Dense Packings of 3k(k+1)+1 Equal Disks in a Circle for k = 1, 2, 3, 4, and 5

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ABSTRACT

For each $k \geq 1$ and corresponding hexagonal number $h(k) = 3k(k+1)+1$, we introduce $m(k) = (k-1)!$ packings of $h(k)$ equal disks inside a circle which we call the *curved hexagonal* packings. The curved hexagonal packing of 7 disks ($k = 1$, $m(1) = 1$) is well known and the one of 19 disks ($k = 2$, $m(2) = 1$) has been previously conjectured to be optimal. New curved hexagonal packings of 37, 61, and 91 disks ($k = 3$, 4, and 5, $m(3) = 2$, $m(4) = 6$, and $m(5) = 24$) were the densest we obtained on a computer using a so-called "billiards" simulation algorithm. A curved hexagonal packing pattern is invariant under a 60° rotation. When $k \geq 3$, the curved hexagonal packings are not mirror symmetric but they occur in $m(k)/2$ image-reflection pairs. For $k \to \infty$, the density (covering fraction) of curved hexagonal packings tends to $\frac{\pi^2}{12}$. The limit is smaller than the density of the known optimum disk packing in the infinite plane. We present packings that are better than curved hexagonal packings for 127, 169, and 217 disks ($k = 6$, 7, and 8).

In addition to new packings for $h(k)$ disks, we present new packings we found for $h(k)+1$ and $h(k)-1$ disks for k up to 5, i.e., for 36, 38, 60, 62, 90, and 92 disks. The additional packings show the "tightness" of the curved hexagonal pattern for $k \leq 5$: deleting a disk does not change the optimum packing and its quality significantly, but adding a disk causes a substantial rearrangement in the optimum packing and substantially decreases the quality.

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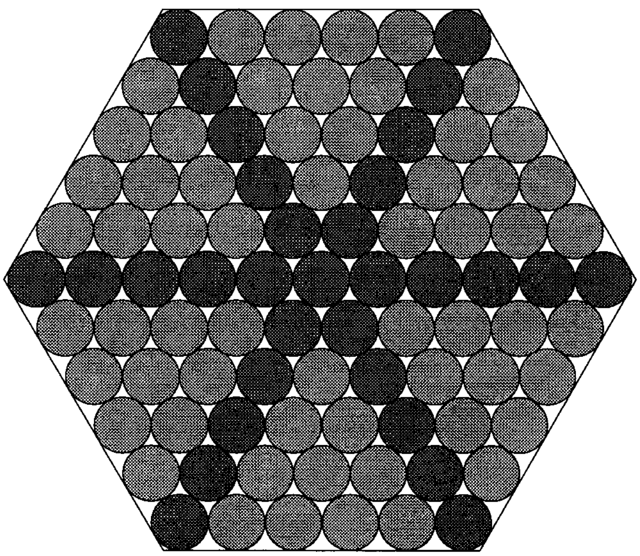
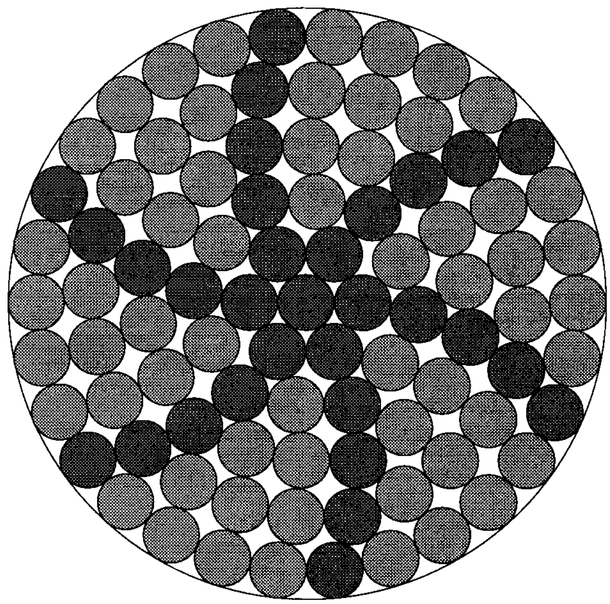
1. Introduction

Patterns of dense geometrical packings are sensitive to the geometry of the enclosing region of space. In particular, dense packings of equal nonoverlapping disks in a circle are different from those in a regular hexagon, as one might expect. In this paper, for each $k \geq 1$ and



91 disks
density = 0.88434871149353
D/d = 11.154700538379



91 disks
density = 0.81499829406214
D/d = 10.566772233506

Figure 1.1: *Left.* The well known best packing of $h(5) = 91$ disks in a regular hexagon. *Right.* One of the 24 best (that we found) packings of $h(5) = 91$ disks in a circle. (The different shading of the disks is auxiliary; it is not a part of the pattern).

corresponding hexagonal number $h(k) = 3k(k+1)+1$, we present a pattern of packings of $h(k)$ equal disks in a circle which can be viewed a "curved" analogue of the densest packing of $h(k)$ disks in a regular hexagon. For a particular k, there exists a set of $m(k) = (k-1)!$ equivalent different *curved hexagonal packings.* A curved hexagonal packing pattern is invariant under a $60°$ rotation. When $k \geq 3$, the curved hexagonal packings are not mirror symmetric but they occur in $m(k)/2$ "image-reflection" pairs. The density (covering fraction) of a curved hexagonal packing tends to $\frac{\pi^2}{12}$ as $k \to \infty$. Because the limit density is smaller than the density of the best (hexagonal) packing of equal disks on an infinite plane, it is natural to expect the curved hexagonal packings not to be optimal for all sufficiently large k. It might come as a surprise, though, that for several initial values of k there seems to be no better packing than the curved hexagonal ones. Indeed, for 7 disks ($k = 1$, $m(1) = 1$) the curved hexagonal packing is well known to be optimal and the one for 19 disks ($k = 2$, $m(2) = 1$) has been previously conjectured as such [K]. For 37, 61, and 91 disks ($k = 3$, 4, and 5, $m(3) = 2$, $m(4) = 6$, $m(5) = 24$), the curved hexagonal packings were the densest we obtained by computer experiments using the so-called "billiards" simulation algorithm.

The "billiards" simulation algorithm [L] [LS] has so far proved to be a reliable method for generating optimal packings of disks in an equilateral triangle [GL1]. Our experiments with this algorithm for packings in a circle (a detailed account will be reported elsewhere [GL2]) either confirmed or improved the best previously reported packings for $n \leq 25$. We are unaware of any published conjectures for packing $n > 25$ disks in a circle, but the "billiards" algorithm kept producing packings for many $n > 25$, specifically, for $n = h(3) = 37$, $n = h(4) = 61$, and $n = h(5) = 91$. The latter three sets of packings happened to have the curved hexagonal pattern and they were the best found for their value of n. As for $n = h(6) = 127$, $n = h(7) = 169$, and $n = h(8) = 217$, the algorithm found the packings which are *better* than the corresponding curved hexagonal ones.

For the values $k \leq 5$, for which the densest packings of $h(k)$ disks in a circle apparently have the curved hexagonal patterns, these packings look "tight." To test our intuition of their "tightness" we compared these packings with packings obtained for $h(k) - 1$ and $h(k) + 1$ disks. Thus, we generated dense packings for 36, 38, 60, 62, 90, and 92 disks and verified that deleting a disk does not change the optimum packing and its quality significantly, but adding a disk causes a substantial rearrangement in the optimum packing and substantially decreases the quality. This tightness may be considered an analogue of the similar tightness property for the infinite classes of packings in an equilateral triangle as noted in [GL1]. Specifically the variations in the packing pattern and quality when one disk is added or subtracted are similar to those observed for packings of $\frac{k(k+1)}{2}$ disks in the triangles.

2. Packings of 7, 19, 37, and 61 disks in a circle

The four best packings are presented in Fig. 2.1. The packing of 7 disks is well known

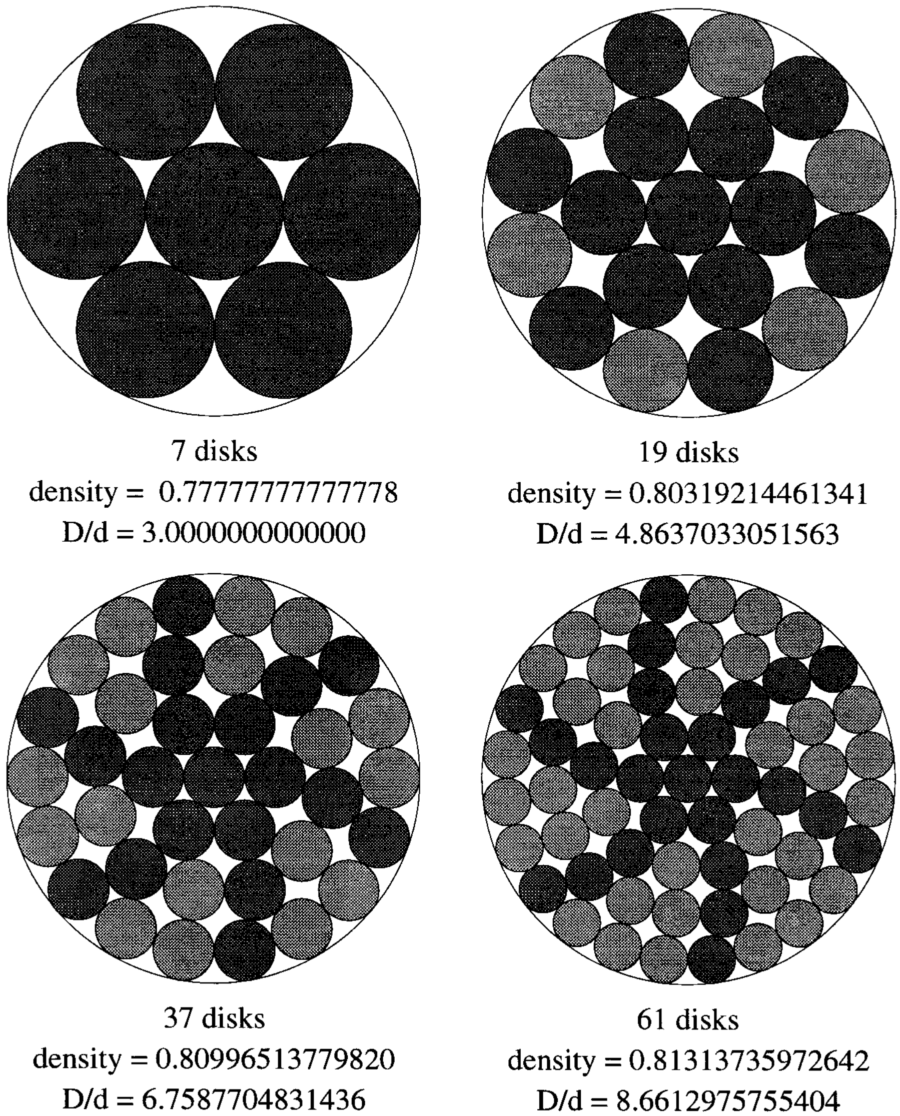


Figure 2.1: *Top.* The well known best packing of $h(1) = 7$ disks and the best previously conjectured packing of $h(2) = 19$ disks. *Bottom.* The best packing (one of the two best that we found) of $h(3) = 37$ disks and the best packing (one of the six best that we found) of $h(4) = 61$ disks.

to be optimal, and that of 19 disks is conjectured as such in [K]. The packings shown of 37 and 61 disks have not been reported before; they are the best we found for these numbers of disks. The density 7/9 of the 7-disk packing is presented in decimal form in conformance with the other three densities; an alternative finite form of the parameters for all the packings in Fig. 2.1 exists and is discussed in Section 3.

3. The curved hexagonal pattern

The pattern can be explained by comparing it with the corresponding hexagonal pattern. Fig. 1.1 depicts the two patterns side-by-side for $h(5) = 91$ disks. Each is composed of six sections which we tried to emphasize by shading more heavily the disks on the boundaries between them. In the true hexagonal packing, the sections are triangular, the boundaries are six straight "paths" that lead from the central disk to the six extreme disks. In the curved hexagonal packing, the triangular sections are "curved" and so are the paths.

To define the entire structure of the curved hexagonal packing, it suffices to define positions of disks on one path. Let the central disk be labeled 0 and the following disks on the path be successively labeled 1, 2, ... k. Consider k straight segments connecting the centers of the adjacent labeled disks: 0 to 1, 1 to 2, ... ,$(k-1)$ to k. Given a direction of rotation (it is clockwise in Fig. 1.1), each following segment is rotated at the same angle, let us call it α, in this direction with respect to the previous segment. Consider α as a parameter. When $\alpha = 0$ we have the original hexagonal packing at the left. If we gradually increase the α, "humps" grow on the six sides of the hexagon. When α reaches the value $\alpha_k = \frac{\pi}{3k}$, all the disks of the last, kth, layer are at the same distance from the central disk.

Following the path of labeled disks as defined above, the distance P in disk diameters (d) from the center of disk 0 to the center of disk k is given by

(1) $$P = |1 + e^{i\alpha_k} + e^{2i\alpha_k} + \ \dots \ + e^{(k-1)i\alpha_k}|$$

Since $\alpha_k = \frac{\pi}{3k}$, this simplifies to

$$P = \frac{1}{2\sin\frac{\pi}{6k}}$$

It follows that the ratio D/d of the enclosing circle diameter to the disk diameter for the curved hexagonal packing is

(2) $$\frac{D}{d} = 1 + \frac{1}{\sin\frac{\pi}{6k}}$$

The packing density, i.e., the fraction of the enclosing circle area which is covered by disks, can then be found as

(3) $$density = h(k)(d/D)^2$$

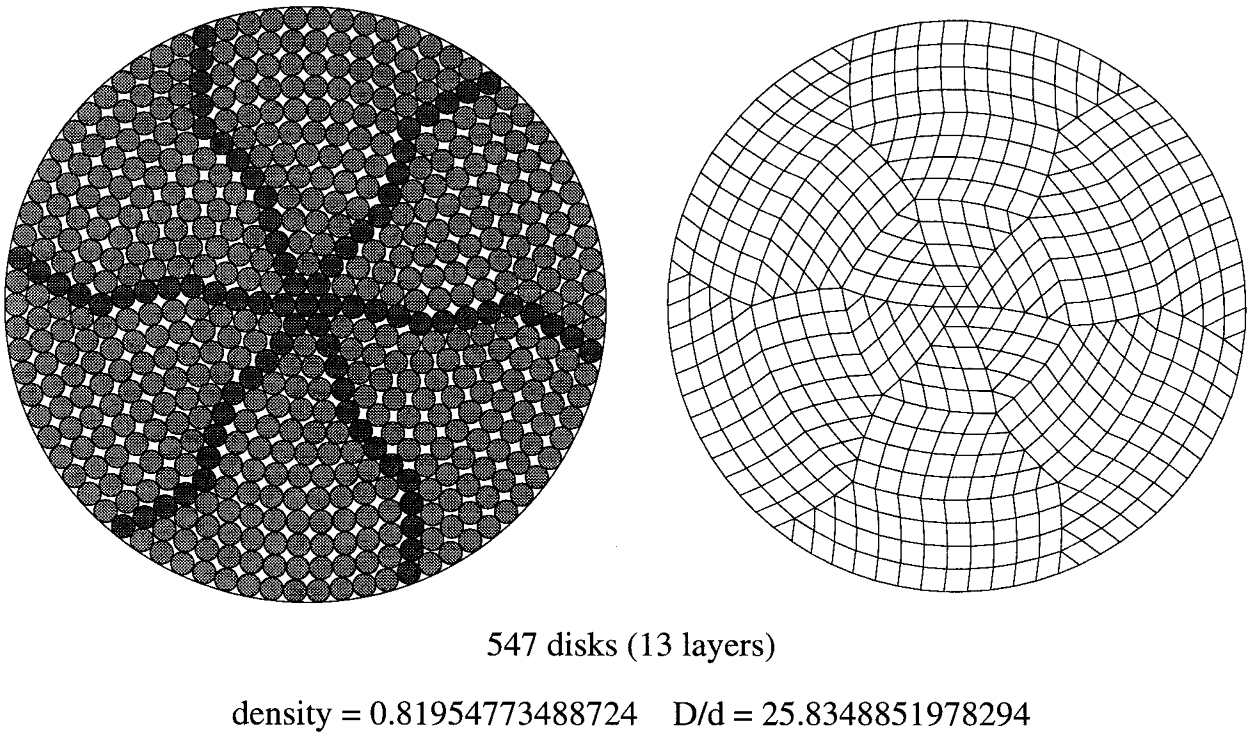
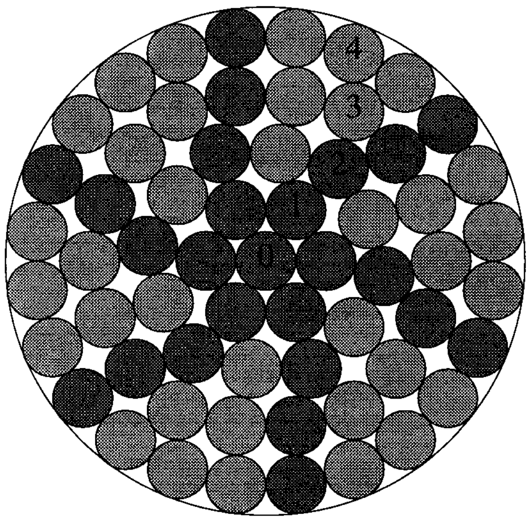


Figure 3.1: *Left.* A curved hexagonal packing of $h(13) = 547$ disks in a circle. In this packing the sense of rotation is flipped for layers 6,7,8, and 9. *Right.* The bond diagram for the packing at the left. The diagram contains a straight segment between centers of any pair of touching disks.

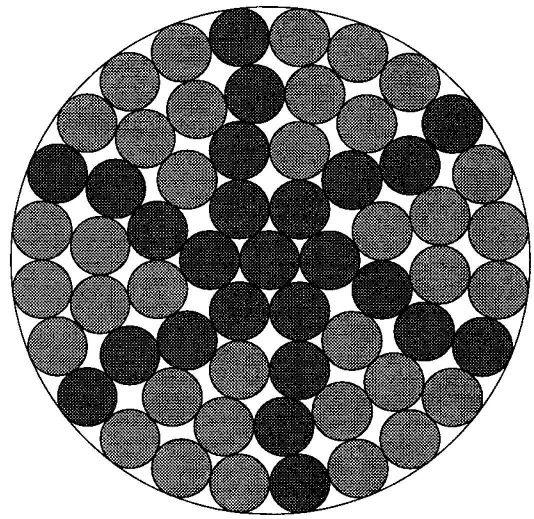
The density tends to the limit $\frac{\pi^2}{12} = 0.822467033...$ as $k \to \infty$.

The limit density of the curved hexagonal packing is exactly the square of the density of the hexagonal packing of the infinite plane. The latter density is $\frac{\pi}{2\sqrt{3}} = 0.906899682....$ The fact, that the latter density is larger than the former hints that the curved hexagonal pattern may be non-optimal for large k. Indeed, in following Section 5 we present better packings for $k = 6$, 7, and 8.

A curved hexagonal packing of $h(k)$ disks can be constructed for any $k \geq 1$. Fig. 3.1 depicts an instance for $k = 13$. We believe there are total of $m(k) = (k-1)!$ different equivalent curved hexagonal patterns of $h(k)$ disks. A curved hexagonal packing pattern is invariant under a 60° rotation. When $k \geq 3$, the curved hexagonal packings are not mirror symmetric but they occur in $m(k)/2$ image-reflection pairs. A method to generate different curved hexagonal packings of a k-layered pattern, that is, for $h(k)$ disks, is as follows. Take the basic pattern as described above and choose a subset among layers 2, 3, ..., $k-1$. Flip the sense of rotation of the chosen layers (or of the corresponding segments on the labeled path). The flip is equivalent to making the mirror reflection of these layers. Note that layers 1 and k are not subject to the flip, because the disks are positioned on them invariantly for all curved hexagonal packings for the given k. The resulting set of 2^{k-2} packings consists of 2^{k-3} image-reflection pairs. We call these modifications "regular" curved hexagonal packings and Fig. 3.1 displays one of them.



61 disks; permutations 1, 3, 2 and 3, 1, 2



61 disks; permulations 2, 3, 1 and 2, 1, 3

Figure 3.2: Two out of the three existing different equivalent image-reflection pairs of dense packings of 61 disks in a circle; the third pair which corresponds to permutations 1, 2, 3 and 3, 2, 1 is given in Fig. 2.1. The role of permutations is explained in the text. The packing at the left is regular which is demonstrated by the path of disks labeled 0, 1, 2, 3, 4, each of which has a triangular hole attached. The packing at the right is irregular.

The regular packings do not exhaust all the variants since $2^{k-2} < (k-1)!$ for $k \geq 4$. We believe that the packings in the full set can be identified by different permutations in the order of summands $e^{i\alpha_k}, e^{2i\alpha_k}, \ldots, e^{(k-1)i\alpha_k}$ in the expression (1) for P, or simply by permutations in the sequence

$$1, 2, \ldots, k-1. \tag{4}$$

For each permuted sequence, a path of $k+1$ disks can be constructed and this path, when completed with layers, happens to form a curved hexagonal packing which is equivalent to the basic one. Permutation $i_1, \ldots, i_{k-1}$ of sequence (4) produces the mirror reflection to the pattern produced by permutation $k - i_1, \ldots, k - i_{k-1}$.

The basic packing of 61 disks depicted in Fig.2.1 corresponds to the original sequence 1, 2, 3 or its reflection 3, 2, 1. The packings of 61 disks that correspond to permuted sequences 1, 3, 2 and 2, 3, 1 or their reflections 3, 1, 2 and 2, 1, 3, respectively, are shown in Fig. 3.2. A characteristic feature of a regular packing is the existence of a path any disk on which has a triangular hole attached. On the *bond diagram* which has a straight segment between centers of any pair of touching disks (see an example in Fig. 3.1), this path is seen of a (broken) line of k segments that leads from the center to the periphery each segment on which is a side of a triangle. We identified such a path in the packing at the left in Fig. 3.2 by labeling its disk components. Hence this packing is regular. The one at the right is irregular, because no required path can be found for it. The basic packing of 61 disks shown in Fig. 2.1 is also regular. This amounts to six different packings of $h(4) = 61$ disks, four packings of which are regular. Indeed, $m(4) = 6$ and $4 = 2^{4-2}$. Our belief is that for $n = 61$ there is no better packing than the six presented. Also we believe that there is no seventh packing *equal* in quality to the six presented but distinct from any of them.

Similarly, there are $m(5) = 24$ equivalent curved hexagonal packings of $h(5) = 91$ disks, $8 = 2^{5-2}$ packings of which are regular. Two regular packings are represented in Fig. 1.1; they correspond to the original sequence 1, 2, 3, 4 or its reflection 4, 3, 2, 1. The six other regular packings (not shown) correspond to permuted sequences (1, 2, 4, 3), (1, 4, 2, 3), and (1, 4, 3, 2) or their reflection, respectively, (4, 3, 1, 2), (4, 1, 3, 2), and (4, 1, 2, 3). 16 irregular packings are represented in Figs. 3.3 and 3.4 one in each image-reflection pair. Each packing is accompanied by its two generating sequences and its bond diagram. The bond diagrams clearly distinguish the packings and also show that the packings are irregular. Again, we believe there is no packing better than those 24 and there is no 25th packing equal in quality to those 24 but distinct from any one of them.

For $k > 5$ there are packings of $h(k)$ disks that are better than the curved hexagonal, but we believe there is no $(m(k) + 1)$st packing equal in quality to a curved hexagonal and distinct from any one of $m(k)$ of them.

Because each internal ith layer of disks has triangles (as seen on bond diagrams) that connect it to the corresponding outer layer $i + 1$, and because the outermost kth layer is obviously rigid, a curved hexagonal packing is rigid. The rigidity means that the only continuous motion of a subset of disks which is possible without violating the no-overlap condition is rotation of the entire assembly as a whole.

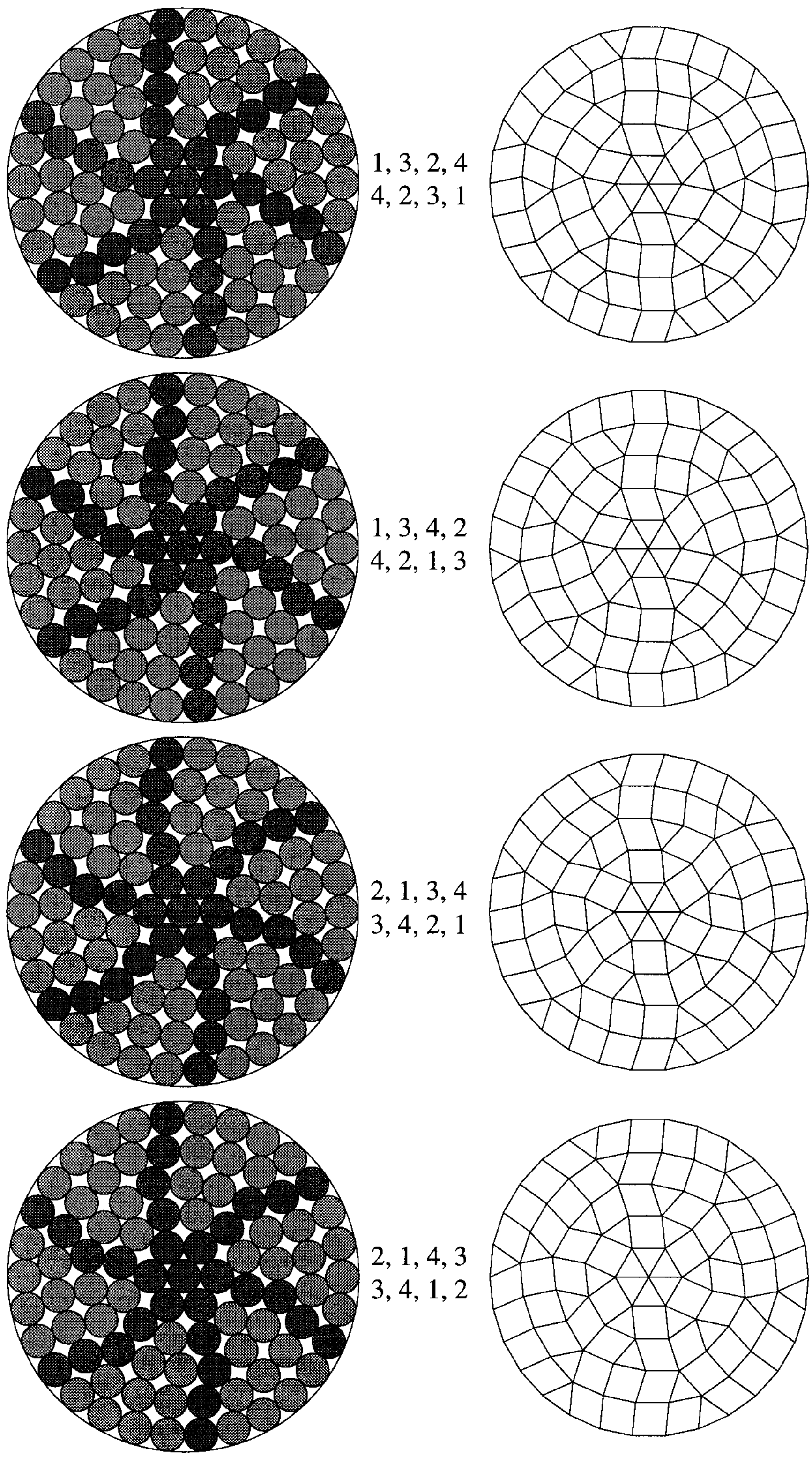


Figure 3.3: First four irregular densest (that we found) image-reflection pairs of packings of 91 disks. Each pattern shown at the left represents a pair. It is accompanied by its two generating sequences in the middle and the bond diagram at the right.

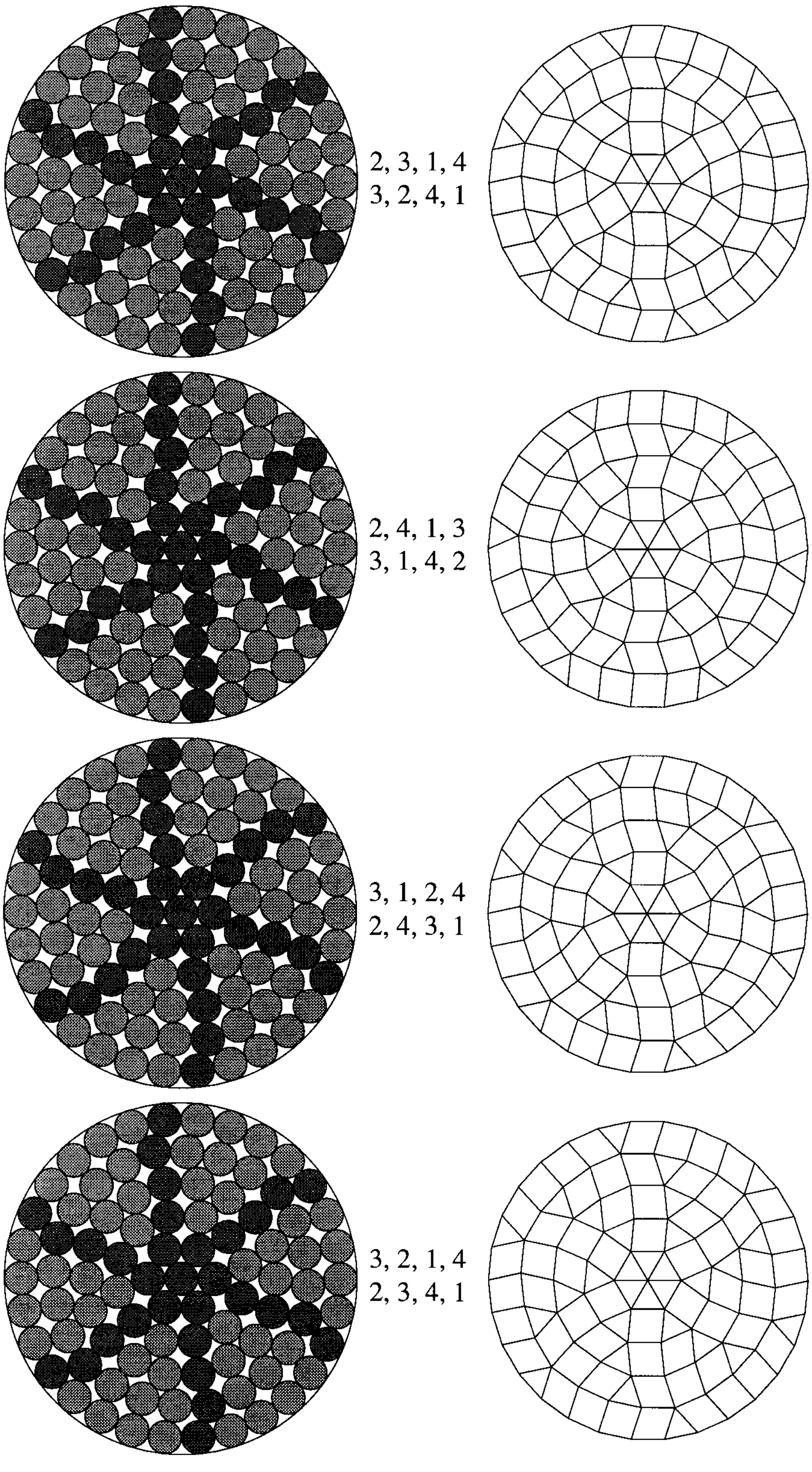


Figure 3.4: Second four irregular densest (that we found) image-reflection pairs of packings of 91 disks. Each packing shown at the left is accompanied by its two generating sequences and the bond diagram as described in the text.

4. How the "billiards" algorithm produces packings

A detailed description of the philosophy, implementation and applications of this event-driven algorithm can be found in [L], [LS]. Essentially, the algorithm simulates a system of n perfectly elastic disks. In the absence of gravitation and friction, the disks move along straight lines, colliding with each other and the region walls according to the standard laws of mechanics, all the time maintaining a condition of no overlap. To form a packing, the disks are uniformly allowed to gradually increase in size, until no significant growth can occur. Fig. 4.1 displays four successive snapshots in an experiment with 61 disks. We took the snapshots beginning the time when a local order begins to form and till the time when disk positions are fixed.

The latest snapshot shown in Fig. 4.1, the one at 441704 collisions, looks dense and, within the drawing resolution, it is identical (up to a mirror-reflection) to the final snapshot (see Fig. 2.1). However, numerically there are gaps of the order of 10^{-5} to 10^{-3} of the disk diameter in disk-disk and disk-wall pairs that appear to be touching each other in Fig. 4.1. Accordingly, only the first 3 decimal digits of D/d corresponding to the latest shown in Fig. 4.1 snapshot are identical with the correct D/d. To close these gaps and to achieve full convergence of D/d and of the density, it usually takes 10 - 20 million further collisions. We consider D/d and the density to have converged when their values do not change with full double precision for several million collisions.

Of course, as is typical in numerical iterative convergent procedures, if the computations were performed with the infinite precision, the convergence would be never achieved and the ever diminishing gaps would always be there. The "experimental" converged values agree with the "theoretical" ones computed by formulas (2) and (3) to 14 or more significant digits. Moreover, when we initialize the disk positions differently, then the final parameters achieved are either quite distinct and significantly smaller than those achieved in the run presented in Fig. 4.1 – and then the corresponding pattern is different from a curved hexagonal packing – or they are identical to 14 or more significant digits – and then the corresponding final pattern is one of the six known curved hexagonal ones. This makes us suspect that we have found the best possible packing and that its parameters D/d and density are correct to 14 significant digits.

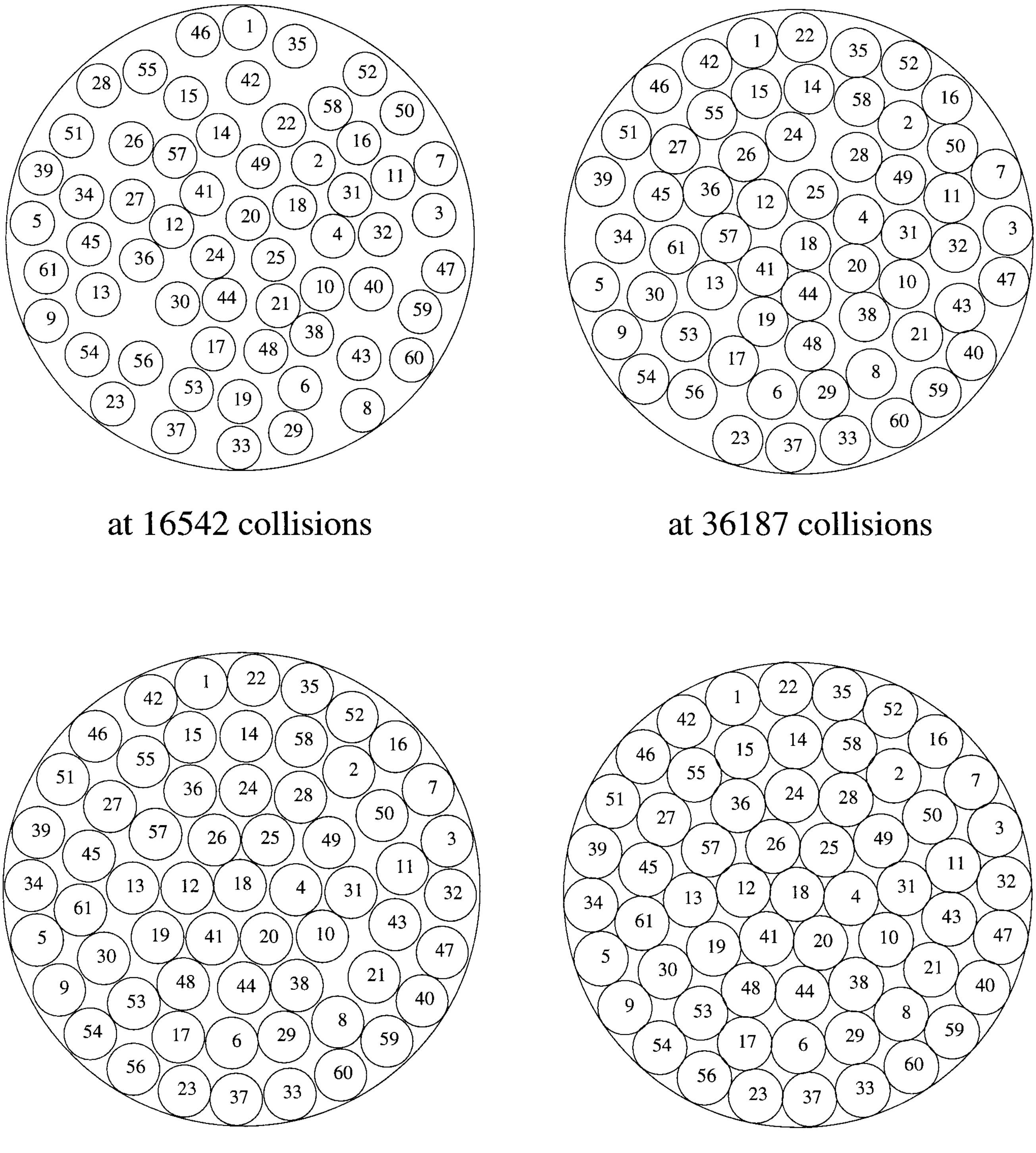


Figure 4.1: Successive snapshots of simulating expansion of 61 disks inside a circle. The progress is monitored by counting collisions. Disks are labeled 1 to 61 arbitrarily but the same disk carries the same label in all four snapshots. The last pattern is a mirror-reflection of the packing of 61 disks in Fig. 2.1

This algorithm does not equally favor the existing curved hexagonal packings. For the chosen algorithm parameter settings, including the slow disk expansion (the ratio of disk expansion speed to the average linear motion speed is 0.001), the overwhelming majority of produced curved hexagonal packings were of the regular patterns. For example, out of our 960 runs with $n = h(5) = 91$ disks, curved hexagonal packings were obtained 89 times (9%). Among those, the eight existing regular patterns (33%) were seen in 80 runs (90%), with about the same frequency each. Only 6 out of the existing 16 irregular patterns were seen in the remaining 9 runs and those 6 were also favored non equally: pairs 1324-4231 and 1342-4213 were seen in 4 runs each, and pair 3124-2431 was seen once. We gave up waiting for the other 10 irregular patterns, shown in Figs. 3.3 and 3.4, to be generated spontaneously, i.e., from random initial configurations. Instead, we constructed those from their path sequences using the method discussed in Section 3.

5. Packings of 127, 169, and 217 disks

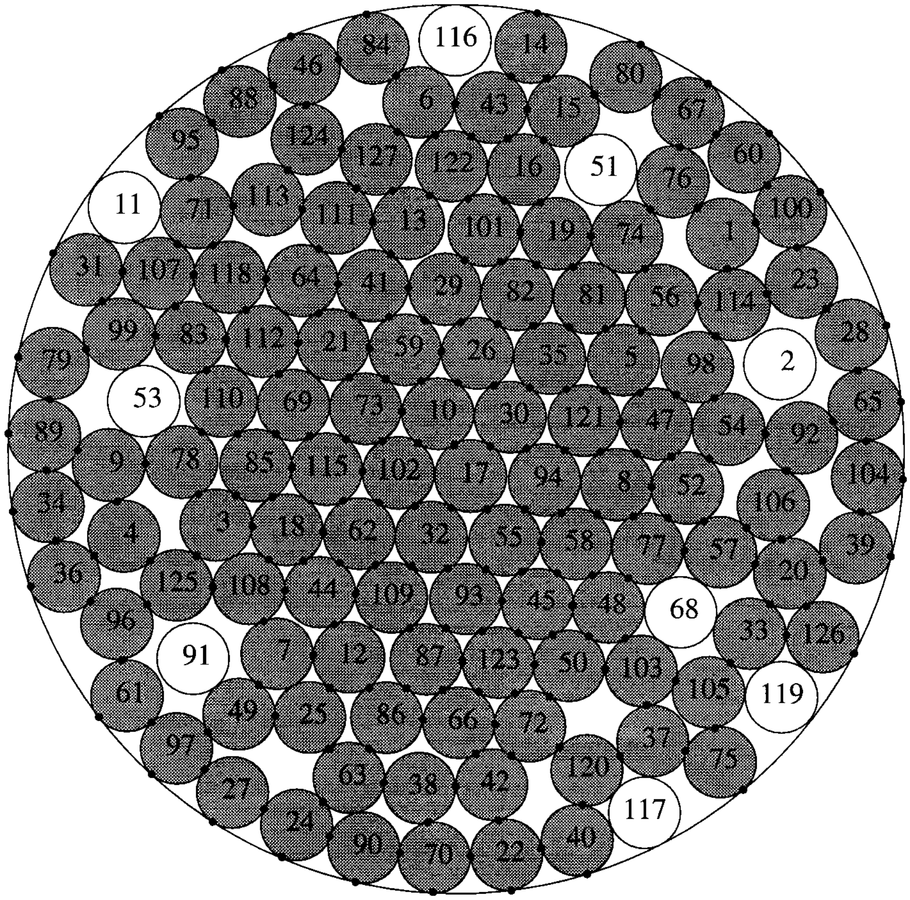
We ran the "billiards" algorithm for $n = h(6) = 127$, $n = h(7) = 169$, and $n = h(8) = 217$ disks. For these n the algorithm produced better packings than the curved hexagonal ones. The patterns of those packings can all be described as a (possibly disturbed) hexagonal disk assembly in the middle surrounded with irregularly placed disks at the circular border. For larger n this common pattern becomes more evident. As an example, we show in Fig. 5.1 the best packing we obtained for 127 disks.

We believe the packings achieved for $n = 127$, 169, and 217 are stable. But we do not think they are the best, because of a large number of local minima for these n. For example, the packing shown in Fig. 5.1 was the best among 111 independent tries. Each try resulted in a packing which is distinct from the others and had distinct parameters D/d and density. 14 out of 111 packings were better than the curved hexagonal.

These runs for $h(k)$ disks, $k > 5$, were in contrast to the runs for $n = h(5) = 91$ and smaller n. For example, for 91 disks the best (curved hexagonal) packings were obtained 89 times out of 960 with parameters D/d and density agreeing to 15 significant places. The results of packings for $h(k)$, $k > 5$, disks are summarized in Table 5.1.

Table 5.1: Curved hexagonal packings vs. the experimental ones for k =6, 7, and 8

k	6	7	8
n	127	169	217
curv. hex. density	0.81622935362082	0.81710701192903	0.81776562948873
curv. hex. D/d	12.4737132456699	14.3814899996548	16.2897882986785
experimental density	0.81755666415904	0.82672262170717	0.83499393075147
experimental D/d	12.4635835402127	14.2976098376868	16.1208600418865
better packings	14 out of 111	all 70	all 62



127 disks (244 bonds; 9 rattlers)
density = 0.81755666415904
D/d = 12.4635835402127

Figure 5.1: A packing of $h(6) = 127$ disks in a circle that is better than a corresponding curved hexagonal one. Little black dots are "bonds"; a bond indicates that the corresponding distance is less than 10^{-13} of the disk diameter. Where a pair disk-disk or disk-wall are apparently in contact but no bond is shown, the computed distance is at least 10^{-5} of the disk diameter. The shaded disks can not move given their bond positions, the non-shaded are "rattlers" that are free to move within their confines.

6. Tightness of curved hexagonal packings

Fig. 7.1 and Fig. 7.2 depict the best found packings of $n = h(k) - 1$ and $h(k) + 1$ disks for $k = 2$, 3, 4, and 5, that is, for $n = 18$, 20, 36, 38, 60, 62, 90, and 92. The known packings of $n = 6$ and 8 disks have to the same tendencies, namely:

(a) the pattern of dense packing of $n = h(k) - 1$ disks is obtained by removing one disk from the pattern of dense packing of $h(k)$ disks (which is a curved hexagonal packing for the considered $k \leq 5$) and its parameter D/d is either not changed (for $k = 1$ and 2) or is decreased only slightly (for $k = 3$, 4, and 5);

(b) the pattern of dense packing of $n = h(k) + 1$ disks differs significantly from the pattern of dense packing of $h(k)$ disks and its parameter D/d is increased substantially for all $k = 1$, 2, 3, 4, and 5).

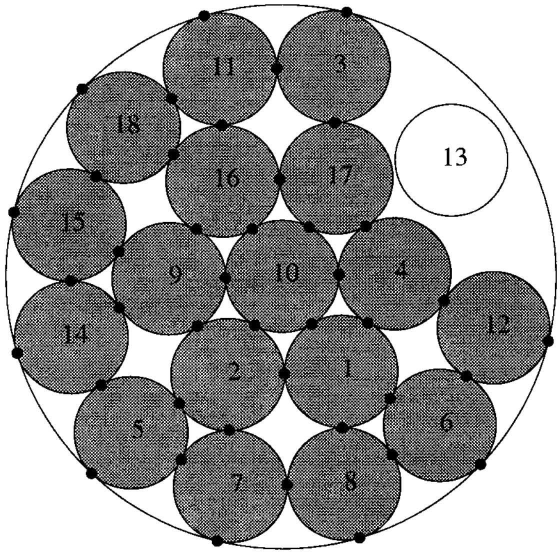
The changes in D/d are "slight" and "substantial" only in comparison to each other. The ratio of the decrease of D/d for $n = h(k) - 1$ over the increase of D/d for $n = h(k) + 1$ is given in Table 6.1.

Table 6.1: The ratio of the decrease of D/d for $n = h(k) - 1$ over its increase for $n = h(k) + 1$.

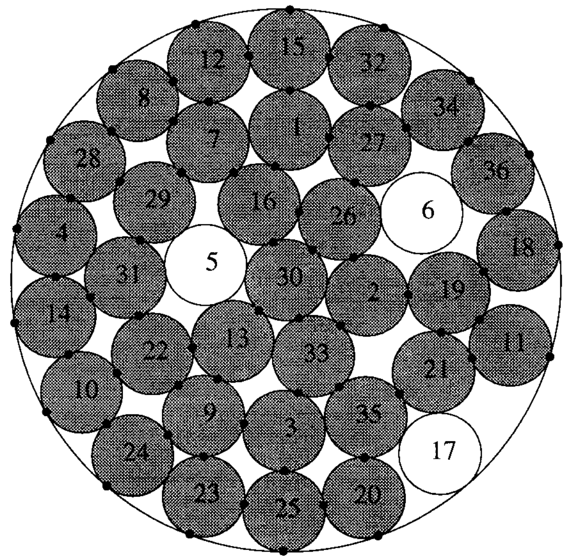
k	1	2	3	4	5
$n = h(k)$	7	19	37	61	91
ratio	0	0	0.0592	0.0895	0.1702

7. Discussion

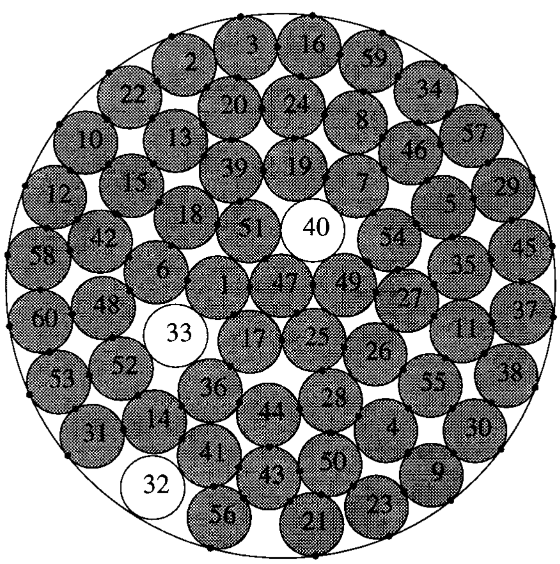
It is natural to expect for a sufficiently large n the dense packing of n equal disks in a circle to be of an "uninteresting" pattern like that in Fig. 5.1, with a large, perhaps disturbed, core of hexagonally packed disks and irregularly placed disks along the periphery. The fraction of the peripheral irregularity disks and the perturbation in the hexagonally packed core would diminish with n. On the other hand, for $n \leq 25$ symmetric patterns of dense packing have been previously observed that do not obey the general description for "uninteresting" packings given above for large n. Our computer experiments show that at least for a particular class of $n = h(k) = 3k(k + 1) + 1$, the transition from the structured "interesting" pattern to the "uninteresting" core-hexagonal one occurs between $n = 91$ and $n = 127$.



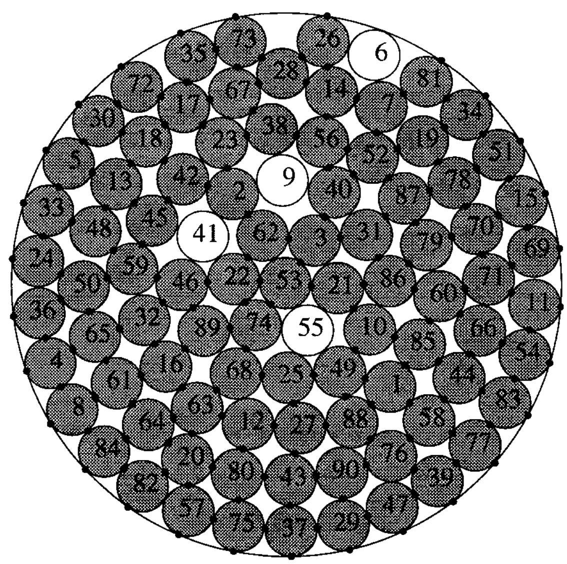
18 disks (41 bonds, 1 rattler)
density = 0.76091887384428
D/d = 4.86370330515637



36 disks (66 bonds, 3 rattlers)
density = 0.79088397691361
D/d = 6.74675379342416

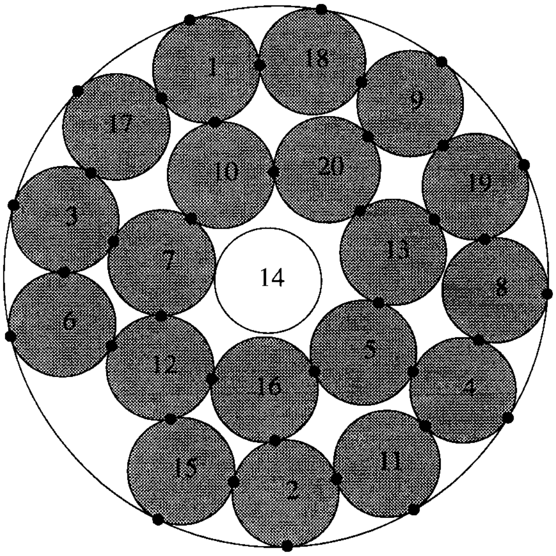


60 disks (115 bonds, 3 rattlers)
density = 0.80259916264035
D/d = 8.64621984545790

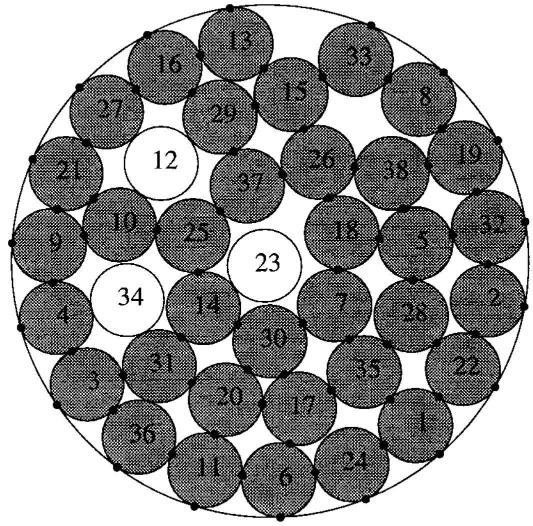


90 disks (172 bonds, 4 rattlers)
density = 0.80921006858796
D/d = 10.54606917795370

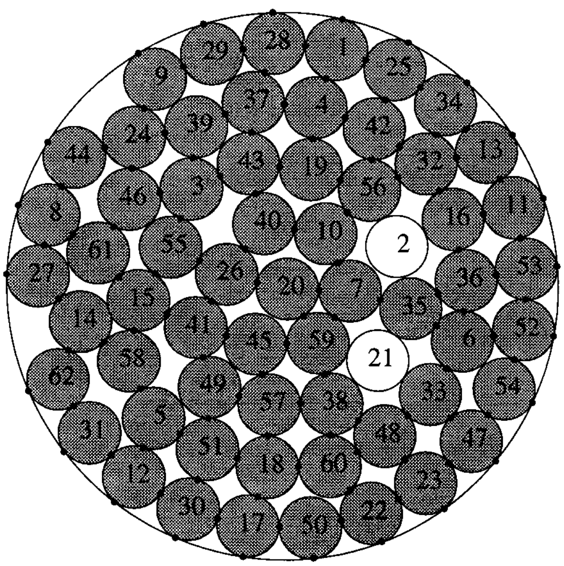
Figure 7.1: The best found packings of $n = h(k) - 1$ disks for $k = 2, 3, 4$, and 5 ($n = 18, 38, 60$, and 90).



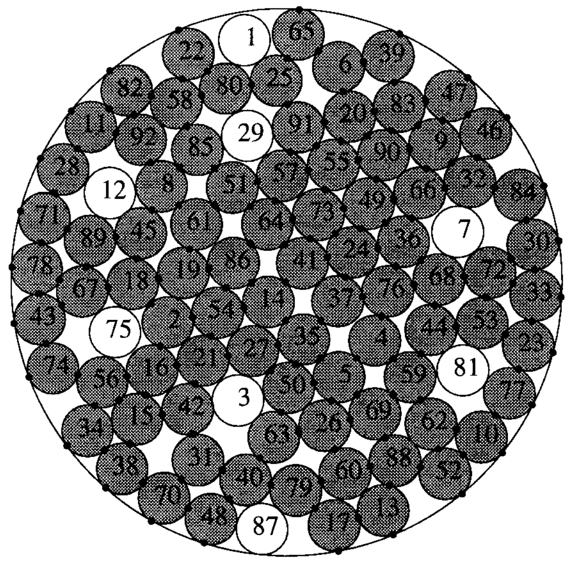
20 disks (38 bonds, 1 rattler)
density = 0.76224828956498
D/d = 5.12232073699150



38 disks (70 bonds, 3 rattlers)
density = 0.78402454881084
D/d = 6.96188696522814



62 disks (120 bonds, 2 rattlers)
density = 0.79523110498328
D/d = 8.82976540897204



92 disks (170 bonds, 8 rattlers)
density = 0.80530522287188
D/d = 10.68842356335587

Figure 7.2: The best found packings of $n = h(k) + 1$ disks for $k = 2$, 3, 4, and 5 ($n = 20$, 38, 62, and 92).

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