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Juggling Drops and Descents*†

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1 Introduction

As circus and vaudeville performers have known for a long time, juggling is fun. In the last twenty years or so this has led to a surge in the number of amateur jugglers. It has been observed that scientists, and especially mathematicians and computer scientists, are disproportionately represented in the juggling community. It is difficult to explain this connection in any straightforward way, but music has long been known to be popular among scientists; juggling, like music, combines abstract patterns and mind-body coordination in a pleasing way. In any event, the association between mathematics and juggling may not be as recent as it appears, since it is believed that the tenth century mathematician Abu Sahl started out juggling glass bottles in the Bagdad marketplace ([3], p. 79).

In the last fifteen years there has been a corresponding increase in the application of mathematical and scientific ideas to juggling ([1], [2], [7], [11], [13], [18]), including, for instance, the construction of a juggling robot ([8]). In this article we discuss some of the mathematics that arises out of a recent juggling idea, sometimes called "site swaps." It is curious that these idealized juggling patterns lead to interesting mathematical questions, but are also of considerable interest to "practical" jugglers. The basic idea seems to have been discovered independently by a number of people; we know of three groups or individuals that developed the idea around 1985: Bengt Magnusson and Bruce Tiemann ([12], [11]), Paul Klimek in Santa Cruz, and one of us (C. W.) in conjunction with other members of the Cambridge University Juggling Association. A precursor of the idea can be found in [14].

Although our interests here are almost entirely mathematical, the reader interested in actual juggling or its history might start by looking at [21] and [19]; a leisurely discussion of site swaps, aimed at jugglers, can be found in [12].

In the first section we describe the basic ideas, and in the second section we prove the basic combinatorial result that counts the number of site swaps with a given period and a given number of balls. This theorem has a non-obvious generalization to arbitrary posets ([6]). Special cases of that result can be interpreted in terms of an interesting generalization of site swaps; we find it delightful that a question arising from juggling leads to new mathematics which in turn may say something about patterns that jugglers might want to consider.

2 Juggling

As mathematicians are in the habit of doing, we start by throwing away irrelevant detail. In a juggling pattern we will ignore how many people or hands are involved, ignore which objects are being used, and ignore the specific paths of the thrown objects. We will assume that there are a fixed number of objects (occasionally referred to as "balls" for convenience) and will pay attention only to the times at which they are thrown, and will assume that the throw times are periodic. Although much of the interest of actual juggling comes from peculiar throws (behind the back, off the head, etc.), peculiar objects (clubs, calculus texts, chain saws, etc.), and peculiar rhythms, we will find that the above idealization is sufficiently interesting.

Suppose that you are juggling b balls in a constant rhythm. Since the throws occur at discrete equally-spaced moments of time, and since in our idealized world you have been juggling forever and will continue to do so, we identify the times t

of throws with integers $t \in \mathbf{Z} := \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

Since it would be silly to hold onto a ball forever, we assume that each ball is thrown repeatedly. We also assume that only one ball is thrown at any given time. With these conventions, a juggling pattern with b balls is described, for our purposes, by b doubly-infinite disjoint sequences of integers.

The three ball cascade is perhaps the most basic juggling trick. Balls are thrown alternately from each hand and travel in a figure eight pattern. The balls are thrown at times

ball 1:	$\ldots -6, -3, 0, 3, 6, \ldots$
ball 2:	$\ldots -5, -2, 1, 4, 7, \ldots$
ball 3:	$\ldots -4, -1, 2, 5, 8, \ldots$

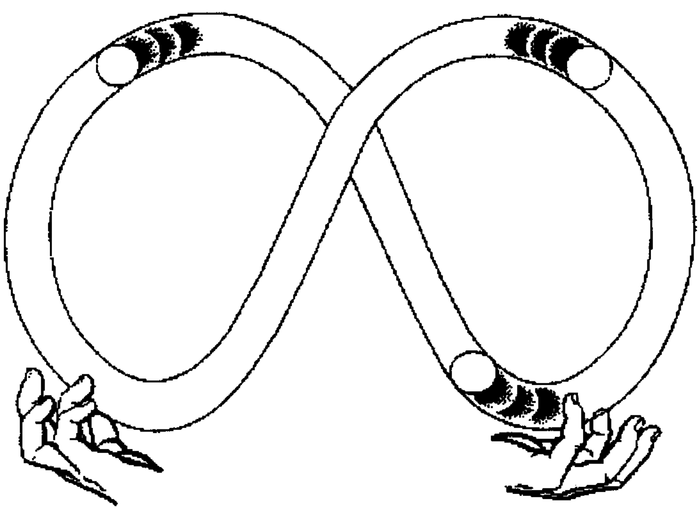

Figure 1: A cascade

This pattern has a natural generalization for any odd number of balls, but can't be done in a natural way with an even number of balls — even if simultaneous throws were allowed, in a symmetrical cascade with an even number of balls there would be a collision at the center of the figure eight.

Another basic pattern, sometimes called the fountain or waterfall, is most commonly done with an even number of balls and consists of two disjoint circles of balls.

The four ball waterfall gives rise to the four sequences $\{4n+a : n \in \mathbf{Z}\}$ of throw times, for $a = 0, 1, 2, 3$.

The last truly basic juggling pattern is called the shower (Figure 3, following page). In a shower the balls travel in a circular pattern, with one hand throwing a high throw and the other throwing a low horizontal throw. The shower can be done with any number of balls; most people find that the three ball shower is significantly harder than the three ball cascade. The three ball shower corresponds

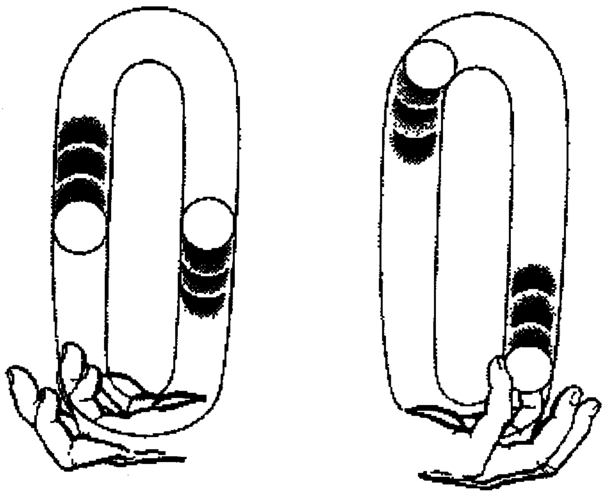

Figure 2: A fountain (waterfall)

to the sequences

ball 1:	$\ldots -6, -5, 0, 1, 6, 7 \ldots$
ball 2:	$\ldots -4, -3, 2, 3, 8, 9 \ldots$
ball 3:	$\ldots -2, -1, 4, 5, 10, 11 \ldots$

We should mention that although non-jugglers are often sure that they have seen virtuoso performers juggle 17 or 20 balls, the historical record for a sustained ball cascade seems to be nine. Enrico Rastelli, sometimes considered the greatest juggler of all time, was able to make twenty catches in a 10-ball waterfall pattern. Rings are somewhat easier to juggle in large numbers, and various people have been able to juggle 11 and 12 rings.

Now we return to our idealized form of juggling. Given lists of throw times of b balls define a function $f: \mathbf{Z} \to \mathbf{Z}$ by

$$f(x) = \begin{cases} y & \text{if the ball thrown at time } x \text{ is next thrown at time } y \\ x & \text{if there is no throw at time } x. \end{cases}$$

This function is a permutation of the integers. Moreover, it satisfies $f(t) \geq t$ for all $t \in \mathbf{Z}$. This permutation partitions the integers into orbits which (ignoring the orbits of size one) are just the lists of throw times.

The function $f(t) = t + 3$ corresponds to the 3-ball cascade, which could be graphically represented as in Figure 4.

Similarly, the function $f(x) = x + 4$ represents the ordinary 4-ball waterfall. The three ball shower corresponds to a function that has a slightly more complicated description. The juggler is usually most interested in the duration $f(t) - t$ between throws which corresponds, roughly, to the height to which balls must be thrown.

Definition: A *juggling pattern* is a permutation $f: \mathbf{Z} \to \mathbf{Z}$ such that $f(t) \geq t$ for all $\in \mathbf{Z}$. The *height* function of a juggling pattern is $df(t) := f(t) - t$.

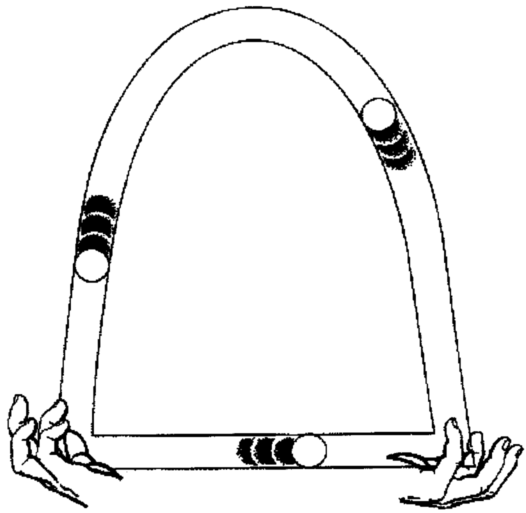

Figure 3: A shower

The three ball cascade has a height function $df(t) = 3$ that is constant. The three ball shower has a periodic height function whose values are $\ldots 5, 1, 5, 1, \ldots$. The juggling pattern in Figure 5 corresponds to the function

$$f(x) = \begin{cases} x+4 & \text{if } x \equiv 0,\ 1 \bmod 3 \\ x+1 & \text{if } x \equiv 2 \bmod 3 \end{cases}$$

which is easily verified to be a permutation. The height function takes on the values $4, 4, 1$ cyclically. This trick is therefore called the "441" among those who use the standard site swap notation. It is not terribly difficult to learn but is not a familiar pattern to most jugglers.

Remark

- We refer to $df(t)$ as the height function even though it more properly is a rough measure of the elapsed time of the throw. From basic physics the height is proportional to the square of the elapsed time. The elapsed time is actually less than $df(t)$ since the ball must be held before being thrown; for a more physical discussion of actual elapsed times and throw heights see [11].
- Although there is nothing in our idealized setup that requires two hands, or even "hands" at all, we note that in the usual two-handed juggling patterns,

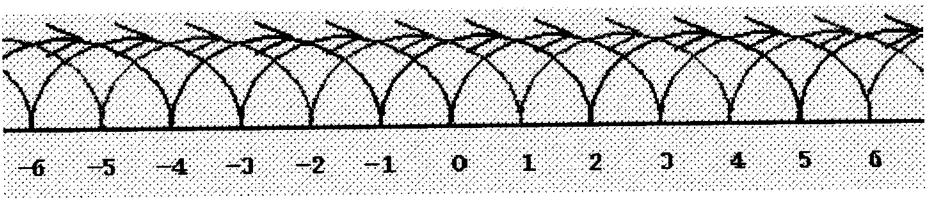

Figure 4: $t \to t+3$

that a throw with odd throw height $df(t)$ goes from one hand to the other, and a throw with even throw height goes from one hand to itself.

- If $f(t) = t$, so that $df(t) = 0$, then no throw takes place at time t. In actual practice this usually corresponds to an empty hand.
- Nothing in our model really requires that the rhythm of the juggling pattern be constant. We only need a periodic pattern of throw times. We retain the constant rhythm terminology in order to be consistent with jugglers' standard model of site swaps.
- The catch times are irrelevant in our model. Thus a throw at time t of height $df(t)$ is next thrown at time $t + df(t) = f(t)$, but in practice it is caught well before that time in order to allow time to prepare for the next throw. A common time to catch such a throw is approximately at time $f(t) - 1.5$ but great variation is possible. A theorem due to Claude Shannon ([13], [7]) gives a relationship between flight times, hold times, and empty times in a symmetrical pattern.

Now let f be a juggling pattern. This permutation of $\mathbf{Z}$ partitions the integers into orbits; since $f(t) \geq t$, the orbits are either infinite or else singletons.

Definition: The number of balls of a juggling pattern f, denoted $B(f)$, is the number of infinite orbits determined by the permutation f.

Our first result says that if the throw height is bounded, which is surely true for even the most energetic of jugglers, then the number of balls is finite and can be calculated as the average value of the throw heights over large intervals.

Theorem If f is a bijection and $df(t) = f(t) - t$ is a non-negative and bounded then the limit

$$\lim_{|I|\to\infty} \frac{\sum_{x\in I} df(x)}{|I|}$$

exists and is equal to $B(f)$, where the limit is over all integer intervals

$$I = \{a, a+1, \ldots, b\} \subset \mathbf{Z}.$$

Proof Suppose that $df(t) \leq B$ for all t. If I is an interval such that $|I| > B$ then any infinite orbit intersects I. The sum of $df(t)$ over the

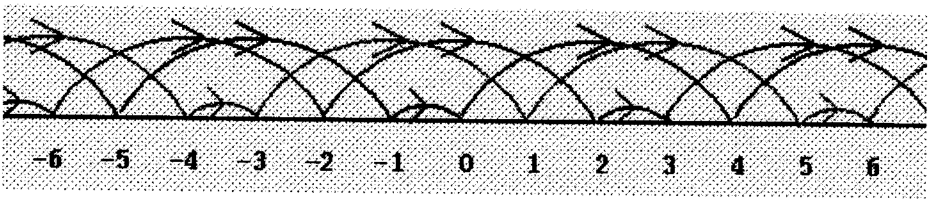

Figure 5: 441

points in I lying in a given infinite orbit is bounded above by I and below by $|I| - 2B$.

If I is large enough then the sum of $df(t)$ for $t \in I$ can be made arbitrarily close to the number of infinite orbits of f; the singleton orbits don't contribute since $df(t) = 0$ for those orbits. Thus in the limit the average of df over an interval $\{a, a+1, \ldots, b\}$ of consecutive integers must become arbitrarily close to the number of infinite orbits of the permutation. ■

Remark

- The limit is clearly a uniform limit in the sense that for all positive ϵ there is an m such that if I is an interval of integers with more than m elements then the average of df over I is within ϵ of $B(f)$.
- As an example illustrating the theorem we note if f is the 441 pattern described earlier, then the height function $df(t)$ is periodic of period 3. The long term average of $df(t)$ over any interval approaches the average over the period, i.e., $(4+4+1)/3 = 3$, which confirms what we already knew: the 441 pattern is a 3-ball trick.
- The hypothesis of bounded throw heights is necessary. Indeed, if $T(0) = 0$ and, for nonzero t, $T(t)$ is the highest power of 2 that divides t then the pattern $f(t) = t + 2 \cdot T(t)$ has unbounded throw height and infinite $B(f)$, as in Figure 6. More vividly: you can juggle infinitely many balls if you can throw arbitrarily high.

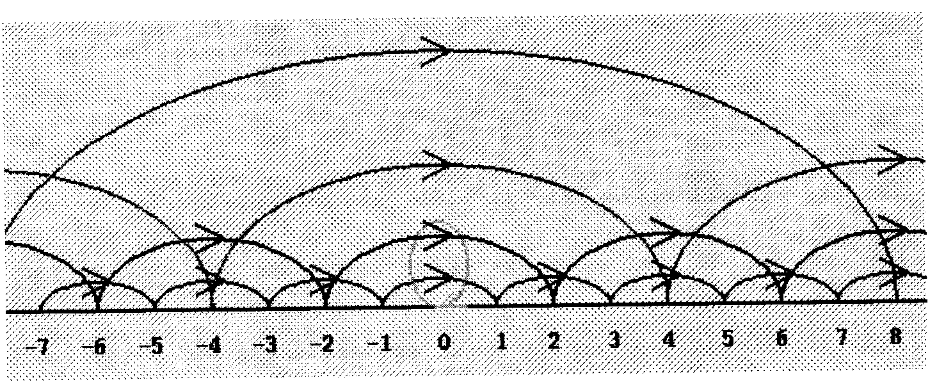

Figure 6: Infinitely many balls

3 Periodic Juggling

From now on we want to juggle periodically. A juggling pattern is perceived to be periodic by an audience when its height function is periodic in the mathematical sense.

Definition: A *period-n juggling pattern* is a bijection $f : \mathbf{Z} \to \mathbf{Z}$ such that $df(t+n) = df(t)$ for all $t \in \mathbf{Z}$.

If df is of period n then it might also have a period m for some divisor m of n. If n is the smallest period of df then any other period is a multiple of n; in this case we will say that f is a pattern of **exact** period n.

A period-n juggling pattern can be described by giving the finite sequence of non-negative integers $df(t)$ for $t = 0, 1, \ldots, n-1$. Thus the pattern 51414 denotes a period-5 pattern; by Theorem 1 it is a 3-ball pattern since the "period average" of the height function $df(t)$ is 3.

Which finite sequences correspond to juggling patterns? Certainly a necessary condition is that the average must be an integer. However this isn't sufficient. The sequence 354 has average 3 but does not correspond to a juggling pattern—if you try to draw an arrow diagram for a map f as above you'll find that no such map exists. This is also easy to see directly, for if $df(1) = 5$ and $df(2) = 4$ then

$$f(1) = 1 + df(1) = 6 = 2 + df(2) = f(2)$$

and such a map isn't a bijection.

Lemma If f is a period-n juggling pattern then

$$s \equiv t \bmod n \implies f(s) \equiv f(t) \bmod n.$$

Proof If $df(t)$ is periodic of period n then the function $f(t) = t + df(t)$ is of period n modulo n. ■

The Lemma implies that a juggling pattern f induces a well-defined injective, and hence bijective, mapping on the integers modulo n. Let $[n]$ denote the set $\{0, 1, \ldots, n-1\}$ and let S_n denote the symmetric group consisting of all permutations (bijections) of the set $[n]$. Then for every period n juggling pattern f there is a well-defined permutation $\pi_f \in S_n$ that is defined by the condition

$$f(t) \equiv \pi_f(t) \bmod n, \qquad 1 \leq t \leq n.$$

Theorem A sequence $a_0 a_1 \cdots a_{n-1}$ of non-negative integers satisfies $df(t) = a_t$ for some period-n juggling pattern f if and only if $a_t + t \bmod n$ is a permutation of $[n]$.

Proof Suppose that f is a juggling pattern and $a_t = df(t)$. Then $f(t) \equiv \pi_f(t) \bmod n$ so there is an integer-valued function $g(t)$ such $f(t) = \pi_f(t) + n \cdot g(t)$ and

$$df(t) = f(t) - t = \pi_f(t) - t + n \cdot g(t)$$

and

$$a_t + t \equiv df(t) + t \equiv \pi_f(t) \bmod n$$

and the stated condition is satisfied.

Conversely, suppose that

$$a_0 a_1 \cdots a_{n-1}$$

is such that $a_t + t$ is a permutation of $[n]$. If we define a_t for all integers t by extending the sequence periodically and then define $f(t) = a_t + t$ then f is the desired juggling pattern. To see that f is injective note that if $f(t) = f(u)$ then $t \equiv u \bmod n$ since $f(t)$ is injective modulo n. Then $a_t = a_u$. From $f(t) = a_t + t = f(u) = a_u + u$ it follows that $t = u$ and f is injective as claimed. To show that f is surjective, suppose that $u \in \mathbf{Z}$. Since $t + a_t \bmod n$ is a permutation of $[n]$ we can find a a t such that $f(t) = t + a_{\{t\}} \equiv u \bmod n$. By adding a suitable multiple of n we can find a t' such that $f(t') = u$. This finishes the proof of the fact that any sequence satisfying the stated condition comes from a juggling pattern. ■

To see if 345 corresponds to a juggling pattern we add t to the t-th term and reduce modulo 3. The result is 021, which is a permutation, so 345 is indeed a juggling pattern (in fact a somewhat difficult one that is quite amusing). On the other hand, the sequence 354 leads, by the same process, to 000 which certainly isn't a permutation of [3].

3.1 Remarks for Jugglers Only

- The above description is geared towards the standard model: two hands throwing alternately, in constant rhythm. In fact there could be any number of hands and it is not necessary to assume that the rhythm is constant.
- The practical meaning of the throw heights 0, 1, and 2 in the standard model requires a little thought. A throw height of 0 corresponds to an empty hand. A throw height of 1 corresponds to a rapid shower pass from one hand to another that is thrown again immediately. A throw height of 2 would ordinarily indicate a very low throw from a hand to itself that is thrown again by that hand immediately. This is actually rather unnatural in practice; the conventional interpretation ([11], [12]) is that a throw height of 2 is a held ball.
- The paradigm for categorizing juggling patterns here is very interesting in practice, although many of the patterns require considerable proficiency. Several jugglers who have spent time in working on site swaps describe the same gain in flexibility and conceptual power that mathematicians seem to report from the use of well-chosen abstractions. The simplest non-obvious site-swap seems to be 441; it is similar to, but **not** the same as, the common 3-ball pattern of throwing balls up on the side while passing a ball back and forth underneath in a shower pass from hand to hand. (The latter pattern is not commonly performed with an even rhythm; if it is, it is 810.) The 3-ball 45141 pattern is also amusing, and the 4-ball 5551 pattern looks very much like the 5-ball cascade. The range of feasible and interesting tricks seems to

be unlimited; we mention the following sample: 234, 504, 345, 5551, 40141, 561, 633, 55514, 7562, 7531, 566151, 561, 663, 771, 744, 753, 426, 459, 9559, 831.

- A number of programs are available that simulate site swaps on a computer screen, sometimes with quite impressive graphics. These programs take a finite sequence of non-negative integers as input and dynamically represent the pattern. The Internet news group rec.juggling is a source of information on site swaps and various juggling animation software.

In order to find out which finite sequences represent juggling patterns we start by noting that a period-n pattern induces a permutation on the first n integers.

4 Counting Periodic Juggling Patterns

Let $N(b,n)$ denote the number of period-n juggling patterns f with $B(f) = b$. Our next goal is to calculate this number. From the juggler's point of view it might be more useful to count the number of patterns of exact period n and to count cyclic shifts of a pattern as being essentially the same as the original pattern. Later we will see that this more natural question can be answered easily once we know $N(b,n)$.

The basic idea in the determination of $N(b,n)$ is to fix a permutation $\pi \in S_n$ and count the number of patterns f such that $\pi_f = \pi$. From the proof of the previous theorem we have the formula

$$f(t) = \pi_f(t) + n \cdot g(t) = \pi(t) + n \cdot g(t), \qquad 0 \le t < n.$$

Thus we must count the number of functions $g\colon [n] \to \mathbf{Z}$ such that if f is defined by the above formula then $df(t) \ge 0$ and $B(f) = b$.

The number of balls of such a pattern f is equal to the average of $df(t)$ over $[n]$. Thus

$$B(f) = \frac{1}{n}\sum_{t=0}^{n-1} df(t) = \frac{1}{n}\sum_{t=0}^{n-1} \left(\pi(t) - t + n \cdot g(t)\right).$$

Since $\pi(t)$ is a permutation of $[n]$ we see that this reduces to

$$B(f) = \sum_{t=0}^{n-1} g(t).$$

Thus a function g determines a pattern with $B(f) = b$ if the sum of its values is equal to b.

The condition that $df(t) \ge 0$ is a little bit more intricate. Since

$$df(t) = \pi(t) - t + n \cdot g(t)$$

we see that $g(t)$ must be non-negative and also must be strictly positive whenever $\pi(t) < t$.

Definition: An integer $t \in [n]$ is a drop for the permutation $\pi \in S_n$ if $\pi(t) < t$; moreover, we define

$$d_\pi(t) = \begin{cases} 1 & \text{if } t \text{ is a drop for } \pi \\ 0 & \text{if } t \text{ is not a drop for } \pi. \end{cases}$$

Write $G(t) = g(t) - d_\pi(t)$ so that

$$f(t) = \pi(t) + n \cdot d_\pi(t) + n \cdot G(t).$$

Let k be the number of drops of π. Then $B(f) = b$ if and only if the sum of the values of G is equal to $b - k$.

We can summarize this discussion so far as follows. The number $N(b, n)$ of period-n juggling patterns with b balls is equal to the sum over all permutations $\pi \in S_n$ of the number of non-negative functions $G(t)$ on $[n]$ whose value-sum is $b - k$, where k is the number of drops of π.

A standard combinatorial idea can be used to count the number of sequences of non-negative integers with a given sum.

Lemma The number of non-negative n-tuples with sum x is

$$\binom{x+n-1}{n-1}.$$

Proof A standard "stars and bars" argument (in Feller's terminology, e.g., p. 38 of [9]) gives the answer. The number of such sequences is equal to the number of ways of arranging $n-1$ bars and x stars in a row if we interpret the size of each contiguous sequence of stars as a component of the n-tuple and the bars as separating components. The number of such sequences of bars and stars is the same as the number of ways to chose $n-1$ locations for the bars out of a total of $x+n-1$ locations, which is just the stated binomial coefficient. ■

Let $\delta_n(k)$ be the number of permutations in S_n that have k drops. By combining the earlier remark with the lemma we arrive at

$$N(b,n) = \sum_{k=0}^{n-1} \delta_n(k) \binom{n+b-k-1}{n-1}.$$

Later it will be convenient to consider the number of period-n juggling patterns with fewer than b balls. If this number is denoted $N_<(b,n)$ then, using a familiar binomial coefficient identity, we find that

$$\begin{aligned} N_<(b,n) &= \sum_{a=0}^{b-1} N(a,n) = \sum_{a=0}^{b-1}\sum_{k=0}^{n-1} \delta_n(k) \binom{n+a-k-1}{n-1} \\ &= \sum_{k=0}^{n-1} \delta_n(k) \sum_{a=0}^{b-1} \binom{n+a-k-1}{n-1} = \sum_{k=0}^{n-1} \delta_n(k) \binom{n+b-k-1}{n} \end{aligned}$$

In order to simplify this further we recall the idea of a descent of a permutation and show that even though drops and descents aren't the same thing, the number of permutations with k drops is the same as the number with k descents.

Definition: If $\pi \in S_n$ then $i \in [n]$ is a **descent** of π if $\pi(i) > \pi(i+1)$ where $0 \le i < n-1$. The number of elements of S_n with k descents is denoted

$$\left\langle {n \atop k} \right\rangle$$

and is called an **Eulerian number**.

We will write permutations as a list of n integers in which the i-th element is $\pi(i)$, e.g.,

$$\pi(0)\pi(1)\ldots\pi(n-1).$$

A descent in π is just a point in this finite sequence in which the next term is lower than the current term.

Example. The permutation 10432 in S_5 has three descents and two drops.

If π is a permutation then it can also be written in cycle form in the usual way. In order to specify this form uniquely we write each cycle with its largest element first and arrange the cycles so that the leading elements of the cycles are in increasing order, where we include the singleton cycles.

Definition: If $\pi \in S_n$ let $\hat{\pi}$ be the permutation that results from writing π in cycle form, as above, and then erasing parentheses.

Example. The permutation $\pi \in S_8$ corresponding to the sequence 16037425 has a cycle decomposition (0162)(475) that has the canonical form (3)(6201)(765). Therefore $\hat{\pi}$ is 36201754.

Note that the map taking π to $\hat{\pi}$ is bijective since π can be uniquely reconstructed from $\hat{\pi}$ by inserting left parentheses before every left-to-right maximum and then inserting matching right parentheses. This permutation of S_n is certainly bizarre at first glance, but it plays a surprisingly crucial role in various situations (see [5] or [15]).

Lemma The number of permutations of $[n]$ with k descents is equal to the number with k drops, i.e.,

$$\delta_n(k) = \left\langle {n \atop k} \right\rangle.$$

Proof A descent of $\hat{\pi}$ must lie inside a cycle of π since our conventions guarantee that the last element in a cycle is followed by a larger integer. By the meaning of the cycle decomposition π (namely, that elements within cycles are mapped to the next element in the cycle) we see that a descent of $\hat{\pi}$ corresponds to a drop of π. Conversely, a drop in π must occur within a cycle (i.e., not in passing from the last element of a cycle to the first) and corresponds to a descent in $\hat{\pi}$. Thus the number of permutations with k descents is equal to the number $\delta_n(k)$ with k drops. ■

Example, again. The permutation $\pi = 16037425$ has drops at $t = 2, 5, 6, 7$, and the permutation $\hat{\pi} = 36201754$ has descents at $i = 1, 2, 5, 6$.

The Eulerian numbers $\delta_n(k) = \left\langle {n \atop k} \right\rangle$ play a role in a variety of combinatorial questions beyond drops and descents ([10], [15], [16]), although no notation seems to be standard yet. We recall some of their basic properties. If a permutation $\pi = \pi(0)\pi(1)\ldots\pi(n-1)$ has k descents then its reversal $\pi' = \pi(n-1)\pi(n-2)\ldots\pi(0)$ has $n-k-1$ descents. Thus

$$\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-k-1} \right\rangle. \tag{1}$$

By relating permutations of $[n]$ to permutations of $[n-1]$ in the usual way, a more involved combinatorial argument shows that

$$\left\langle {n \atop k} \right\rangle = (k+1)\left\langle {n-1 \atop k} \right\rangle + (n-k)\left\langle {n-1 \atop k-1} \right\rangle. \tag{2}$$

Using this recursion, it is easy to tabulate Eulerian numbers.

Finally, the Eulerian numbers arise as coefficients of the linear relations connecting the polynomials x^n with the polynomials $\binom{x+k}{n}$.

Worpitzky's Identity.

$$x^n = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \binom{x+k}{n}.$$

This identity can by readily proved by induction using equation (2). It apparently first appeared in [20] (see also [10] and [16]); in [15] it appears as a special case of a much more general statement.

Theorem The number of period-n juggling patterns with fewer than b balls is b^n, i.e.,

$$N_<(b,n) = b^n.$$

Proof Our previous formula for $N_<(b,n)$ was

$$N_<(b,n) = \sum_{k=0}^{n-1} \delta_n(k) \binom{n+b-k-1}{n} = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \binom{n+b-k-1}{n}.$$

Replace k by $n-k-1$ and use (2) to get

$$N_<(b,n) = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \binom{b+k}{n}.$$

The claim is then an immediate consequence of Worpitzky's identity.
■

The simplicity of the final result is surprising. The astute reader will note that we could have avoided introducing the concept of descents by proving equations (1) and (2) directly for the counting function $\delta_n(k)$ for drops. It is a pleasant exercise to provide a direct combinatorial argument. We took the slightly longer route above because it is amusing and useful in proving the much more general result in [6].

By the theorem there are $(b+1)^n - b^n$ patterns of period n with exactly b balls if cyclic shifts are counted as distinct. Let $M(n,b)$ be the number of patterns of exact period n with exactly b balls, where cyclic shifts are not counted as distinct. Thus $M(n,b)$ is probably the number that is of most interest to a juggler.

If d is a divisor of n then each pattern of exact period d will be occur d times as pattern of length n. Thus

$$(b+1)^n - b^n = \sum_{d|n} dM(d,b).$$

By Möbius inversion we obtain the following corollary to the previous theorem.

Corollary 1

$$M(n,b) = \frac{1}{n}\sum_{d|n} \mu(n/d)((b+1)^d - b^d).$$

For instance, there are 12 genuinely distinct patterns with period three with three balls. The reader may find it instructive to list all of them explicitly.

Several people have reproved Theorem 3 from other points of view. Richard Stanley sent us a proof using results in [15]. Jeremy Kahn sent us a bijective proof using a different labeling function for juggling patterns. Walter Stromquist sent us an interesting bijective proof that uses a very curious relabeling of site swap patterns. Adam Chalcraft ([4]) sent us a proof using ideas similar to those of Stromquist. It is striking that the result seems to be of considerable interest to a number of people.

Several of these proofs are shorter than ours, and some are much closer to being more transparent "bijective" proofs. However, the proof given here, in addition to using some interesting combinatorics, is the special case of the proof of the much more general result in [6]. The basic motivation of that result is to replace the set $[n]$ with an arbitrary poset. For some posets we can give a natural interpretation of that more general result in terms of juggling patterns in which more than one ball can be thrown at once, but we still haven't been able to give a juggling interpretation for arbitrary posets. After hearing of our results from Richard Stanley, E. Steingrímsson reproved ([17]) the general results about posets using results from his thesis. Among many other things, he generalizes the notions of descents and drops (actually, in his terminology, a mirror notion he calls "exceedances") to certain wreath products of symmetric groups.

NOTE ADDED IN PROOF: In their recent preprint, "Juggling and applications to q-analogues," Richard Ehrenborg and Margaret Readdy give a q-analogue of our main result. In addition they generalize the ideas to multiplex patterns (in which a hand can catch and throw more than one ball at once) and give applications to q-Stirling numbers and the Poincare series of an affine Weyl group.

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5 Appendix

The main result of the preceding article is that there are b^n juggling patterns with period n and fewer than b balls. After we described this in talks 5 years ago, several people who heard the result came forward with combinatorial proofs that gave explicit correspondences between such juggling patterns and sequences of length n of letters from a b-element alphabet. The ideas seem to have been discovered (and rediscovered) repeatedly in the last few years. To our knowledge, the first such proofs were due to Walter Stromquist and Richard Stanley, but Richard Ehrenborg and Margaret Readdy, Adam Chalcraft, Michael Kleber and Jeremy Kahn, Jeremy Rickard, and Hendrik Lenstra, Jr. have also all provided arguments of related (or more general) results.

The OMP conference, and the reprinting of the article in this volume, give us a welcome opportunity to try to synthesize some of these ideas and give an example of a more combinatorial proof. In fact, the argument given here is intentionally "visual," and we leave the task of finding a more algebraic expression of the proof to the interested reader.

We will actually consider a more general situation than in the paper. It seems inflexible and unrealistic to assume that a juggler will juggle a periodic pattern forever without variation. So we drop the assumption of periodicity and merely assume that the juggler has been juggling forever, and will continue to juggle forever. More mathematically, we will consider juggling patterns $f: \mathbf{Z} \to \mathbf{Z}$ that are permutations (bijections), satisfying $f(t) > t$, such that $f(t) - t$ is not necessarily periodic. These will be put into one-to-one correspondence with certain 2-way infinite sequences. The fact that there are b^n periodic juggling patterns will be seen to be an easy corollary. We will also later briefly outline direct combinatorial arguments for that case.

We should also remark that in the article juggling patterns f were allowed to have fixed points, i.e., times t such that $f(t) = t$. This correspond to a temporarily empty hand, so that there was no throw at time t. It is easy to convert the results here to that slightly more general class of permutations satisfying $g(t) \geq t$, since as f ranges over all permutations that satisfy $f(t) > t$, the functions $g(t) = f(t) - 1$ range over all permutations satisfying $g(t) \geq t$.

If $f: \mathbf{Z} \to \mathbf{Z}$ is a permutation then we let $O(f)$ denote the number, possibly infinite, of orbits of f. Let $\mathbf{J}$ denote the set of permutations $f: \mathbf{Z} \to \mathbf{Z}$ such that $O(f)$ is finite, and such that $f(t) > t$ for all $t \in \mathbf{Z}$. We think of $\mathbf{J}$ as all juggling patterns, not necessarily periodic, that have a throw at every instance in time (i.e., do not contain 0-throws in the usual site swap notation), extend infinitely in both directions in time, and involve finitely many juggling balls.

We wish to describe, or name, elements of $\mathbf{J}$ by certain 2-way infinite sequences $s: \mathbf{Z} \to \mathbf{Z}^+$ of positive integers. We let $\mathbf{T}$ denote the set of all bounded functions $s: \mathbf{Z} \to \mathbf{Z}^+$ which take on their maximum value infinitely often in both directions. More precisely, if $b = \sup(s) := \sup\{s(t) : t \in \mathbf{Z}\}$ is the maximum term in the *sequence then the set* $\{t : s(t) = b\}$ *has no maximum or minimum. The set* $\mathbf{T}$ *can be thought of as the set of all "throw sequences" in that the sense that s tells us which ball to throw: as we will see,* $s(t) = k$ *means that at time t the* k^{th} *most recently thrown ball is thrown again.*

Theorem *There is a natural one-to-one correspondence*

$$\mathbf{J} \Longleftrightarrow \mathbf{T}$$

such that if f corresponds to s then $O(f) = \sup(s)$.

To prove this theorem we will exhibit mutually inverse mappings between $\mathbf{J}$ *and* $\mathbf{T}$. *Suppose that* $f \in \mathbf{J}$ *is a juggling pattern with b balls (orbits). One way to visualize f, used in the article, is to put integer points* $t \in \mathbf{Z}$ *on a number line and connect t to* $f(t)$ *by an arrow. An orbit is then a doubly unending connected sequence of such (forward) arrows (e.g., see Figures 4 or 5 in the article). The first step in showing how to associate a throw sequence s to f is to construct a more canonical picture of f by arranging the positions of the arrows more carefully.*

We observe that the permutation f determines a natural ordering of the orbits between two throw times t and $t+1$, *namely, by how recently the corresponding ball has been thrown. To give some mild sop to those seeking an algebraic formulation devoid of references to juggling, we could say that an orbit containing* t' *is the* k^{th} *most recent orbit between t and* $t+1$ *if* $f(t') > t$ *and the interval* $[t', t]$ *intersects k orbits of f.*

The most recently appearing orbit is clearly the one containing t, and will be drawn as the bottom arrow or "track." The next most recent orbit will be then one containing the largest $t' < t$ *that is not in the orbit containing t (e.g.,* $t-1$ *unless* $f(t-1) = t$*). In any event, we put these orbits into b tracks lying above the real line between integers and connect the tracks at the integers as required by the permutation. The result is a canonical representation of our permutation that we will call a "track diagram"; for instance the track diagram for the 3-ball site swap* 3162 *is as follows.*

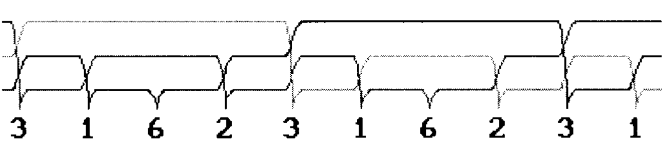

Figure 7: **The site swap 3162.**

At each time $t \in \mathbf{Z}$ *(i.e., the times at which the juggling balls are thrown) the ordering changes in a very specific way: one track becomes the most recent and the previously more recent tracks have their level bumped up by one. In isolation, the diagram of a typical transition might look like*

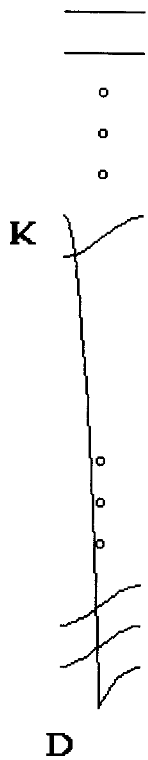

Figure 8: **The transition D_k**

where the orbit that is k^{th} most recent (between $t-1$ and t) connects, or "plunges" to the point t on the line. The idea of these transitions seems to be due to Ehrenborg and Readdy; we will call the transition in which the k^{th} level plunges the diagram D_k. With this canonical image of a juggling pattern, we can describe the desired map from $\mathbf{J}$ to $\mathbf{T}$. If $f \in \mathbf{J}$ is a juggling pattern then define the corresponding doubly infinite throw sequence s_f by setting its t^{th} element $s_f(t)$ to be equal to k if the transition at t has diagram D_k. Thus $1 \leq k \leq b := O(f)$ and the sequences contains positive integers bounded by b. Thus the s corresponding to the 3162 site swap pictured above is as follows.

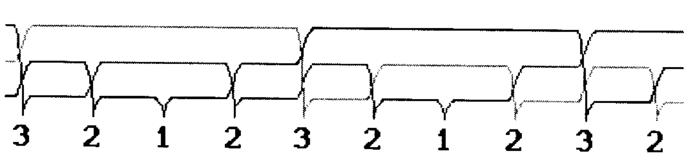

Figure 9: **Throw sequence** 31213121...

As suggested above, non-jugglers could define the throw sequence more algebraically by

$$s_f(t) = \#\{\text{orbits intersecting } [f^{-1}(t), t]\}.$$

We need to check that the sequence s_f produced from f takes on the maximum value b infinitely often in both directions. This is easy: the least recent orbit (highest track) must plunge sooner or later and it can only plunge when $s_f(t) = b$. Similarly, the highest track rose to become the highest track at some point in the past when $s_f(t) = b$. Thus the sequence s_f does contain b and must contain it infinitely often in both directions.
Now we construct a map from $\mathbf{T}$ to $\mathbf{J}$. Suppose that $s \in \mathbf{T}$ is a throw sequence. Imagine the tracks with all of the transition diagrams removed.

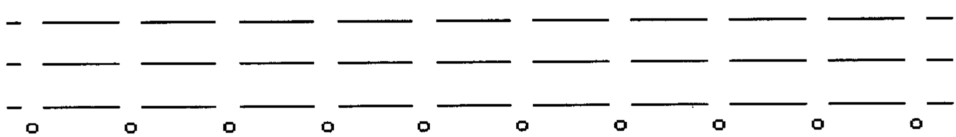

Figure 10: **A track diagram with transitions removed**

The construction of f from a given throw sequence s is immediate: if $s(t) = k$ then fill in transition diagram D_k at time t. The result is a canonical picture of a permutation f_s in $\mathbf{J}$! Indeed, for a given t the track that starts up at t must ultimately plunge since $s(t') = b$ happens infinitely often to the right (at such points the track must either move up one or after at most b such steps, plunge). If this track plunges to the ground at time t', then $f(t) = t$.
We have now constructed a map $f \mapsto s_f$ from $\mathbf{J}$ to $\mathbf{T}$ and $s \mapsto f_s$ from $\mathbf{T}$ to $\mathbf{J}$. These are mutually inverse. Indeed, if s is a sequence then the transition of f_s at time t is, by construction, D_k for $k = s(t)$; thus the throw sequence corresponding to f_s is just s. On the other hand, if $f \in \mathbf{J}$ is a juggling pattern and we record the transitions at each time t and then glue them into the incomplete diagram in the previous figure then we certainly recover f. This finishes our "visual" proof of the theorem.
Now we return to the very special case of periodic juggling: suppose that $f(t) - t$ has period n. This implies that the track diagram is invariant under the transformation $t \to t + n$, and that the throw sequence s_f is periodic with period n. There are b^n sequences of length n on the alphabet $\{1, \cdots, b\}$. Of those, $(b-1)^n$ of them do not contain the value b. This recovers the "b^n" result of the paper (slightly rephrased, to account both for fact that there are exactly b balls, and to for the lack of throws of height 0).

Corollary 2 *There are $b^n - (b-1)^n$ juggling patterns $f\colon \mathbf{Z} \to \mathbf{Z}$ that satisfy $f(t) > t$ and have period n and b balls.*

The argument presented at the OMP conference was the simplification of the proof of the theorem obtained by, loosely speaking, "wrapping" the visualization around a *circle; several points in the argument actually become easier, and the upshot is that periodic juggling patterns are visualized as a collection of orbits around a central planet. A closely related idea had been expressed completely combinatorially in an elegant post by Jeremy Rickard to the Usenet news group sci.math.research. In this posting periodic site swaps* f *were represented by sequences* $\{a_i\}$ *of length* n*, where* $a_i = f(i) - i$*. These sequences had average strictly less than* b *and were shown to be in one-to-one correspondence with arbitrary sequences* $\{s_i\}$ *of length* n*, with values in* $\{0, \cdots, b-1\}$.

The argument proceeded by by showing that both are in one-to-one correspondence with an auxiliary set of "walks" which we will now describe. Place n *buckets in a circle with balls in front of each. Walk around this circle exactly* b *times, picking up balls whenever you don't already have one, and placing balls in empty buckets whenever you choose. We require that all balls have been placed in buckets by the end of the* b^{th} *circuit. The distance that each ball travels gives a valid* $\{a_i\}$*. If we write on each bucket the number of the circuit on which it received its ball, then one gets a valid* $\{b_i\}$.

We leave it to the reader to verify details, and to verify that this is essentially equivalent to a wrapped version of the proof of the theorem.

Finally, we raise an open question. How can these ideas be used to describe, or "name," juggling patterns with infinitely many balls? If we drop the requirement of bijectivity then there is a natural theorem, communicated to us by Hendrik Lenstra, Jr. There is a correspondence between injections $f\colon \mathbf{Z} \to \mathbf{Z}$ *such that* $f(t) > t$ *(but with no requirement on finiteness of orbits) and (possibly unbounded) throw sequences* $s\colon \mathbf{Z} \to \mathbf{Z}^+$ *such that for all* t *there is a* $t' > t$ *such that* $s(t) \geq s(t')$*. Injections could be thought of as modeling a juggler whose skill increases with time; as time goes on the juggler is allowed to pick up more and more juggling balls and toss them into the pattern.*