is proved that $N(4) = 5$ and this is attributed to Mrs. E. Klein. This is tied to the social and intellectual climate in Budapest in the thirties which has been described both by Paul Erdős and Szekeres on several occasions (see e.g. [7]), and with names like the Happy End Theorem.

II. The following two questions related to statement of Theorem 2.1 are explicitly formulated:

(a) Does the number $N(n)$ exist for every n?
(b) If so, estimate the value of $N(n)$.

It is clear that the estimates were considered by Erdős from the very beginning. This is evident at several places in the article.

III. The first proof proves the existence of $N(n)$ by applying Ramsey's theorem for partitions of quadruples. It is proved that $N(n) \leq r(2, 4, \{5, n\})$. This is still a textbook argument. Another proof based on Ramsey's theorem for partitions of triples was found by A. Tarsi (see [8]). So far no proof has emerged which is based on the graph Ramsey theorem only.

IV. The authors give "a new proof of Ramsey's theorem which differs entirely from the previous ones and gives for $m_i(k, \ell)$ slightly smaller limits". Here $m_i(k, \ell)$ denotes the minimal value of $|X|$ such that every partition of i-element subsets of X into two classes, say α and β, each k-element contains an i-element subset of class α or each ℓ-element subset contains an i-element subset of class β.

Thus, $m_i(k, \ell)$ is the Ramsey number for 2-partitions of i-element subsets. These numbers are denoted today by $r(2, i, \{k, \ell\})$ or $r_i(k, \ell)$. The proof is close to the standard textbook proofs of Ramsey's theorem. Several times P. Erdős attributed it to G. Szekeres.

Erdős and Szekeres explicitly state that $(r_2(k+1, \ell+1) =) m_2(k+1, \ell+1) \leq \binom{k+\ell}{2}$ and this value remained for 50 years essentially the best available upper bound for graph Ramsey numbers until the recent (independent) improvements by Rödl and Thomason. The current best upper bound (for $k = \ell$) is [9] essentially

$$\binom{2k}{k} / \sqrt{k} \, .$$

V. It is not as well known that [1] contains yet another proof of the graph theoretic formulation of Ramsey's theorem (in the above notation, $i = 2$) which is stated for its particular simplicity. We reproduce its formulation here.

Theorem. *In an arbitrary graph let the maximum number of independent points be k; if the number of points is $N \geqq m(k, \ell)$ then three exists in our graph a complete graph of order ℓ.*

Proof. For $\ell = 1$, the theorem is trivial for any k, since the maximum number of independent points is k and if the number of points is $(k + 1)$, there must be an edge (complete graph of order 1).

Now suppose the theorem proved for $(\ell - 1)$ with any k. Then at least $\frac{N-k}{k}$ edges start from one of the independent points. Hence if

$$\frac{N-k}{k} \geqq m(k, \ell - 1) \, ,$$

i.e.,

$$N \geqq k \cdot m(k, \ell - 1) + k \, ,$$

then, out of the end points of these edges we may select, in virtue of our induction hypothesis, a complete graph whose order is at least $(\ell - 1)$. As the points of this graph are connected with the same point, they form together a complete graph of order ℓ.

This indicates that Erdős and Szekeres were well aware of the novelty of the approach to Ramsey's theorem. Also this is the formulation of Ramsey's problem which motivated some of the key pieces of Erdős' research. First an early use of the averaging argument and then the formulation of Ramsey's theorem in a "high off-diagonal" form: If a graph G has a bounded clique number (for example, if it is triangle-free) then its independence number has to be large. The study of this phenomenon led Erdős so key papers [10], [11], [12] which will be discussed in the next section in greater detail.

VI. The paper [1] contains a second proof of Theorem 2.1. This is a more geometrical proof which yields a better bound

$$N(n) \leq \binom{2n-4}{n-2} + 1$$

and it is conjectured (based on the exact values of $N(n)$ for $n = 3, 4, 5$) that $N(n) = 2^{n-2} + 1$. This is still an unsolved problem. The second proof (which 50 years later very nicely fits to a computational geometry context) is based on yet another Ramsey-type result.

Theorem 2.2 (the ordered pigeon hole principle). *Let m, n be positive integers. Then every set of $(m-1)(n-1)+1$ distinct integers contains either a monotone increasing n-set or monotone decreasing m-set.*

The authors of [1] note that the same problem was considered by R. Rado. The stage has been set.

The ordered pigeon-hole principle has been generalized in many different directions (see e.g., [13], [14]).

All this is contained in this truly seminal paper. Viewed from a contemporary perspective, the Erdős-Szekeres paper did not solve any well-known problem at the time and did not contribute to Erdős' instant mathematical fame (as a number theorist). But the importance of the paper [1] for the later development of combinatorial mathematics cannot be overestimated. To illustrate this development is one of the aims of this paper.

Apart from the problem of a good estimation of the value of N there is a peculiar structural problem related to [1]:

Call a set $Y \subseteq X$ an *n-hole* in X if Y is the set of vertices of a convex n-gon which does not contain other points in X.

Problem. Does there always exist $N^*(n)$ such that if X is any set of at least $N^*(n)$ points in the plane in general position then X contains an n-hole.

It is easy to prove that $N^*(n)$ exists for $n \leq 5$ (see Harborth (1978) where these numbers are determined). Horton (1983) showed that $N^*(7)$ does not exist. Thus only the existence of $N^*(6)$ is an open problem (see [15], [16] for recent related problems).

3. Estimating Ramsey numbers

Today it seems that the first question in this area which one might be tempted to consider is the problem of determining the actual sizes of the sets which are guaranteed by Ramsey's theorem (and other Ramsey-type theorems). But one should try to resist this temptation since it is "well-known" that Ramsey numbers (of all sorts) are difficult to determine and even good asymptotic estimates are difficult to find.

It seems that these difficulties were known to both Erdős and Ramsey. But Erdős considered them very challenging and addressed this question in several of his key articles. In many cases his estimations obtained decades ago are still the best available. Not only that, his innovative techniques became standard and whole theories evolved from his key papers.

Here is a side comment which may partly explain this success: Erdős was certainly one of the first number theorists who took an interest in combinatorics in the contemporary sense (being preceded by isolated events, for example, V. Jarnik's work on the minimum spanning tree problem and the Steiner problem see e.g. [17], [18]. Incidentally Jarnik was one of the first coauthors of Erdős). Together with Turán, Erdős brought to the "slums of topology" not only his brilliance but also his expertise and "good taste". It is our opinion that these facts profoundly influenced further development of the whole field. Thus it is not perhaps surprising that if one would isolate a single feature of Erdős' contribution to Ramsey theory then it is perhaps his continuing emphasis on estimates of various Ramsey-related questions. From the large number of results and papers we decided to cover several key articles and comment on them from a contemporary point of view.

I. 1947 paper [10]. In a classically clear way, Erdős proved

$$2^{k/2} \leq r(k) < 4^k \tag{3.1}$$

for every $k \geq 3$.

His proof became one of the standard textbook examples of the power of the probabilistic method. (Another example perhaps being the strikingly simple proof of Shannon of the existence of exponentially complex Boolean functions.)

The paper [10] proceeds by stating (3.1) in an inverse form: Define $A(n)$ as the greatest integer such that given any graph G of n vertices, either it or its complementary graph contains a complete subgraph of order $A(n)$. Then for $A(n) \geq 3$,

$$\frac{\log n}{2\log 2} < A(n) < \frac{2\log n}{\log 2} .$$

Despite considerable efforts over many years, these bounds have been improved only slightly (see [9], [20]). We commented on the upper bound improvements above. The best current lower bound is

$$r(n) \geq (1 + O(1))\frac{\sqrt{2}e}{n}2^{n/2}$$

which is twice the Erdős bound (when computed from his proof).

The paper [10] was one of 23 papers which Erdős published within 3 years in the *Bull. Amer. Math. Soc.* and already here it is mentioned that although the upper bound for $r(3, n)$ is quadratic, the present proof does not yield a nonlinear lower bound. That had to wait for another 10 years.

II. The 1958 paper [11] — Graph theory and probability. The main result of this paper deals with graphs, circuits, and chromatic number and as such does not seem to have much to do with Ramsey theory.

Yet the paper starts with the review of bounds for $r(k, k)$ and $r(3, k)$ (all due to Erdős and Szekeres). Ramsey numbers are denoted as in most older Erdős papers by symbols of $f(k)$, $f(3, k)$, $g(k)$. He then defines analogously the function $h(k, \ell)$ as "the least integer so that every graph of $h(k, \ell)$ vertices contains either a closed circuit of k or fewer lines or the graph contains a set of ℓ independent points. Clearly $h(3, \ell) = f(3, \ell)$".

The main result of [11] is that $h(k, \ell) > \ell^{1+1/2k}$ for any fixed $k \geq 3$ and ℓ sufficiently large. The proof is one of the most striking early use of the probabilistic method. Erdős was probably aware of it and this may explain (and justify) the title of the paper. It is also proved that $h(2k + 1, \ell) < c\ell^{1+1/k}$ and this is proved by a variant of the greedy algorithm by induction on ℓ. Now after this is claimed, it is *remarked* that the above estimation (3.1) leads to the fact that there exists a graph G with n vertices which contain no closed circuit of fewer than k edges and such that its chromatic number is $> n^\epsilon$.

This side remark is in fact perhaps the most well known formulation of the main result of [11]:

Theorem 3.1. *For every choice of positive integers k, t and ℓ there exists a k-graph G with the following properties:*

(1) The chromatic number of $G > t$.
(2) The graph of $G > \ell$.

This is one of the few true combinatorial classics. It started in the 40's with Tutte [21] and Zykov [26] for the case $k = 2$ and $\ell = 2$ (i.e., for triangle-free graphs). Later, this particular case was rediscovered and also conjectured several times [22], [23]. Kelly and Kelly [23] proved the case $k = 2$, $\ell \leqq 5$, and conjectured the general statement for graphs. This was settled by Erdős in [11] and the same probabilistic method has been applied by Erdős and Hajnal [27] to yield the general result.

Erdős and Rado [29] proved the extension of $k = 2$, $\ell = 2$ to transfinite chromatic numbers while Erdős and Hajnal [28] gave a particularly simple construction of triangle-free graphs, so called shift graphs $G = (V, E)$: $V = \{(i,j);\ 1 \le i < j \le n\}$ and $E = \{(i,j),(i',j');\ i < j = i' < j'\}$. G_n is triangle-free and $\chi(G_n) = \lceil \log n \rceil$.

For many reasons it is desirable to have a constructive proof of Theorem 3.1. This has been stressed by Erdős on many occasions. This appeared to be difficult (see [25]) and a construction in full generality was finally given by Lovász [30]. A simplified construction has been found in the context of Ramsey theory by Nešetřil and Rödl [31]. The graphs and hypergraphs with the above properties (i), (ii) are called *highly chromatic (locally) sparse graphs*, for short.

Their existence could be regarded as one of the true paradoxes of finite set theory and it has always been felt that this result is one of the central results in combinatorics.

Recently it has been realized that sparse and complex graphs may be used in theoretical computer science for the design of fast algorithms. However, what is needed there is not only a construction of these "paradoxical" structures but also their reasonable size. In one of the most striking recent developments, a program for constructing complex sparse graphs has been successfully carried out. Using several highly ingenious constructions which combine algebraic and topological methods it has been shown that there are complex sparse graphs, the size of which in several instances improves the size of random objects. See Margulis [32], Alon [34] and Lubotzky et al. [33].

Particularly, it follows from Lubotzky et al. [33] that there are examples of graphs with girth ℓ, chromatic number t and the size at most $t^{3\ell}$. A bit surprisingly, the following is still open:

Problem. Find a primitive recursive construction of highly chromatic locally sparse k-uniform hypergraphs. Indeed, even triple systems (i.e., $k = 3$) present a problem.

III. $r(3,n)$ **[12].** The paper [12] provides the lower bound estimate on the Ramsey number $r(3,n)$.

Using probabilistic methods Erdős proved

$$r(3,n) \ge \frac{n^2}{\log^2 n} \tag{3.2}$$

(while the upper bound $r(3,n) \le \binom{n+1}{2}$ follows from [1]).

The estimation of Ramsey numbers $r(3,n)$ was Erdős' favorite problem for many years. We find it already in his 1947 paper [10] where he mentioned that he cannot prove the nonlinearity of $r(3,n)$. Later he stressed this problem (of estimating $r(3,n)$) on many occasions and conjectured various forms of it. He certainly felt the importance of this special case. How right he was is clear from the later developments, which read as a saga of modern combinatorics. And as isolated as this may seems, the problem of estimating $r(3,n)$ became a cradle of many methods and results, perhaps far exceeding the original motivation.

In 1981 Ajtai, Komlós and Szemerédi in their important paper [35] proved by a novel method

$$r(3,n) \le c\frac{n^2}{\log n}\ . \tag{3.3}$$

This bound and their method of proof has found many applications. The Ajtai, Komlós and Szemerédi proof was motivated by yet another Erdős problem from combinatorial number theory.

In 1941 Erdős and Turán [37] considered problem of dense Sidon sequences (or B_2-sequences). An infinite sequence $S = a_1 < a_2 < \cdots$ of natural numbers is called *Sidon sequence* if all pairwise sums $a_i + a_j$ are distinct. Define

$$f_S(n) = \max\{x : a_x \le n\}$$

and for a given n, let $f(n)$ denote the maximal possible value of $f_S(n)$. In [37], Erdős and Turán prove that for *finite* Sidon sequences $f(n) \sim n^{1/2}$ (improving Sidon's bound of $n^{1/4}$; Sidon's motivation came from Fourier analysis [38]). However for every *infinite* Sidon sequence S growth of $f_S(n)$ is a more difficult problem and as noted by Erdős and Turán,

$$\underline{\lim} f_S(n)/n^{1/2} = 0 \ .$$

By using a greedy argument it was shown by Erdős [36] that $f_S(n) > n^{1/3}$. (Indeed, given k numbers $x_1 < \cdots < x_k$ up to n, each triple $x_i < x_j < x_k$ kills at most 3 other numbers x, $x_i + x_j = x_k + x$, $x_i + x_k = x_j + x$ and $x_j + x_k = x_i + x$ and thus if $k + 3\binom{k}{3} < ck^2 < n$ we can always find a number $x < n$ which can be added to S.) Ajtai, Komlós and Szemerédi proved using a novel "random construction" the existence of an infinite Sidon sequence S such that

$$f_S(n) > c \cdot (n \log n)^{1/3} \ .$$

An analysis of independent sets in triangle-free graphs is the basis of their approach and this yields as a corollary the above mentioned upper bound on $r(3, n)$. (The best upper bound for $f_S(n)$ is of order $c \cdot (n \log n)^{1/2}$.) It should be noted that the above Erdős-Turán paper [37] contains the following still unsolved problem: Let $a_1 < a_2 < \cdots$ be an arbitrary sequence. Denote by $f(n)$ the number of representations of n as $a_i + a_j$. Erdős and Turán prove that $f(n)$ cannot be a constant for all sufficiently large n and conjectured that if $f(n) > 0$ *for* all sufficiently large n then $\limsup f(n) = \infty$. This is still open. Erdős provided a multiplicative analogue of this conjecture (i.e., for the function $g(n)$, the number of representation of n as $a_i a_j$); this is noted already in [37]). One can ask what this has to do with Ramsey theory. Well, not only was this the motivation for [35] but a simple proof of the fact that $\limsup g(n) = \infty$ was given by Nešetřil and Rödl in [39] just using Ramsey's theorem.

We started this paper by listing the predominance of Erdős's first works in number theory. But in a way this is misleading since the early papers of Erdős stressed elementary methods and often used combinatorial or graph-theoretical methods. The Erdős-Turán paper is such an example and the paper [40] even more so.

The innovative Ajtai-Komlós-Szemerédi paper was the basis for a further development (see, e.g., [41]) and this in turn led somewhat surprisingly to the recent remarkable solution of Kim [42], who proved that the Ajtai-Komlós-Szemerédi bound is up to a constant factor, the best possible, i.e.,

$$r(n, 3) > c\frac{n^2}{\log n} \ .$$

Thus $r(n, 3)$ is the only nontrivial infinite family of (classical) Ramsey numbers with known asymptotics.

IV. Constructions. It was realized early by Erdős the importance of finding explicit constructions of various combinatorial objects whose existence he justified by probabilistic methods (e.g., by counting). In most case such constructions have not yet found but yet even constructions producing weaker results (or bounds) formed an important line of research. For example, the search for an explicit graph of size (say) $2^{n/2}$ which would demonstrate this Ramsey lower bound has been so far unsuccessful. This is not an entirely satisfactory situation since it is believed that such graphs share many properties with random graphs and thus they could be good candidates for various lower bounds, for example, in theoretical computer science for lower bounds for various measures of complexity. (See the papers [43] and [44] which discuss properties of pseudo- and quasi-random graphs.)

The best constructive lower bound for Ramsey numbers $r(n)$ is due to Frankl and Wilson. This improves on an earlier construction of Frankl [46] who found a first constructive superpolynomial lower bound.

The construction of Frankl-Wilson graphs is simple:

Let p be a prime number, put $q = p^3$. Define the graph $G_p = (V, E)$ as follows:

$$V = \binom{[q]}{p^2 - 1} = \{F \subseteq \{1, \ldots, p^3\} : |F| = p^2 - 1\} ,$$

$$\{F, F'\} \in E \text{ iff } |F \cap F'| \equiv -1 (\text{mod } q) .$$

The graph G_p has $\binom{p^3}{p^2-1}$ vertices. However, the Ramsey properties of the graph G_p are not trivial to prove: It follows only from deep extremal set theory results due to Frankl and Wilson [45] that neither G_p nor its complement contain K_n for $n \geq \binom{p^3}{p-1}$. This construction itself was motivated by several extremal problems of Erdős and in a way (again!) the Frankl-Wilson construction was a byproduct of these efforts.

We already mentioned earlier the developments related to Erdős paper [11]. The constructive version of bounds for $r(3, n)$ led Erdős to geometrically defined graphs. An early example is Erdős-Rogers paper [47] where they prove that there exists a graph G with ℓ^{1+c_k} vertices, which contains no complete k-gon, but such that each subgraph with ℓ vertices contains a complete $(k-1)$-gon.

If we denote by $h(k, \ell)$ the minimum integer such that every graph of $h(k, \ell)$ vertices contains either a complete graph of k vertices or a set of ℓ points not containing a complete graph with $k-1$ vertices, then

$$h(k, \ell) \leq r(k, \ell) .$$

However, for every $k \geq 3$ we still have $h(k, \ell) > \ell^{1+c_k}$.

This variant of the Ramsey problem is due to A. Hajnal. The construction of the graph G is geometrical: the vertices of G are points on an n-dimensional sphere with unit radius, and two points are joined if their Euclidean distance exceeds $\sqrt{2k/(k-1)}$.

Graphs defined by distances have been studied by many people (e.g., see [48]). The best constructive lower bound on $r(3, n)$ is due to Alon [49] and gives $r(3, n) \geq cn^{3/2}$. See also a remarkable elementary construction [50] giving a weaker result.

4. Ramsey Theory

It seems that the building of a theory *per se* was never Erdős's preference. He is a life long problem solver, problem poser, admirer of mathematical miniatures and beauties. THE BOOK is an ideal. Instead of developing the whole field he seemed always to prefer consideration of particular cases. However, many of these cases turned out to be key cases and somehow theories emerged.

Nevertheless, one can say that Erdős and Rado systematically investigated problems related to Ramsey's theorem with a clear vision that here was a new basis for a theory. In their early papers [51], [52] they investigated possibilities of various extensions of Ramsey's theorem. It is clear that these papers are a result of a longer research and understanding of Ramsey's theorem.

As if these two papers summarized what was known, before Erdős and Rado went on with their partition calculus projects reflected by the grand papers [53] and [54]. But this is beyond the scope of this paper. [51] contains an extension of Ramsey's theorem for colorings by an infinite number of colors. This is the celebrated Erdős-Rado canonization lemma:

Theorem 4.1 ([51]). *For every choice of positive integers p and n there exists $N = N(p,n)$ such that for every set X, $|X| \geq N$, and for every coloring $c : \binom{X}{p} \to \mathbb{N}$ (i.e., a coloring by arbitrarily many colors) there exists an n-element subset Y of X such that the coloring c restricted to the set $\binom{Y}{p}$ is "canonical".*

Here a coloring of $\binom{Y}{p}$ is said to be canonical if there exists an ordering $Y = y_1 < \cdots < y_n$ and a subset $w \subseteq \{1,\ldots,p\}$ such that two n-sets $\{z_1 < \cdots < z_p\}$ and $\{z_1' < \cdots < z_p'\}$ get the same color if and only if $z_i = z_i'$ for exactly $i \in w$. Thus there are exactly 2^p canonical colorings of p-tuples. The case $w = \phi$ corresponds to a monochromatic set while $w = \{1,\ldots,p\}$ to a coloring where each p-tuple gets a different color (such a coloring is sometimes called a "rainbow" or totally multicoloring).

Erdős and Rado deduced Theorem 4.1 from Ramsey's theorem. For example, the bound $N(p,n) \leq r(2p, 2^{2p}, n)$ gives a hint as to how to prove it. One of the most elegant forms of this argument was published by Rado [55] in one of his last papers.

The problem of estimating $N(p,n)$ was recently attacked by Lefman and Rödl [56] and Shelah [57]. One can see easily that Theorem 4.1 implies Ramsey's theorem (e.g., $N(p,n) \geq r(p, n-2, n)$) and the natural question arises as to how many exponentiations one needs. In [56] this was solved for graphs ($p = 2$) and Shelah [57] solved recently this problem in full generality: $N(p,n)$ is the lower function of the same height $r(p,4,n)$ i.e., $(p-1)$ exponentiations.

The Canonization Lemma found many interesting applications (see, e.g., [58]) and it was extended to other structures. For example, the canonical van der Waerden theorem was proved by Erdős and Graham [59].

Theorem 4.2 ([59]). *For every coloring of positive integers one can find either a monochromatic or a rainbow arithmetic progression of every length. (Recall: a rainbow set is a set with all its elements colored differently.)*

This result was extended by Lefman [60] to all regular systems of linear equations (see also [78]).

One of the essential parts of the development of the "new Ramsey theory" age was the stress on various structural extensions and structure analogies of the original results. A key role was played by Hales-Jewett theorem (viewed as a combinatorial axiomatization of van der Waerden's theorem), Rota's conjecture (the vector-space analogue of Ramsey's theorem), Graham-Rothschild parameter sets, all dealing with new structures. These questions and results displayed the richness of the field and attracted so much attention to it.

It seems that one of the significant turns appeared in the late 60's when Erdős, Hajnal and Galvin started to ask questions such as "which graphs contain a monochromatic triangle in any 2-coloring of its edges". Perhaps the essential parts of this development can be illustrated with this particular example.

We say that a graph $G = (V,E)$ is t-Ramsey for the triangle (i.e., K_3) if for every coloring of E by t-colors, one of the colors contains a triangle. Symbolically we denote this by $G \to (K_3)^2_t$. This is a variant of the Erdős-Rado partition arrow. Ramsey's theorem gives us $K_6 \to (K_3)^2_2$ (and $K_{r(2,t,3)} \to (K_3)^2_t$). But there are other essentially different examples. For example, a 2-Ramsey graph for K_3 need not contain K_6. Graham [62] constructed the unique minimal graph with this property: The graph $K_3 + C_5$ (triangle and pentagon completely joined) is the smallest graph G with $G \to (K_3)^2_2$ which does not contain a K_6. Yet $K_3 + C_5$ contains K_5 and subsequently van Lint, Graham and Spencer constructed a graph G not containing even a K_5, with $G \to (K_3)^2_2$. Until recently, the smallest example was due to Irving [63] and had 18 vertices. Very recently, two more constructions appeared by Erickson [64] and Bukor [65] who found examples with 17 and 16 vertices (both of them use properties of Graham's graph).

Of course, the next question which was asked is whether there exists a K_4-free graph G with $G \to (K_3)^2_2$. This question proved to be considerably harder and it is possible to say that it has not yet been solved completely satisfactorily.

The existence of a K_4-free graph G which is t-Ramsey for K_3 was settled by Folkman [66] ($t = 2$) and Nešetřil and Rödl [67]. The proofs are complicated and the graphs constructed are very large. Perhaps just to be explicit Erdős [68] asked whether there exists a K_4-free graph G which arrows triangle with $< 10^{10}$ vertices. This question proved to be very accurate and it was finally shown by Spencer [69] that there exists such a graph with 3×10^8 vertices. Of course, it is possible that such a graph exists with only 100 vertices!

The proof of this statement is probabilistic. The probabilistic methods were not only applied to get various bounds for Ramsey numbers. Recently, the Ramsey properties of the Random Graph $K(n,p)$ were analyzed by Rödl and Ruciński and the threshold probability for p needed to guarantee $K(n,p) \to (K_3)^2_t$ with probability tending to 1 as $n \to \infty$, was determined (see [70]).

Many of these questions were answered in a much greater generality and this seems to be a typical feature for the whole area. On the other side these more general statements explain the unique role of original Erdős problem. Let us be more specific. We need a few definitions: An ordered graph is a graph with a linearly ordered set of vertices. Isomorphism of ordered graphs means isomorphism preserving orderings. If A, B are ordered graphs (for now we will find it convenient to denote graphs by $A, B, C, \ldots$) then $\binom{B}{A}$ will denote the set of all induced subgraphs of B which are isomorphic to A. We say that a class $\mathcal{K}$ of graphs is *Ramsey* if for every choice of ordered graphs A, B from $\mathcal{K}$ there exists $C \in \mathcal{K}$ such that $C \to (B)^A_2$. Here, the notation $C \to (B)^A_2$ means: for every coloring $c : \binom{C}{A} \to \{1,2\}$ there exists $B' \in \binom{C}{B}$ such that the set $\binom{B'}{A}$ is monochromatic (see, e.g., [71].) Similarly we say that a class $\mathcal{K}$ of graphs is *canonical* if for every choice of ordered graphs A, B from $\mathcal{K}$ there exists $C \in \mathcal{K}$ with the following property: For every coloring $c : \binom{C}{A} \to \mathbb{N}$ there exists $B' \in \binom{C}{B}$ such that the set $\binom{B'}{A}$ has a canonical coloring.

Denote by $Forb(K_k)$ the class of all K_k-free graphs. Now we have the following

Theorem 4.3. *For a hereditary class $\mathcal{K}$ of graphs the following statements are equivalent:*

1. *$\mathcal{K}$ is Ramsey*
2. *$\mathcal{K}$ is canonical*
3. *$\mathcal{K}$ is a union of the following 4 types of classes: the class $Forb(K_k)$, the class of complements of graphs from $Forb(K_k)$, the class of Turán graphs (i.e., complete multipartite graphs) and the class of equivalences (i.e., complements of Turán graphs).*

(1. $\Leftrightarrow$ 3. is proved in [71], 2. $\Rightarrow$ 1. is easy, and one can prove 1. $\Rightarrow$ 2. directly along the lines of Erdős-Rado proof of canonization lemma.) Thus, as often in Erdős' case, the triangle-free graphs were not just any case but rather the typical case.

From today's perspective it seems to be just a natural step to consider. Ramsey properties of geometrical graphs. This was initiated in a series of papers by Erdős, Graham, Montgomery, Rothschild, Spencer and Straus, [72], [73], [74]. Let us call a finite configuration C of points in $\mathbb{E}^n$ *Ramsey* if for every r there is an $N = N(r)$ is that in every r-coloring of the points of $\mathbb{E}^N$, a monochromatic congruent copy of C is always formed. For example, the vertices of a unit simplex in $\mathbb{E}^n$ is Ramsey (with $N(r) = n(r-1)+n$), and it is not hard to show that the Cartesian product of two Ramsey configurations is also Ramsey. More recently, Frankl and Rödl [75] showed that any simplex in $\mathbb{E}^n$ is Ramsey (a simplex is a set of $n+1$ points having a positive n-volume).

In the other direction, it is known [72] that any Ramsey configuration must lie on the surface of a sphere (i.e., be "spherical"). Hence, 3-collinear points do not form a Ramsey configuration, and in fact, for any such set C_3, $\mathbb{E}^N$ can always be 16-colored so as to avoid a monochromatic congruent copy of C_3. It is not known if the value 16 can be reduced (almost certainly it can). The major open question is to characterize the Ramsey configurations. It is natural to conjecture that they are exactly the class of spherical sets. Additional evidence of this was found by Kříž [76] who showed for example, that the set of vertices of any regular polygon is Ramsey. A fuller discussion of this interesting topic can be found in [77].

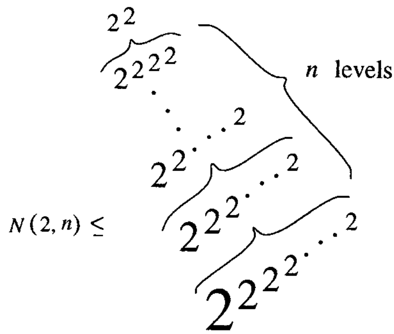
5. Adventures in Arithmetic Progressions

Besides Ramsey's theorem itself the following result provided constant motivation for Ramsey Theory:

Theorem 5.1 (van der Waerden [79]). *For every choice of positive integers k and n, there exists a least $N(k,n) = N$ such that for every partition of the set $\{1, 2, \ldots, N\}$ into k classes, one of the classes always contains an arithmetic progression with n terms.*

The original proof of van der Waerden (which developed through discussions with Artin and Schreier — see [80] for an account of the discovery) and which is included in an enchanting and moving book of Khinchine [81] was until recently essentially the only known proof. However, interesting modifications of the proof were also found, the most important of which is perhaps the combinatorial formulation of van der Waerden's result by Hales and Jewett [82].

The distinctive feature of van der Waerden's proof (and also of Hales-Jewett's proof) is that one proves a more general statement and then uses double induction. Consequently, this procedure does not provide a primitive recursive upper bound for the size of N (in van der Waerden's theorem). On the other hand, the best bound (for n prime) is (only!) $W(n+1) \geq n2^n$, n prime (due to Berlekamp [83]). Thus, the question of whether such a huge upper bound was also necessary, was and remains to be one of the main research problems in the area. In 1988, Shelah [84] gave a new proof of both van der Waerden's and the Hales-Jewett's theorem which provided a primitive recursive upper bound for $N(k,n)$. However the bound is still very large being of the order of fifth function in the Ackermann hierarchy — "tower of tower functions". Schematically,



Even for the solution of the modest looking conjecture $N(2,n) \leq 2^{\cdot^{\cdot^{\cdot^{2}}}} \Big\} n$, the first author of this paper is presently offering \$1000.

The discrepancy between the general upper bound for van der Waerden numbers and the known values is the best illustrated for the first nontrivial value: while $N(2,3) = 9$ the Shelah proof gives the stack of 2's of height 2^{16}.

These observations are not new and were considered already in the Erdős and Turán 1936 paper [85]. For the purpose of improving the estimates for the van der Waerden numbers, they had the idea of proving a stronger — now called a *density* — statement. They considered (how typical!) the particular case of 3-term arithmetic progressions and for a given positive integer N, defined $r(N)$ (their notation) to denote the maximum number elements of a sequence of numbers $\leq N$ which does not contain a 3-term arithmetic progression. They observed the subadditivity of function $r(N)$ (which implies the existence of a limiting value of $r(N)/N$) and proved $r(N) \leq \left(\frac{3}{8} + \epsilon\right) N$ for all $N \geq N(\epsilon)$.

After that they remarked that probably $r(N) = o(N)$. And in the last few lines of their short paper they define numbers $r_\ell(N)$ to denote the maximum number of integers less than or equal to N such that no ℓ of them form an arithmetic progression. Although they do not ask explicitly whether $r_\ell(N) = o(N)$ (as Erdős did many times since), this is clearly in their mind as they list consequences of a good upper bound for $r_\ell(N)$: long arithmetic progressions formed by primes and a better bound for the van der Waerden numbers.

As with the Erdős-Szekeres paper [1], the impact of the modest Erdős-Turán note [85] is hard to overestimate. Thanks to its originality, both in combinatorial and number theoretic contexts, and to Paul Erdős' persistence, this led eventually to beautiful and difficult research, and probably beyond Erdős' expectations, to a rich general theory. We wish to briefly mention some key points of this development.

Good lower estimates for $r(N)$ were obtained soon after by Salem and Spencer [86] and Behrend [87] which still gives the best bound. These bounds recently found a surprising application in a least expected area, namely in the fast multiplication of matrices (Coppersmith, Winograd [88]).

The upper bounds and $r(N) = o(N)$ appeared to be much harder. In 1953 K. Roth [89] proved $r_3(N) = o(N)$ and after several years of partial results, E. Szemerédi in 1975 [91] proved the general case

$$r_\ell(N) = o(N) \quad \text{for every} \quad \ell\,.$$

This is generally recognized as the single most important Erdős solved problem, the problem for which he has paid the largest amount. By now there are more expensive problems (see Erdős' article in these volumes) but they have not yet been solved. And taking inflation into account, possibly none of them will ever have as an expensive solution. Szemerédi's proof changed Ramsey theory in at least two aspects. First, several of its pieces, most notably so called Regularity Lemma, proved to be useful in many other combinatorial situations (see e.g., [90], [91], [92]). Secondly, perhaps due to the complexity of Szemerédi's combinatorial argument, and the beauty of the result itself, an alternative approach was called for. Such an approach was found by Hillel Furstenberg [93], [94] and developed further in many aspects in his joint work with B. Weiss, Y. Katznelson and others. Let us just mention two results which in our opinion best characterize the power of this approach: In [95] Furstenberg and Katznelson proved the density version of Hales-Jewett theorem. More recently, Bergelson and Leibman [96] proved the following striking result (conjectured by Furstenberg):

Theorem 5.2 ([96]). *Let $p_1, \ldots, p_k$ be polynomials with rational coefficients taking integer values on integers and satisfying $p_i(0) = 0$ for $i = 1, \ldots, k$. Then every set X of integers of positive density contains for every choice of numbers $\nu_1, \ldots, \nu_k$, a subset*

$$\mu + p_1(d)\nu_1,\ \mu + p_2(d)\nu_2, \ldots, \mu + p_k(d)\nu_k$$

for some μ and $d > 0$.

Choosing $p_i(x) = x$ and $\nu_i = i$ we get the van der Waerden theorem. Already, the case $p_i(x) = x^2$ and $\nu_i = i$ was open for several years (this gives long arithmetic progressions in sets of positive density with their differences being some square).

For none of these results are there known combinatorial proofs. Instead, they are all proved by a blend of topological dynamics and ergodic theory methods, proving countable extensions of these results. For this part of Ramsey theory this setting seems to be most appropriate. However, it is a long way from the original Erdős-Turán paper.

Let us close this section with a more recent example. In 1983 G. Pisier formulated (in a harmonic analysis context) the following problem: A set of integers $x_1 < x_2 < \cdots$ is said to be *independent* if all finite subsums of distinct elements are distinct. Now let X be an infinite set and suppose for some $\epsilon > 0$ that every finite subset $Y \subseteq X$ contains a subsubset Z of size $\geq \epsilon|Z|$ which is independent. Is it then true that X is a finite union of independent sets?

Despite many efforts and partial solutions the problem is still open. It was again Paul Erdős who quickly realized the importance of the Pisier problem and Erdős, Nešetřil and Rödl recently [97], [98] studied "Pisier type problems". For various notions of an independence relation, the following question was considered: Assume that an infinite set X satisfies for some $\epsilon > 0$, some hereditary density condition (i.e., we assume that every finite set Y contains an independent subsubset of size $\geq \epsilon|Y|$). Is it then true that X can be partitioned into finitely many independent sets?

Positive instances (such as collinearity, and linear independence) as well as negative instances (such as Sidon sets) were given in [97], [98]. Also various "finitization versions" and analogues of the Pisier problem were answered in the negative. But at present the original Pisier problem is still open. In a way one can consider Pisier type problems as dual to the density results in Ramsey theory: One attempts to prove a positive Ramsey type statement under a strong (hereditary) density condition. This is exemplified in [98] by the following problem which is perhaps a fitting conclusion to this paper surveying 60 years of Paul Erdős' service to Ramsey theory.

The Anti-Szemerédi Problem [98]

Does there exist a set X of positive integers such that for some $\epsilon > 0$ the following two conditions hold simultaneously:

(1) For every finite $Y \subseteq X$ there exists a subset $Z \subseteq X$, $|Z| \geq \epsilon|Y|$, which does not contain a 3-term arithmetic progression;

(2) Every finite partition of X contains a 3-term arithmetic progression in one of its classes.

References

1. P. Erdős and G. Szekeres, A combinatorial problem in geometry, Composito Math. 2 (1935), 464–470.
2. I. Schur, Über die Kongruens $x^m + y^m = z^m (\text{mod } p)$, Jber. Deutch. Math. Verein 25 (1916), 114–117.
3. I. Schur, Gesammelte Abhandlungen (eds. A. Brauer, H. Rohrbach), 1973, Springer.
4. B. L. van der Waerden, Berver's einer Baudetschen Vermutung, Nieuw. Arch. Wisk. 15 (1927), 212–216.
5. R. Rado, Studien zur Kombinatorik, Math. Zeitschrift 36 (1933), 242–280.
6. F. P. Ramsey, On a problem of formal logic, Proc. London Math. Soc. 48 (1930), 264–286.
7. P. Erdős, Art of Counting, MIT Press.
8. R. L. Graham, B. L. Rothschild, and J. Spencer, Ramsey theory, Wiley, 1980, 2nd edition, 1990.
9. A. Thomason, An upper bound for some Ramsey numbers, J. Graph Theory 12 (1988), 509–517.
10. P. Erdős, Some remarks on the theory of graphs, Bull. Amer. Math. Soc. 53 (1947), 292–294.
11. P. Erdős, Graph theory and probabilities, Canad. J. Math. 11 (1959), 34–38.
12. P. Erdős, Graph theory and probability II, Canad. J. Math. 13 (1961), 346–352.
13. V. Chvátal and J. Komlos, Some combinatorial theorems on monotonicity, Canad. Math. Bull. 14, 2 (1971).
14. J. Nešetřil and V. Rödl, A probabilistic graph theoretical method, Proc. Amer. Math. Soc. 72 (1978), 417–421.
15. P. Valtr, Convex independent sets and 7-holes in restricted planar point sets, Discrete Comput. Geom. 7 (1992), 135–152.
16. J. Nešetřil and P. Valtr, A Ramsey-type result in the plane, Combinatorics, Probability and Computing 3 (1994), 127–135.
17. V. Jarnik and M. Kössler, Sur les graphes minima, contenant n points donnés, Čas. Pěst. Mat. 63 (1934), 223–235.
18. P. Hell and R. L. Graham, On the history of the minimum spanning tree problem, Annals of Hist. Comp. 7 (1985), 43–57.
19. D. Hilbert, Über die irreduribilität ganzer rationaler funktionen mit ganssahligen koeffieienten, J. Reine und Angew. Math. 110 (1892), 104–129.
20. J. H. Spencer, Ramsey's theorem — a new lower bound, J. Comb. Th. A, 18 (1975), 108–115.
21. Blanche Descartes, A three colour problem, Eureka 9 (1947), 21, Eureka 10 (1948), 24. (See also the solution to Advanced problem 1526, Amer. Math. Monthly 61 (1954), 352.)

22. G. A. Dirac, The structure of k-chromatic graphs, Fund. Math. 40 (1953), 42–55.
23. J. B. Kelly and L. M. Kelly, Paths and Circuits in critical graphs, Amer. J. Math. 76 (1954), 786–792.
24. J. Mycielski, Sur le coloriage des graphes, Collog. Math. 3 (1955), 161–162.
25. J. Nešetřil, Chromatic graphs without cycles of length ≤ 7, Comment. Math., Univ. Carolina (1966).
26. A. A. Zykov, On some properties of linear complexes, Math. Sbornik 66, 24 (1949), 163–188.
27. P. Erdős and A. Hajnal, On chromatic number of set systems, Acta Math. Acad. Sci. Hungar. 17 (1966), 61–99.
28. P. Erdős and A. Hajnal, Some remarks on set theory IX, Mich. Math. J. 11 (1964), 107–112.
29. P. Erdős and R. Rado, A construction of graphs without triangles having preassigned order and chromatic number, J. London Math. Soc. 35 (1960), 445–448.
30. L. Lovász, On the chromatic number of finite set-systems, Acta Math. Acad. Sci. Hungar. 19 (1968), 59–67.
31. J. Nešetřil and V. Rödl, A short proof of the existence of highly chromatic graphs without short cycles, J. Combin. Th. B, 27 (1979), 525–52?.
32. G. A. Margulis, Explicit constructions of concentrators, Problemy Peredachi Informatsii 9, 4 (1975), 71–80.
33. A. Lubotzky, R. Phillips, and P. Sarnak, Ramanujan Graphs, Combinatorica 8(3) (1988), 261–277.
34. N. Alon, Eigenvalues, geometric expanders, sorting in sounds and Ramsey theory, Combinatorica 3 (1986), 207–219.
35. M. Ajtai, J. Komlós and E. Szemerédi, A dense infinite Sidon sequence, European J. Comb. 2 (1981), 1–11.
36. P. Erdős, Problems and results in additive number theory, Colloque sur la Theorie des Numbres, Bruxelles (1955), 127–137.
37. P. Erdős and P. Turán, On a problem of Sidon in additive number theory and on some related problems, J. London Math. Soc. 16 (1941), 212–215.
38. S. Sidon, Ein Satz über trigonometrische Polynome und seine Anwendungen in der Theorie der Fourier-Reihen, Math. Ann. 106 (1932), 539.
39. J. Nešetřil and V. Rödl, Two proofs in combinatorial number theory, Proc. Amer. Math. Soc. 93, 1 (1985), 185–188.
40. P. Erdős, On sequences of integers no one of which divides the product of two others and on some related problems, Izv. Nanc. Ise. Inset. Mat. Mech. Tomsrk 2 (1938), 74–82.
41. N. Alon and J. Spencer, Probabilistic methods, Wiley, New York, 1992.
42. J. H. Kim, The Ramsey number $R(3,t)$ has order of magnitude $t^2/\log t$, Random Structures and Algorithms (to appear).
43. F. R. K. Chung, R. L. Graham and R. M. Wilson, Quasirandom graphs, Combinatorica 9 (1989), 345–362.
44. A. Thomason, Random graphs, strongly regular graphs and pseudorandom graphs, In: Survey in Combinatorics, Cambridge Univ. Press (1987), 173–196.
45. P. Frankl and R. M. Wilson, Intersection theorems with geometric consequences, Combinatorica 1 (1981), 357–368.
46. P. Frankl, A constructive lower bound fo Ramsey numbers, Ars Combinatorica 2 (1977), 297–302.
47. P. Erdős and C. A. Rogers, The construction of certain graphs, Canad. J. Math. (1962), 702–707.
48. D. Preiss and V. Rödl, Note on decomposition of spheres with Hilbert spaces, J. Comb. Th. A 43 (1) (1986), 38–44.

49. N. Alon, Explicit Ramsey graphs and orthonormal labellings, Electron. J. Combin. 1, R12 (1994), (8pp).
50. F. R. K. Chung, R. Cleve, and P. Dagum, A note on constructive lower bound for the Ramsey numbers $R(3,t)$, J. Comb. Theory 57 (1993), 150–155.
51. P. Erdős and R. Rado, A combinatorial theorem, J. London Math. Soc. 25 (1950), 249–255.
52. P. Erdős and R. Rado, Combinatorial theorems on classifications of subsets of a given set, Proc. London Math. Soc. 3 (1951), 417–439.
53. P. Erdős and R. Rado, A partition calculus in set theory, Bull. Amer. Math. Soc. 62 (1956), 427–489.
54. P. Erdős, A. Hajnal and R. Rado, Partition relations for cardinal numbers, Acta Math. Hungar. 16 (1965), 93–196.
55. R. Rado, Note on canonical partitions, Bull. London Math. Soc. 18 (1986), 123–126. Reprinted: Mathematics of Ramsey Theory (ed. J. Nešetřil and V. Rödl), Springer 1990, pp. 29–32.
56. H. Lefman and V. Rödl, On Erdős-Rado numbers, Combinatorica (1995).
57. S. Shelah, Finite canonization, 1994 (preprint) (to appear in Comm. Math. Univ. Carolinae).
58. J. Pelans and V. Rödl, On coverings of infinite dimensional metric spaces. In Topics in Discrete Math., vol. 8 (ed. J. Nešetřil), North Holland (1992), 75–81.
59. P. Erdős and R. L. Graham, Old and New Problems and Results in Combinatorial Number Theory, L' Enseignement Math. 28 (1980), 128 pp.
60. H. Lefman, A canonical version for partition regular systems of linear equations, J. Comb. Th. A 41 (1986), 95–104.
61. P. Erdős, J. Nešetřil and V. Rödl, Selectivity of hypergraphs, Colloq. Math. Soc. János Bolyai 37 (1984), 265–284.
62. R. L. Graham, On edgewise 2-colored graphs with monochromatic triangles and containing no complete hexagon, J. Comb. Th. 4 (1968), 300.
63. R. Irving, On a bound of Graham and Spencer for a graph-coloring constant, J. Comb. Th. B, 15 (1973), 200–203.
64. M. Erickson, An upper bound for the Folkman number $F(3,3,5)$, J. Graph Th. 17 (6), (1993), 679–68.
65. J. Bukor, A note on Folkman number $F(3,3,5)$, Math. Slovaka 44 (4), (1994), 479–480.
66. J. Folkman, Graphs with monochromatic complete subgraphs in every edge coloring, SIAM J. Appl. Math. 18 (1970), 19–24.
67. J. Nešetřil and V. Rödl, Type theory of partition properties of graphs, In: Recent Advances in Graph Theory (ed. M. Fiedler), Academia, Prague (1975), 405-412.
68. P. Erdős, Problems and result on finite and infinite graphs, In: Recent Advances in Graph Theory (ed. M. Fiedler), Academia, Prague (1975), 183–192.
69. J. Spencer, Three hundred million points suffice, J. Comb. Th. A 49 (1988), 210–217.
70. V. Rödl and A. Ruciński, Threshold functions for Ramsey properties, J. Amer. Math. Soc. (1995), to appear.
71. J. Nešetřil, Ramsey Theory, In: Handbook of Combinatorics, North Holland (1995), to appear.
72. P. Erdős, R. L. Graham, P. Montgomery, B. L. Rothschild, J. H. Spencer, and E. G. Straus, Euclidean Ramsey Theorem, J. Combin. Th. (A) 14 (1973), 341–363.
73. P. Erdős, R. L. Graham, P. Montgomery, B. L. Rothschild, J. H. Spencer, and E. G. Straus, Euclidean Ramsey Theorems II, In A. Hajnal, R. Rado and V.Sós, eds., Infinite and Finite Sets I, North Holland, Amsterdam, 1975, pp. 529–557.

74. P. Erdős, R. L. Graham, P. Montgomery, B. L. Rothschild, J. H. Spencer, and E. G. Straus, Euclidean Ramsey Theorems III, In A. Hajnal, R. Rado and V. Sós, eds., Infinite and Finite Sets II, North Holland, Amsterdam, 1975, pp. 559–583.
75. P. Frankl and V. Rödl, A partition property of simplices in Euclidean space, J. Amer. Math. Soc. 3 (1990), 1–7.
76. I. Kříž Permutation groups in Euclidean Ramsey theory, Proc. Amer. Math. Soc. 112 (1991), 899–907.
77. R. L. Graham, Recent trends in Euclidean Ramsey theory, Disc. Math. 136 (1994), 119–127.
78. W. Deuber, R. L. Graham, H. J. Prömel and B. Voigt, A canonical partition theorem for equivalence relations on $\mathbb{Z}^t$, J. Comb. Th. (A) 34 (1983), 331–339.
79. B. L. van der Waerden, Beweis einer Baudetschen Vermutung, Nieuw Arch. Wisk. 15 (1927), 212–216.
80. B. L. van der Waerden, How the Proof of Baudet's Conjecture was found, in Studies in Pure Mathematics (ed. L. Mirsky), Academic Press, New York, 1971, pp. 251–260.
81. A. J. Khinchine, Drei Perlen der Zahlen Theorie, Akademie Verlag, Berlin 1951 (reprinted Verlag Harri Deutsch, Frankfurt 1984).
82. A. W. Hales and R. I. Jewett, Regularity and positional games, Trans. Amer. Math. Soc. 106 (1963), 222-229.
83. E. R. Berlekamp, A construction for partitions which avoid long arithmetic progressions, Canad. Math. Bull 11 (1968), 409–414.
84. S. Shelah, Primitive recursive bounds for van der Waerden numbers, J. Amer. Math. Soc. 1 (1988), 683–697.
85. P. Erdős and P. Turán, On some sequences of integers, J. London Math. Soc. 11 (1936), 261–264.
86. R. Salem and D. C. Spencer, On sets of integers which contain no three terms in arithmetic progression, Proc. Nat. Acad. Sci. 28 (1942), 561–563.
87. F. A. Behrend, On sets of integers which contain no three in arithmetic progression, Proc. Nat. Acad. Sci. 23 (1946), 331–332.
88. D. Coppersmith and S. Winograd, Matrix multiplication via arithmetic progressions, J. Symb. Comput. 9 (1987), 251–280.
89. K. Roth, On certain sets of integers, J. London Math. Soc. 28 (1953), 104–109.
90. V. Chvátal, V. Rödl, E. Szemerédi, and W. Trotter, The Ramsey number of graph with bounded maximum degree, J. Comb. Th. B, 34 (1983), 239–243.
91. J. Nešetřil and V. Rödl, Partition theory and its applications, in Surveys in Combinatorics, Cambridge Univ. Press, 1979, pp. 96–156.
92. V. Rödl and A. Ruciński, Threshold functions for Ramsey properties, J. Amer. Math. Soc. (1995), to appear.
93. H. Furstenberg, Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions, J. Anal. Math. 31 (1977), 204–256.
94. H. Furstenberg, Recurrence in Ergodic Theory and Combinatorial Number Theory, Princeton Univ. Press, Princeton, 1981.
95. H. Furstenberg and Y. Katznelson, A density version of the Hales-Jewett theorem, J. Analyze Math. 57 (1991), 61–85.
96. V. Bergelson and A. Leibman, Polynomial extensions of van der Waerden's and Szemerédi's theorems, J. Amer. Math. Soc. (1995), to appear.
97. P. Erdős, J. Nešetřil, and V. Rödl, On Pisier Type Problems and Results (Combinatorial Applications to Number Theory). In: Mathematics of Ramsey Theory (ed. J. Nešetřil and V. Rödl), Springer Verlag (1990), 214–231.
98. P. Erdős, J. Nešetřil, and V. Rödl, On Colorings and Independent Sets (Pisier Type Theorems) (preprint).