ELSEVIER
Discrete Mathematics 142 (1995) 49–63
DISCRETE MATHEMATICS

A tight lower bound for the Steiner ratio in Minkowski planes

Biao Gao[a,b,1], Ding-Zhu Du[a,b,1], Ronald L. Graham[c,*]

[a]*Computer Science Department, University of Minnesota, Minneapolis, MN 55455, USA*
[b]*Institute of Applied Mathematics, Chinese Academy of Sciences, Beijing, China*
[c]*AT&T Bell Laboratories, 600 Mountain Ave., Room 2T-102, Murray Hill, NJ 07974, USA*

Received 19 April 1993; revised 13 October 1993

Abstract

A minimum Steiner tree for a given set X of points is a network interconnecting the points of X having minimum possible total length. The Steiner ratio for a metric space is the largest lower bound for the ratio of lengths between a minimum Steiner tree and a minimum spanning tree on the same set of points in the metric space. In this note, we show that for any Minkowski plane, the Steiner ratio is at least 2/3. This settles a conjecture of Cieslik (1990) and also Du et al. (1991).

1. Introduction

Given a compact, convex, centrally symmetric domain D in the Euclidean plane E^2, one can define a norm $\|\cdot\|_D : E^2 \to R$ by setting $\|\bar{x}\|_D = \lambda$ where $\bar{x} = \lambda\bar{u}$ and $\bar{u} \in \partial D$, the boundary of D. We can then define a metric d_D on E^2 by taking

$$d_D(\bar{x}, \bar{y}) = \|\bar{x} - \bar{y}\|_D.$$

Thus, $\partial D = \{\bar{x} \mid \|\bar{x}\|_D = 1\}$. The resulting metric space $M = M(D) = (E^2, d_D)$ is often called a *Minkowski* or *normed* plane with unit disc D. We will usually suppress the explicit dependence of various quantities on D. For a finite subset $X \subset E^2$, a minimum spanning tree $S = S(X)$ consists of a collection of segments AB with $A, B \in X$, which spans all the points of X, and such that the sum of all the lengths $\|AB\|_D$ is a minimum. We denote this minimum sum by $L_M(X)$. Further, we define

$$L_S(X) = \inf_{Y \supseteq X} L_M(Y),$$

---

[*] Corresponding author.

[1] Support in part by the NSF under grant CCR-9208913.

SSDI 0012-365X(95)00005-4

where Y ranges over all finite subsets of E^2 containing X. It is not hard to show that there always exists $X' \supseteq X$ with $|X'| \leqslant 2|X|-2$ having $L_S(X) = L_M(X')$. When equality holds we say that the Steiner tree $T(X)$ $(=S(X'))$ is a *full* Steiner tree for X. The minimum spanning tree $S(Y)$ will be called a *minimum Steiner tree* $T(X)$ for X. The points of $Y \backslash X$ are usually called *Steiner* points of $T(X)$; the points of X are known as *regular* points of $T(X)$.

Minimum Steiner trees have been the subject of extensive investigations during the past 25 years or so (see [4,9,11,16]). Most of this research has dealt with the Euclidean metric, with much of the remaining work concerned with the L_1 metric, or more generally, the usual L_p metric or norm (see [3, 6]). It has been shown, for example, that the determination of $L_S(X)$ in general is an NP-complete problem, both for the Euclidean as well as the L_1 case (cf. [9,10]).

In this note, we study the *Steiner ratio* $\rho(D)$ for $M(D)$, defined by

$$\rho(D) := \inf_X \frac{L_S(X)}{L_M(X)}.$$

Thus, $\rho(D)$ is a measure of how much the total length of a minimum spanning tree can be decreased by allowing additional (Steiner) points. It is known that for the L_1 metric (so that D is the square with vertices $(\pm 1,0),(0,\pm 1)$), $\rho(D)=2/3$ [13] and for the Euclidean (or L_2) metric, $\rho(D)=\sqrt{3}/2$ [7]. More recently, Cieslik [3] and Du et al. [6] independently conjectured that for any normed plane,

$$2/3 \leqslant \rho(D) \leqslant \sqrt{3}/2.$$

Cieslik [3] showed that for any normed plane,

$$0.612 < \rho(D) < 0.9036,$$

while Du et al. [6] proved that for any normed plane,

$$0.623 < \rho(D) < 0.8686.$$

We will prove here that for any normed plane,

$$\rho(D) \geqslant 2/3.$$

Since the L_1 metric has $\rho = 2/3$ then this inequality is therefore best possible.

For prior results on minimum Steiner trees in normed planes, the reader should consult [1,2,8,17,19]. This note is organized in the following way. In Section 2, fundamental properties of minimum Steiner trees are presented. In Section 3, the main result is proved.

2. Preliminaries

A minimum Steiner tree is *full* if every regular point is a leaf (i.e., has degree one). The following lemma states an important property of full minimum Steiner trees, which can be found in [6].

Lemma 1. *Suppose that ∂D is differentiable and strictly convex. Then every full Steiner minimum tree consists of three sets of parallel segments.*

A tree is called a 3-*regular* tree if every vertex which is not a leaf has degree three. A consequence of Lemma 1 is that for strictly convex and differentiable norms, every minimum Steiner tree is a 3-regular tree.

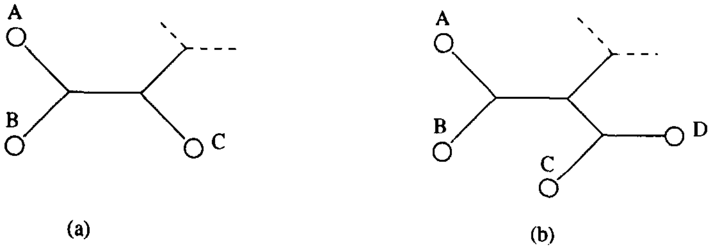
Another consequence of Lemma 1 is the following result. A proof can be found in [5].

Lemma 2. *For strictly convex and differentiable norms, every full minimum Steiner tree on more than three points must have at least one of the local structures shown in Fig.* 1.

Consider a full minimum Steiner tree T in a plane with a strictly convex and differentiable norm. Two regular points are called *adjacent* if one can be reached from the other by always moving in a *clockwise* direction or always moving in a *counterclockwise* direction. Clearly, each regular point has two other adjacent regular points.

We can form a polygon G, called the *characteristic* polygon of T, by joining each pair of adjacent regular points with a straight line segment. Any spanning tree lying inside G is called an *inner* spanning tree. A *minimum* inner spanning tree is one having the least possible total length. A point set P is called *critical* if there is a minimum Steiner tree T for P such that the union of the minimum inner spanning trees (with respect to T) for P divides the characteristic polygon $G=G(T)$ into equilateral triangles. The vertices of these equilateral triangles (which we will call lattice points) lie on a triangular lattice in the normed plane.

Since similar sets have the same ratios of minimum Steiner tree and minimum spanning tree lengths, we need only consider critical sets having equilateral triangles with unit edge length. Clearly, for any critical set, a minimum inner spanning tree is in fact a minimum spanning tree; its length is just $n-1$ where n is the number of its (regular) vertices. Note that any two adjacent regular points have mutual distance 1.



(a) (b)

Fig. 1. Local structures in full minimum Steiner trees.

Define

$$\rho_n(D) := \min_{|P|=n} \frac{L_S(P)}{L_M(P)}.$$

If $\rho_{n-1} > \rho_n$, then n is called a *jump* value. In [7], Du and Hwang prove the following.

Lemma 3. *In a plane with a strictly convex and differentiable norm, if n is a jump value then ρ_n is achieved by some critical set.*

Remark. The proof of Du and Hwang for the Gilbert–Pollak conjecture was a proof by contradiction. In their argument, n is assumed to be the smallest integer such that a counterexample to the conjecture on n points exists. For this n, it is proved that ρ is achieved by some critical set and then shown that the Gilbert–Pollak conjecture holds for every critical set. Actually, if we just assume that n is a jump value, then the arguments of Du and Hwang still apply, and the above lemma follows.

3. The main result

Theorem 1. *For any convex and centrally symmetric D,*

$$\rho(D) \geqslant \frac{2}{3}.$$

Moreover, if $\rho_k(D) = 2/3$ for some k, then $k = 4$ and ∂D is a parallelogram.

Proof. To begin with, we first assume that the boundary ∂D of unit disc D is strictly convex and differentiable. Thus, we can apply the results of the preceding section.

Assume that the theorem is false. Let n denote the least value so that $\rho_n(D) < 2/3$. Thus, n is a jump value. By Lemma 3 there exists a critical set P of size n such that

$$\frac{L_S(P)}{L_M(P)} < \frac{2}{3},$$

that is,

$$L_S(P) < \frac{2}{3} L_M(P) = \frac{2}{3}(n-1). \tag{1}$$

Let T be a minimum Steiner tree on P which witnesses the criticality of P. We first establish several properties of T.

Lemma 4. *T is a full Steiner tree and every edge of T has length less than 2/3.*

Proof. If T is not a full Steiner tree, then we can decompose it into two edge-disjoint subtrees T_1 and T_2 which are Steiner trees on point sets P_1 and P_2, respectively, where $P_1 \cup P_2 = P$ and each P_i has size less than n. Thus, by the minimality of n,

$$L_S(P) = L_S(P_1) + L_S(P_2) \geqslant \frac{2}{3} L_M(P_1) + \frac{2}{3} L_M(P_2) \geqslant \frac{2}{3} L_M(P),$$

contradicting (1).

If T has some edge e of length at least 2/3, then by removing e, we are left with two vertex-disjoint subtrees T_1 and T_2. Clearly, T_1 and T_2 are Steiner trees on disjoint subsets P_1 and P_2, respectively, where $P_1 \cup P_2 = P$. It follows that

$$L_S(P) \geqslant L_S(P_1) + L_S(P_2) + l(e) \geqslant \frac{2}{3}(L_M(P_1) + L_M(P_2) + 1) \geqslant \frac{2}{3} L_M(P),$$

again contradicting (1), where in general we will let $l(T)$ denote the total length (under D) of any graph T (such as an edge, path or tree). □

Lemma 5. *Suppose T_1 is a 3-regular subtree of T which has f leaves. Then*

$$l(T_1) < \frac{2}{3}(f-1).$$

Proof. Assume that

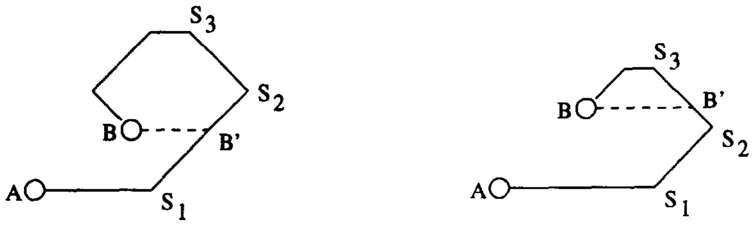
$$l(T_1) \geqslant \frac{2}{3}(f-1)$$

for some subtree T_1 and suppose that T_1 has r leaves which are regular points. Then the removal of T_1 results in $f-r$ subtrees. Suppose that they interconnect sets of $n_1, n_2, \ldots, n_{f-r}$ regular points, respectively. Then $n_1 + n_2 + \cdots + n_{f-r} = n - r$ and

$$\begin{aligned} L_S(P) &\geqslant \frac{2}{3}(n_1 - 1) + \frac{2}{3}(n_2 - 1) + \cdots + \frac{2}{3}(n_{f-r} - 1) + l(T_1) \\ &\geqslant \frac{2}{3}(n-f) + \frac{2}{3}(f-1) = \frac{2}{3}(n-1), \end{aligned}$$

which contradicts (1). □

Let us call a path $AS_1S_2 \ldots S_iB$ joining two adjacent regular points A and B in T *monotone* if it is either a clockwise or counterclockwise path from A to B. We will say that S_1 can be *legally moved* to A if $i \geqslant 3$ and the subpath $S_1S_2S_3$ can be removed from T (disconnecting it into three subtrees) and replaced by a parallel translate



(a) (b)

Fig. 2.

$S'_1S'_2S'_3$ with S'_1 located at point A so that $S'_1S'_2S'_3$ intersects $S_3 \ldots S_iB$. Thus, the two subtrees containing A and B, respectively, are reconnected by $S'_1S'_2S'_3$.

Lemma 6. *Let $AS_1S_2 \ldots S_iB$ be a monotone path in T connecting two regular points A and B. Suppose that S_1 cannot be legally moved to A. Draw a line through B, parallel to AS_1, and intersecting the subpath $S_1S_2S_3$ at B'. Then*

$$l(AS_1S_2S_3)+l(S_2S_3)-l(BB'S_2)\geqslant 1. \tag{2}$$

Proof. Since S_1 cannot be legally moved to A, we have $l(BB')<l(AS_1)$. If B' is on the segment S_1S_2 (see Fig. 2(a)), then

$$(l(AS_1)-l(BB'))+l(S_1B')\geqslant l(AB)=1,$$

that is,

$$l(AS_1S_2)-l(BB'S_2)\geqslant 1.$$

Thus, (2) holds.

On the other hand, if B' is on the segment S_2S_3 (see Fig. 2(b)), then

$$(l(AS_1)-l(BB'))+l(S_1S_2B')\geqslant l(AB)=1,$$

that is,

$$l(AS_1S_2B')+l(S_2B')-l(BB'S_2)\geqslant 1.$$

Thus, (2) also holds in this case, and the lemma is proved. □

It is easy to see that (2) still holds if $l(AS_1)=l(BB')$.

Lemma 7. *Suppose S_1 is a Steiner point in T adjacent to two regular points A and B. Then S_1 can be legally moved to exactly one of A or B.*

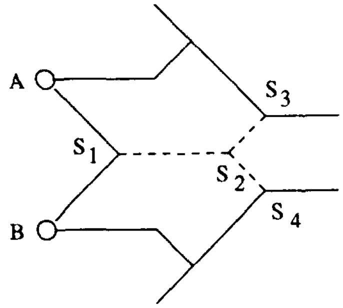


Fig. 3.

Proof. Let S_2 be the Steiner point adjacent to S_1 and let S_3 and S_4 be the two vertices adjacent to S_2.

Suppose that S_1 can be legally moved to both A and B. Then from these two movements, we can obtain a tree of total length at most $l(T)+l(S_1S_2)$, which can be decomposed at A and B (see Fig. 3). Thus,

$$l(T)+l(S_1S_2)\geqslant\frac{2}{3}(n-2)+l(AS_1B).$$

By Lemma 5,

$$l(AS_1B)+l(S_1S_2)<\frac{4}{3}.$$

Therefore,

$$l(T)\geqslant\frac{2}{3}(n-2)+l(AS_1B)-l(S_1S_2)>\frac{2}{3}(n-4)+2l(AS_1B)\geqslant\frac{2}{3}(n-1),$$

contradicting (1).

Suppose now that S_1 cannot be legally moved to either A or B. Let C be the regular point adjacent to A, other than B, and D the regular point adjacent to B, other than A. By Lemma 6,

$$l(AS_1S_2S_3)+l(S_2S_3)-l(CC'S_2)\geqslant 1, \tag{3}$$

$$l(BS_1S_2S_4)+l(S_2S_4)-l(DD'S_2)\geqslant 1, \tag{4}$$

where C' and D' are two points defined in the lemma. Let T' be the 3-regular subtree interconnecting A, B, S_3 and S_4. Adding (3) and (4), we obtain

$$2l(T')-l(AS_1B)-l(CC'S_2D'D)\geqslant 2,$$

that is,

$$l(T')\geqslant 1+\tfrac{1}{2}(l(AS_1B)+l(CC'S_2D'D))\geqslant 2,$$

contradicting Lemma 5. □

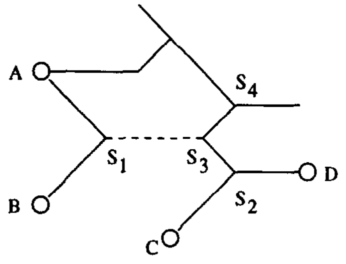


Fig. 4.

We now complete the proof of the first part in the theorem. By Lemma 2, there are two possible local structures we need to consider. We first consider the local structure shown in Fig. 1(b). Then there exists a Steiner point S_3 adjacent to two Steiner points S_1 and S_2 each of which is adjacent to two regular points, say A and B are adjacent to S_1, and C and D are adjacent to S_2. Let S_4 be a vertex adjacent to S_3 (see Fig. 4). We first observe that if $l(BS_1)=l(CS_2)$ then, whether or not S_1 can be legally moved to A, we obtain a contradiction by using the argument given in the proof of Lemma 7. Thus, without loss of generality, we can assume that $l(BS_1)>l(CS_2)$, i.e., S_1 cannot be legally moved to B. Then by Lemma 6, S_1 can be legally moved to A (see Fig. 4). This movement results in a tree of length at most $l(T)+l(S_3S_4)$, which can be decomposed at A into the subtree AS_1B and a subtree interconnecting $n-1$ regular points other than B. Thus,

$$l(T)+l(S_3S_4)\geqslant\frac{2}{3}(n-2)+l(AS_1B)\geqslant\frac{2}{3}(n-1)+\frac{1}{3}.$$

Since $l(T)<\frac{2}{3}(n-1)$, we have

$$l(S_3S_4)>\frac{1}{3}. \tag{5}$$

Moreover, by Lemma 4, $l(BS_1)<2/3$ and $l(DS_2)<2/3$. It follows that

$$l(CS_2)=l(CS_2D)-l(DS_2)>\frac{1}{3}.$$

Note that by Lemma 6,

$$l(BS_1S_3S_2)-l(CS_2)\geqslant 1.$$

Thus,

$$l(S_1S_3S_2)\geqslant 1+l(CS_2)-l(BS_1)>\frac{2}{3}. \tag{6}$$

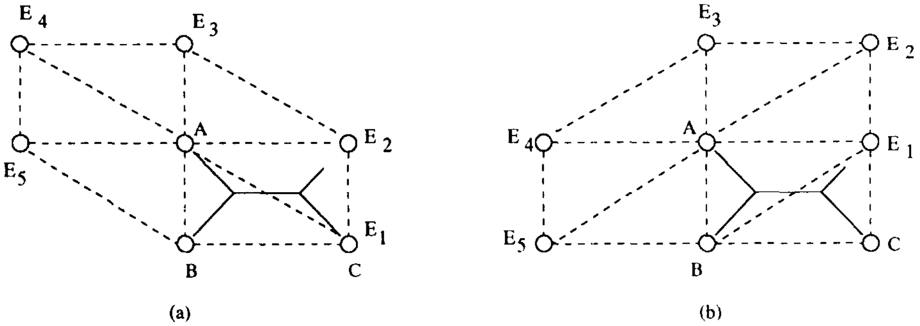


Fig. 5.

Let T' be the 3-regular subtree interconnecting A, B, S_4 and S_2. By (5) and (6),

$$l(T') = l(AS_1B) + l(S_1S_3S_2) + l(S_3S_4) > 2,$$

contradicting Lemma 5.

Next, we consider the local structure shown in Fig. 1(a), i.e., there exists a Steiner point S_2 adjacent to a Steiner point S_1 and a regular point C such that S_1 is adjacent to two regular points A and B. Let S_3 be the vertex adjacent to S_2, other than C and S_1. We claim that

$$l(S_2S_3) < l(BS_1). \tag{7}$$

In fact, if $l(S_2S_3) \geqslant l(BS_1)$, then considering the 3-regular subtree T', interconnecting A, B, C and S_3, we would have

$$l(T') \geqslant l(BS_1S_2C) + l(AS_1B) \geqslant 2,$$

contradicting Lemma 5. Now, let E be the adjacent regular point of A other than B and let $AS_1 \dots S_kE$ be the monotone path connecting A and E. From the definition of a critical set, it is easy to see that $l(AE) = 1$. Let $B, E_1, E_2, \dots, E_5$ denote all the lattice points with distance exactly one to A (see Fig. 5). Then $E \in \{E_1, \dots, E_5\}$. Since $l(AC) < l(AB) + l(BC) = 2$, C is identical to either E_1 or a lattice point which forms an equilateral triangle with B and E_1 (see Fig. 5).

Suppose that E' is a point on the path $S_1S_2S_3$ such that EE' is parallel to AS_1. If E is at E_1, then E' must be on S_2S_3 and $l(S_2E') = l(BS_1)$. It follows that

$$l(S_2S_3) \geqslant l(S_2E') = l(BS_1),$$

contradicting (7). A similar argument can be applied to the case that E is at E_2.

If E is at E_3, then it is easy to see that $k \geqslant 4$. Let E'' be a point on $S_2S_3S_4$ such that EE'' is parallel to S_1S_2 (Fig. 6). Extend BS_1 to F so that EF is parallel to AS_1. Since $l(BE) = 2l(BA)$, we have $l(EF) = 2l(AS_1)$ and $l(S_1F) = l(BS_1)$. Let F'

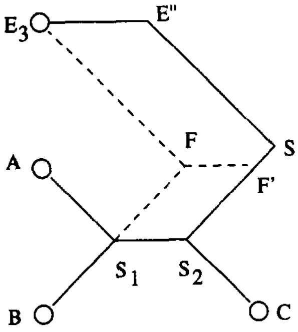


Fig. 6.

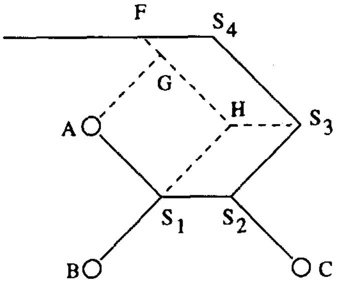


Fig. 7.

be a point on the path $S_2S_3S_4$ so that FF' is parallel to S_1S_2. If F' is on the segment S_2S_3, then

$$l(S_2S_3) \geqslant l(S_1F) = l(BS_1),$$

contradicting (7). If F' is on the segment S_3S_4, then

$$l(S_3S_4) \geqslant l(EF) = 2l(AS_1) \geqslant 2(l(AB) - l(AS_1)) > 2\left(1 - \frac{2}{3}\right) = \frac{2}{3},$$

contradicting Lemma 4.

If E is at E_4 or E_5, then the extension of S_1A must intersect the monotone path $AS_1 \ldots S_kE$. This implies that any line between AS_1 and S_3S_4 and parallel to them must intersect the path $AS_1 \ldots S_kE$. Draw the parallelograms $S_1S_2S_3H$ and AS_1HG and extend HG until it intersects the path $AS_1 \ldots S_kE$, say at F (Fig. 7). (AG cannot intersect the path $AS_1 \ldots S_kE$ since otherwise, removing S_2S_3 and adding AG would result in a tree of length at most $l(T)$ which does not satisfy the condition in Lemma 1 at the intersection of AG and the path $AS_1 \ldots S_kE$). Then FHS_3S_4 is also a parallelogram. It is easy to see that

$$l(GF) \leqslant l(S_3S_4) - l(AS_1).$$

Let T' be the 3-regular subtree interconnecting A, B, C and S_3. Consider the tree $(T\setminus T')\cup AGF$ which interconnects $n-2$ regular points. Then,

$$l(T)-l(T')+l(AGF)\geqslant\frac{2}{3}(n-3).$$

Moreover,

$$\begin{aligned}l(T')-l(AGF)&\geqslant l(T')-(l(S_2S_3)+l(S_3S_4)-l(AS_1))\\&\geqslant l(AS_1B)+l(AS_1S_2C)-l(S_3S_4)\\&>2-\frac{2}{3}=\frac{4}{3}.\end{aligned}$$

Therefore,

$$l(T)\geqslant\frac{2}{3}(n-3)+l(T')-l(AGF)>\frac{2}{3}(n-1),$$

contradicting (1). This completes the proof of the first part of the theorem for strictly convex and differentiable ∂D.

When ∂D is not strictly convex or not differentiable, we can use a sequence of strictly convex and differentiable ones to approach it from its interior. For each norm in the sequence and for any point set P, we know that

$$L_S(P)\geqslant\frac{2}{3}L_M(P). \tag{8}$$

Since $L_S(P)$ and $L_M(P)$ are continuous functions with respect to the norm for fixed P, then letting the sequence approach its limit, we see that (8) holds for the (arbitrary) limiting norm. This completes the proof of the first part of the theorem.

Next, we prove the second part of the theorem. Before doing so, we establish three lemmas.

Lemma 8. *Let $A_1A_2\ldots A_n$ be a path and let OB_{ij}, $1<i<j<n$, be unit vectors based at the origin O and oriented in the directions A_iA_j. if $l(A_1A_2\ldots A_n)=l(A_1A_n)$, then the straight-line segment $B_{12}B_{n-1,n}$ is part of ∂D.*

Proof. We prove that all B_{ij}, $i<j$, are on the same straight line. The lemma is a consequence of this fact.

First, consider $n=3$. Draw the parallelogram $A_1A_2A_3B$. Without loss of generality, assume $l(A_1A_2)\geqslant l(A_1B)=l(A_2A_3)$. Let C be a point on A_1A_2 such that $l(A_1C)=l(A_1B)$. Let E be the intersection point of BC and A_1A_3. Draw line A_2H parallel to BC and intersecting A_1A_3 at H (see Fig. 8). Then

$$l(A_1E)=l(HA_3)$$

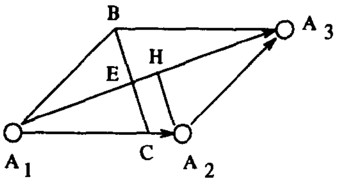


Fig. 8.

and

$$l(A_1H)=l(A_1E)\cdot\frac{l(A_1A_2)}{l(A_1C)}.$$

Thus,

$$\begin{aligned} l(A_1A_3)&=l(A_1H)+l(HA_3)\\ &=l(A_1E)\left(1+\frac{l(A_1A_2)}{l(A_1C)}\right)\\ &=l(A_1E)\left(1+\frac{l(A_1A_2)}{l(A_2A_3)}\right). \end{aligned}$$

Since

$$l(A_1A_3)=l(A_1A_2A_3)=l(A_2A_3)\left(1+\frac{l(A_1A_2)}{l(A_2A_3)}\right),$$

we have

$$l(A_1E)=l(A_2A_3)=l(A_1B)=l(A_1C).$$

This means that quadrilateral A_1CEB is similar to quadrilateral $OB_{12}B_{13}B_{23}$. Therefore, B_{12}, B_{23} and B_{13} are collinear. In addition, B_{13} lies between B_{12} and B_{23}.

Next, consider the case $n=4$. Note that $l(A_1A_2A_3A_4)=l(A_1A_4)$ implies that $l(A_1A_2A_4)=l(A_1A_4)$ because

$$l(A_1A_2A_3A_4)\geqslant l(A_1A_2)+l(A_2A_4)\geqslant l(A_1A_4).$$

From the case $n=3$, B_{14} is on the segment $[B_{12},B_{24}]$. Similarly, B_{14} is on the segment $[B_{13},B_{34}]$, B_{13} is on the segment $[B_{12},B_{23}]$, and B_{24} is on the segment $[B_{23},B_{34}]$ (see Fig. 9). Note that all B_{ij}'s are on ∂D, the boundary of a convex region. Moreover, for any i, j and k, B_{ij}, B_{jk}, and B_{ik} are either all distinct or all identical. It follows that all B_{ij} for $1\leqslant i<j\leqslant 4$ are collinear.

Now consider $n>4$. Note that $l(A_1A_2\ldots A_n)=l(A_1A_n)$ implies that for $4\leqslant j\leqslant n$, $l(A_1A_2A_3A_j)=l(A_1A_j)$ and for $3\leqslant j<k\leqslant n$, $l(A_1A_2A_jA_k)=l(A_1A_k)$. Therefore, for $4\leqslant j\leqslant n$,

$B_{12}, B_{23}, B_{13}, B_{1j}, B_{2j}$ and B_{3j} are collinear

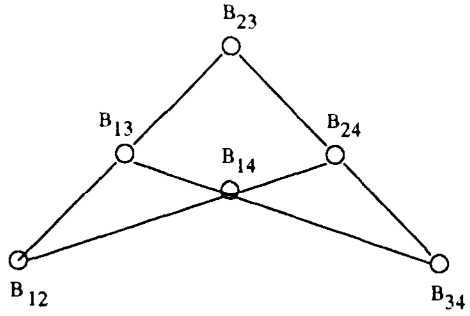


Fig. 9.

and for $3 \leqslant j < k \leqslant n$,

$$B_{12}, B_{1j}, B_{2j}, B_{1k}, B_{2k} \text{ and } B_{jk} \text{ are collinear.}$$

It follows that all B_{ij} for $1 \leqslant i < j \leqslant n$ are collinear. □

Lemma 9. *$\rho_4(D) = 2/3$ iff ∂D is a parallelogram.*

Proof. Note that for any D,

$$\rho_3(D) \geqslant 3/4.$$

Suppose $\rho_4(D) = 2/3$. Thus, 4 is a jump value. Consider $F = \{(A, B, C, E) \mid L_M(A, B, C, E) = 1\}$. Since $L_M(A, B, C, E)$ is continuous with respect to A, B, C and E, then F is a compact set in 8-dimensional space. Clearly,

$$\rho_4(D) = \inf_{(A,B,C,E) \in F} L_S(A, B, C, E).$$

Since $L_S(A, B, C, E)$ is also continuous with respect to A, B, C, and E, there exists a point set $\{A, B, C, E\}$ such that

$$2/3 = \rho_4(D) = L_S(A, B, C, E)/L_M(A, B, C, E). \tag{9}$$

Note that the minimum Steiner tree T for this point set must be full because 4 is a jump value. Suppose that A, B, C and E are arranged in the order as shown in Fig. 10. Let p_{XY} denote the path from X to Y in T. We claim that

$$l(p_{AB}) = l(AB), \tag{10}$$

$$l(p_{CB}) = l(CB), \tag{11}$$

$$l(p_{CE}) = l(CE), \tag{12}$$

$$l(p_{AE}) = l(AE). \tag{13}$$

In fact, if one of them does not hold, then

$$l(AB) + l(CB) + l(CE) + l(AE) < l(p_{AB}) + l(p_{CB}) + l(p_{CE}) + l(p_{AE}) = 2l(T).$$

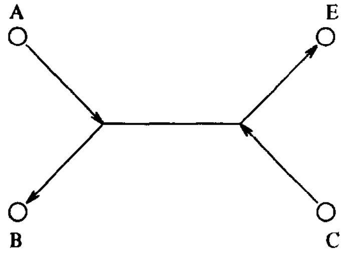


Fig. 10.

So,

$$4L_M(A, B, C, E) \leqslant 3(l(AB)+l(CB)+l(CE)+l(AE)) < 6l(T) = 6L_S(A, B, C, E),$$

contradicting (9). By Lemma 8 and Eqs. (10)–(13), ∂D must be a parallelogram. □

Lemma 10. *Let $d(\partial D, \partial D')$ denote the maximum Euclidean distance between the two intersections of a ray from the origin with ∂D and $\partial D'$, respectively. Then for any $\delta > 0$ and k, there exists $\varepsilon > 0$ such that $d(\partial D, \partial D') < \varepsilon$ implies $|\rho_k(D) - \rho_k(D')| < \delta$.*

Proof. Consider any set of k points as a point in $2k$-dimensional space. Let Ω be the point set in $2k$-dimensional space consisting of 'points' each of which is a set of k points in the plane with a Euclidean minimum spanning tree of length one. Then Ω is a compact set. In addition, it is easy to see that for any D,

$$\rho_k(D) = \inf_{P \in \Omega} \frac{L_S(P)}{L_M(P)}.$$

Thus, $\rho_k(D)$ is continuous with respect to D. □

Now, suppose to the contrary that ∂D is not a parallelogram and $\rho_k(D) = 2/3$ for some fixed value of k. By Lemma 9, $\rho_4(D) > 2/3$. Thus, there exists k', $4 < k' \leqslant k$, such that $\rho_{k'-1}(D) > 2/3$ and $\rho_{k'}(D) = 2/3$. Let P be a set of k' points such that $L_S(P)/L_M(P) = 2/3$. Then every minimum Steiner tree for P is full. By Lemma 10, we can choose a sequence of norms D' with strictly convex and differentiable boundary such that $\rho_{k'-1}(D') < \rho_{k'}(D')$. So, the minimum Steiner tree $T(D')$ for P under each norm D' is still full. By Lemma 1, every $T(D')$ is 3-regular and satisfies the condition that all edges of $T(D')$ lie in three directions. Since the number of 3-regular trees with k' leaves is finite, there is a subsequence of $\{T(D')\}$ which converges to a 3-regular tree and satisfies the same condition. It is easy to see that this tree must be a minimum Steiner tree for P under the norm $\|\cdot\|_D$. By the argument used in the proof of Du and Hwang [7], it follows that P is a critical set. Now, by using the argument in the proof of the first part of the theorem, taking special care of the cases in which equality holds in various inequalities, we eventually obtain a contradiction. This completes the proof of the second part of the claim and the proof is complete. □

4. Discussion

We conjecture that for any norm $\|\cdot\|_D$, there always exists a value k (depending on D) such that $\rho_k(D) = \rho(D)$. A consequence of this conjecture would be that $\rho(D) = 2/3$ iff ∂D is a parallelogram.

The proof techniques used in this paper are different from those in Hwang [13] for proving 2/3 as the Steiner ratio of the rectilinear plane. Graham and Hwang [12] conjectured that m-dimensional rectilinear space has the Steiner ratio $m/(2m-1)$. Although the methods in [13] do not seem to be applicable to proving this conjecture, perhaps the ideas we use here will be of some help. We hope to consider this in the near future.

References

[1] M. Alfaro, M. Conger, K. Hodges, A. Levy, R. Kochar, L. Kuklinski, Z. Mahmood and K. von Haam, The structure of singularities of Φ-minimizing networks in $\mathbb{R}^2$, Pacific J. Math. 149 (1991) 201–210.

[2] G.D. Chakerian and M.A. Ghandehari, The Fermat problem in Minkowski space, Geom. Dedicata 17 (1985) 227–238.

[3] D. Cieslik, The Steiner ratio of Banach–Minkowski planes, in: R. Bodendiek, ed., Contemporary Methods in Graph Theory (BI-Wissenschatteverlag, Mannheim, 1990) 231–248.

[4] E.J. Cockayne, On the Steiner problem, Canad. Math. Bull. 10 (1967) 431–450.

[5] D.Z. Du, On Steiner ratio conjectures, Ann. Oper. Res. 33 (1991) 437–449.

[6] D.Z. Du, B. Gao, R.L. Graham, Z.-C. Liu and P.-J. Wan, Minimum Steiner trees in normed planes, Discrete Comput. Geom. 9 (1993) 351–370.

[7] D.Z. Du and F.K. Hwang, A proof of Gilbert-Pollak's conjecture on the Steiner ratio, Algorithmica 7 (1992) 121–135.

[8] D.Z. Du and F.K. Hwang, Reducing the Steiner problem in a normed space, SIAM J. Comput. 21 (1992) 1001–1007.

[9] M.R. Garey, R.L. Graham and D.S. Johnson, The complexity of computing Steiner minimal trees, SIAM J. Appl. Math. 32 (1977) 835–859.

[10] M.R. Garey and D.S. Johnson, The rectilinear Steiner tree problem is NP-complete, SIAM J. Appl. Math. 32 (1977) 826–834.

[11] E.N. Gilbert and H.O. Pollak, Steiner minimal trees, SIAM J. Appl. Math. 178 (1968) 1–29.

[12] R.L. Graham and F.K. Hwang, Remarks on Steiner minimum trees, Bull. Inst. Math. Acad. Sinica 4 (1976) 177–182.

[13] F.K. Hwang, On Steiner minimal trees with rectilinear distance, SIAM J. Appl. Math. 30 (1976) 104–114.

[14] G. Lawlor and F. Morgan, Paired calibrations applied to soapfilms, immiscible fluids, and surfaces or networks minimizing other norms, Pacific J. Math. 166 (1994) 53–80.

[15] Z.C. Liu and D.Z. Du, On Steiner minimal trees with L_p distance, Algorithmica 7 (1992) 179–191.

[16] Z.A. Melzak, Companion to Concrete Mathematics, Vol. II (Wiley, New York, 1976).

[17] D. Sankoff and P. Rousseau, Locating the vertices of a Steiner tree in an arbitrary metric space, Math. Programming 9 (1975) 240–246.

[18] M. Sarrafzadeh and C.K. Wong, Hierarchical Steiner tree construction in uniform orientations, unpublished manuscripts.

[19] J.M. Smith and M. Gross, Steiner minimal trees and urban service networks, J. Socio. Econ. Plng 16 (1982) 21–38.