NOTES

Graceful Configurations in the Plane

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1. Introduction Let C be a finite configuration of (infinite) lines in the plane. The lines of C partition the plane into regions $R_0, R_1, \ldots, R_m$. Suppose it is possible to label these regions with all the integers $\{0, 1, \ldots, m\}$, say R_k is labeled with $\lambda(R_k)$, so that if R_i and R_j share a common boundary line L of C, then $\lambda(R_i) - \lambda(R_j)$ only depends on L. In this case we say that λ is a *graceful* labeling of C, and that C is a *graceful* configuration. In FIGURE 1, we show a variety of graceful configurations.

The concept of a graceful configuration was introduced by D. E. Knuth [3], who also raised the following general question: What are the graceful configurations? In this note we explore this question. In particular, we describe several infinite families of graceful configurations (Sec. 3), as well as several infinite families of nongraceful configurations (Sec. 4, 5). We conclude with a number of open problems. This topic can be viewed as a type of geometrical analogue to some well-studied questions in graph theory (cf. [1], [2]).

2. Preliminaries The reader may notice that the two labelings of the same configuration C of four lines shown in FIGURE 1(e) and (f) differ in the following way. In (f), when crossing the line L from left to right, the region values change both by -1 (from 3 to 2) and by $+1$ (from 4 to 5). However, in (e), as in all the other labelings in FIGURE 1 (except for (f)), the label differences are constant as we cross *oriented* lines of the configurations in the same direction. We call such labelings *strict* graceful labelings. Graceful labelings such as that in (f) will be called *twisted*. At present no configuration C is known that has a twisted graceful labeling but no nontwisted (i.e., ordinary) one. However, any such C must be rather special, as the following result shows.

THEOREM 1. *Suppose C has a graceful labeling in which the labeling of regions bordering the line L is twisted at a point p. Then p must lie on at least four lines of C.*

Proof. Suppose p lies on just two lines L and L'. Consider a twisted labeling shown in FIGURE 2(a).

Since the labeling is graceful we must have:

$$|a + k - b| = |a - (b + k)|.$$

There are two possibilities:

(i) $a + k - b = a - b - k \Rightarrow k = 0$ ~~X~~ (contradiction)

(ii) $a + k - b = -a + b + k \Rightarrow a = b$ ~~X~~

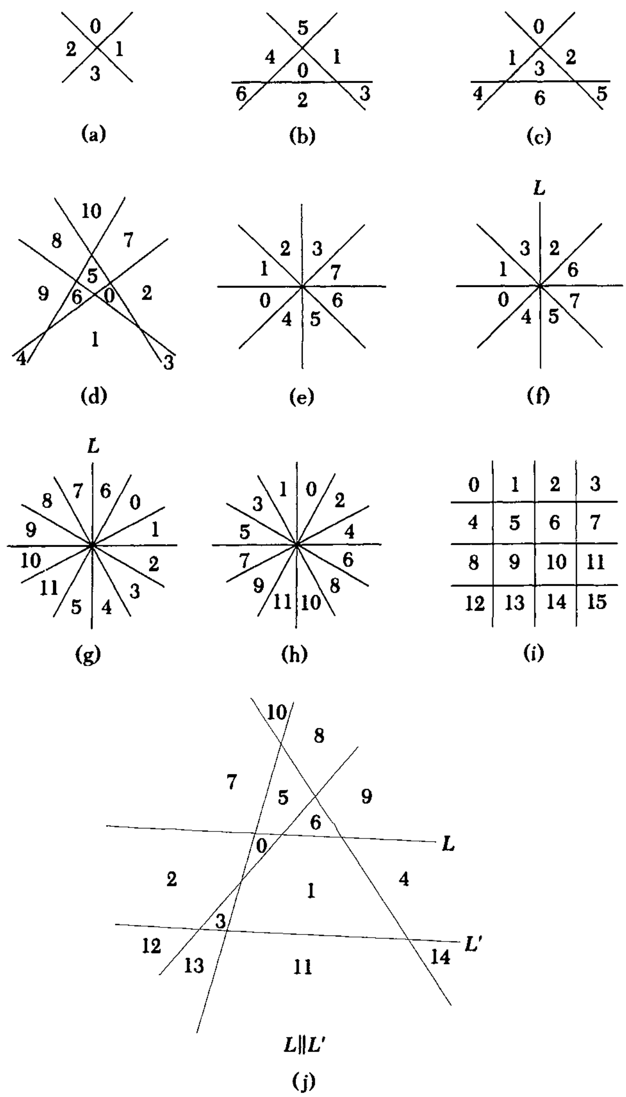


FIGURE 1

Some graceful configurations.

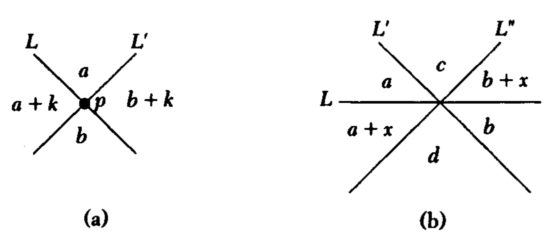


FIGURE 2

Now, suppose p lies on just three lines, L, L', and L''. Consider the twisted labeling shown in FIGURE 2(b). Again, since the labeling is graceful, we have:

$$|c-a| = |b-d|, \quad |b+x-c| = |a+x-d|.$$

There are now four possibilities, each leading to a contradiction.

(i) $$\begin{array}{c} c-a=b-d \\ b+x-c=a+x-d \end{array} \Rightarrow \begin{array}{c} b-c=d-a \\ b-c=a-d \end{array} \Rightarrow a=d \quad \text{X}$$

(ii) $$\begin{array}{c} c-a=b-d \\ b+x-c=d-a-x \end{array} \Rightarrow \begin{array}{c} b-c=d-a \\ x=-x \end{array} \Rightarrow x=0 \quad \text{X}$$

(iii) $$\begin{array}{c} c-a=d-b \\ b+x-c=a+x-d \end{array} \Rightarrow \begin{array}{c} c-d=a-b \\ c-d=b-a \end{array} \Rightarrow a=b \quad \text{X}$$

(iv) $$\begin{array}{c} c-a=d-b \\ b+x-c=d-a-x \end{array} \Rightarrow \begin{array}{c} c=a+d-b \\ c=a-d+b+2x \end{array} \Rightarrow c=a+x \quad \text{X}$$

This proves Theorem 1.

3. Some graceful configurations To begin with, all configurations with at most four lines are graceful. We list these in FIGURE 3.

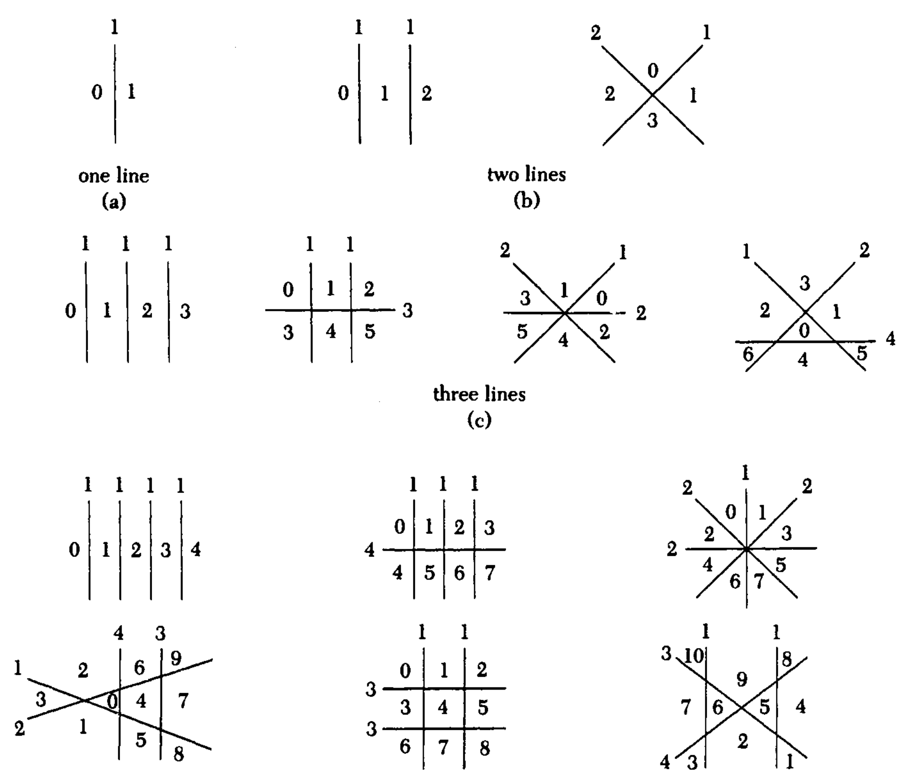
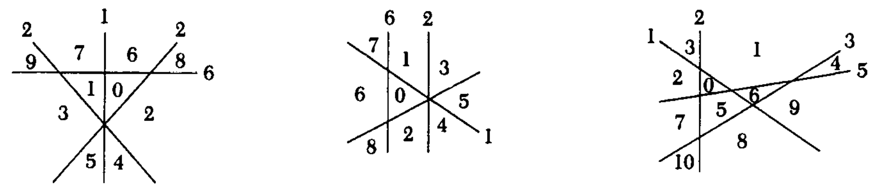


FIGURE 3



Four lines
(d)

FIGURE 3
(Continued).
Configurations with at most four lines.

Below we list several infinite families of graceful configurations.

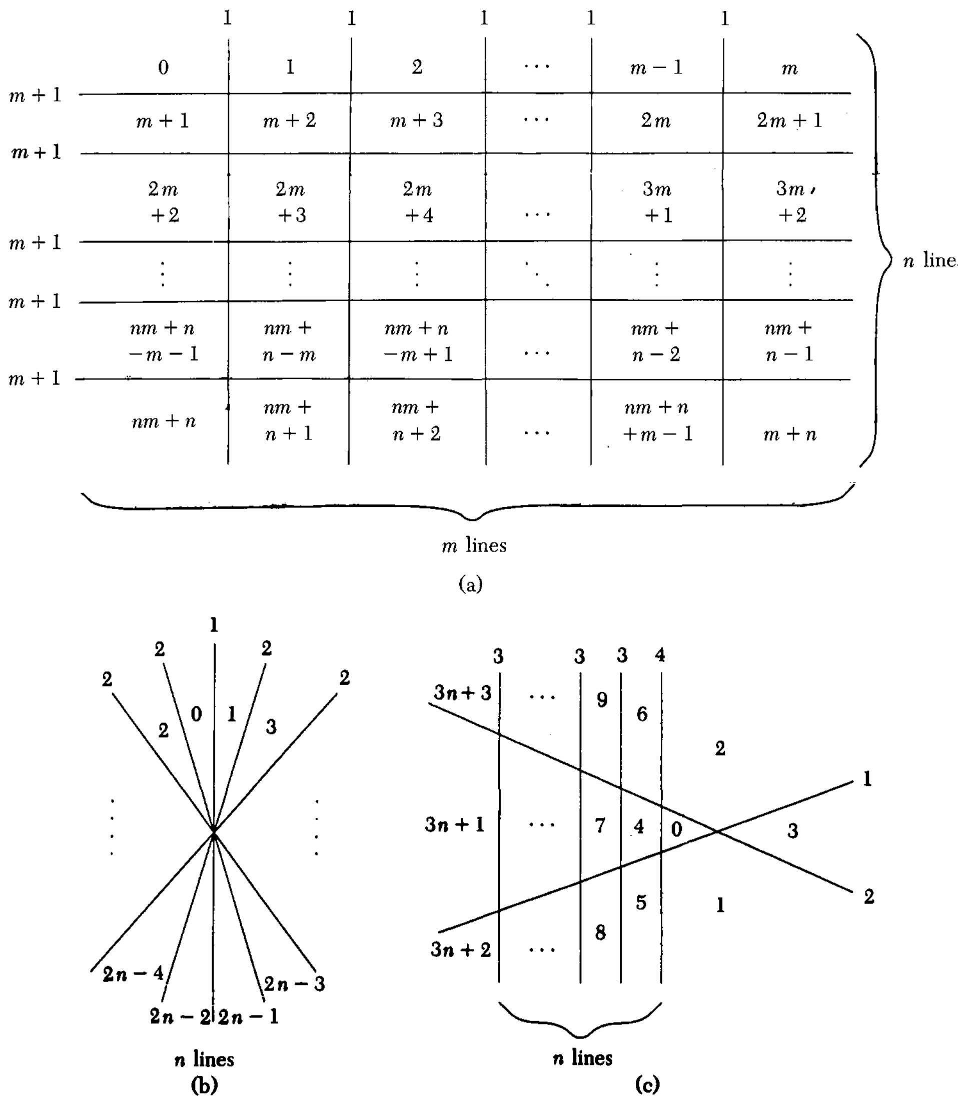
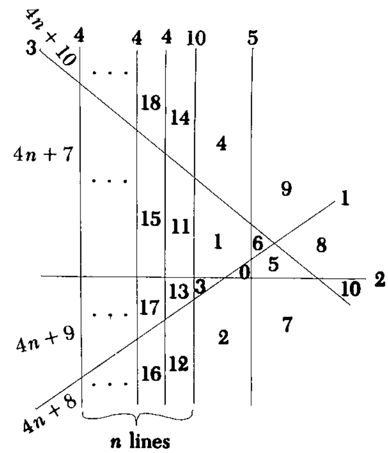
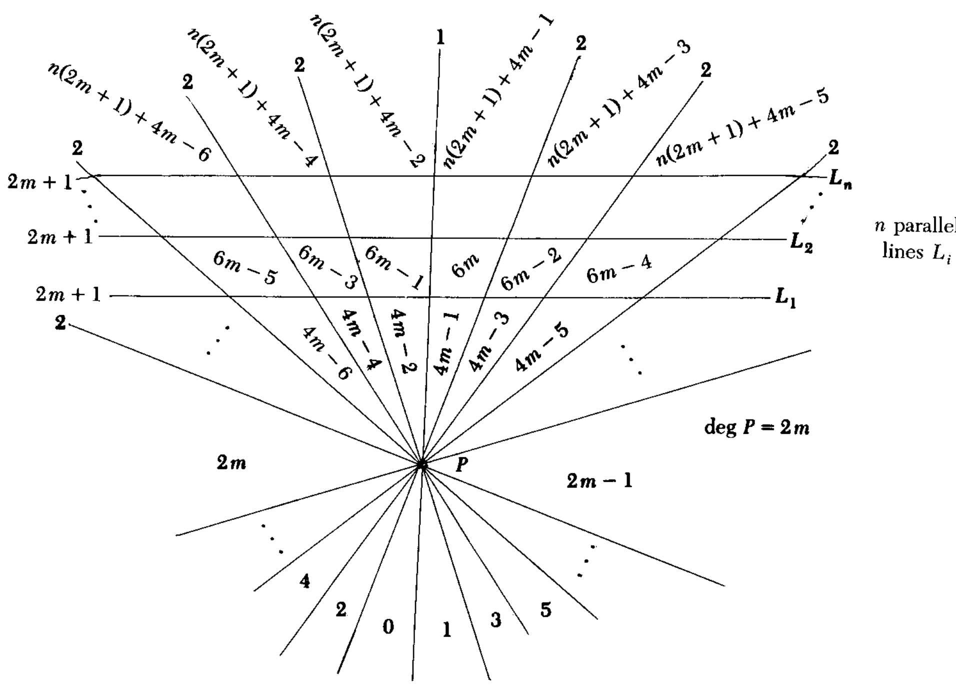


FIGURE 4
Some infinite families of graceful configurations.



(d)



(e)

FIGURE 4
(*Continued*).

4. The smallest nongraceful configuration Suppose the configuration C_5 of five lines shown in FIGURE 5 has a graceful labeling. By subtracting an appropriate value from each region we can assume that the central region is assigned the value 0, where the region values now form an *interval* $x, x+1, \dots, x+15$ with x a non-positive integer. If we assign the values a_i to the five regions adjacent to the central region (see FIGURE 5) then the other 10-region values are as shown. Define S to be the sum of the 16-region values. Thus,

$$S = 6\sum_i a_i = \sum_{j=0}^{15} (x+j) = 16x + 120. \tag{1}$$

Therefore, 3 divides x, say $x = 3y$ and

$$\sum_i a_i = 8y + 20. \tag{2}$$

Note that replacing each a_i by $-a_i$ if necessary, we can assume that $S \geq 0$, i.e., $y \geq -2$. Because 0 is a region value, we must have $y \leq 0$. Thus, there are three possibilities: $y = -2$, -1, and 0. However, each of these three cases can be ruled out by straightforward case enumeration, and we conclude by Theorem 1 that C_5 is not graceful.

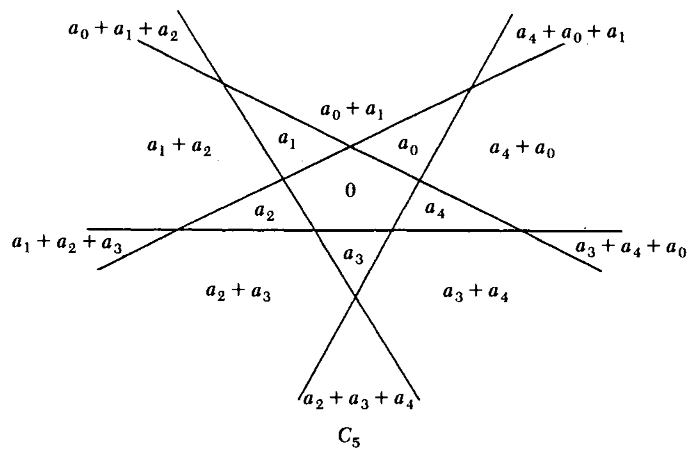


FIGURE 5

5. More families of nongraceful graphs It is possible to generalize the construction of C_5 in the preceding section to the more general configurations C_{2r+1} formed by extending the edges of a regular $(2r+1)$-gon. As before, we assume that C_{2r+1} has been gracefully labeled and we normalize the region values so that the central region has the value 0, with the adjacent regions having the values a_i, $0 \leq i \leq 2r$. An easy calculation shows that the total number R of regions is $1 + (r+1)(2r+1)$. We denote the resulting interval of (normalized) region values by

$$x, x+1, \ldots, x + (r+1)(2r+1).$$

It is not difficult to verify (similar to the previous argument for C_5) that the region values are exactly all the sums $a_i + a_{i+1} + \cdots + a_{i+k}$ for $0 \leq k \leq r$, together with 0, where the index addition is performed modulo $2r+1$.

We next express the sum S of the region values in two ways. On one hand,

$$S = \sum_{j=0}^{R-1} (x+j) = Rx + \binom{R}{2}. \tag{3}$$

On the other,

$$S = \binom{r+2}{2} \sum_{i=0}^{2r} a_i \tag{4}$$

since each a_i occurs exactly $\binom{r+2}{2}$ times. Now, suppose $r = 8t + 6$. Then by (3) and (4),

$$\begin{aligned}\sum_{i=0}^{r} a_i &= \frac{Rx + \binom{R}{2}}{\binom{r+2}{2}} \\ &= \frac{R(2x + R - 1)}{(r+2)(r+1)} \\ &= \frac{(1 + (r+1)(2r+1))(2x + (r+1)(2r+1))}{(r+2)(r+1)} \qquad (5) \\ &= \frac{(64t^2 + 108t + 46)(2x + (8t+7)(16t+13))}{(8t+7)(8t+8)} \\ &= \frac{(32t^2 + 54t + 23)(2x + (8t+7)(16t+13))}{2(8t+7)(t+1)}.\end{aligned}$$

However, this is clearly impossible since the numerator is odd, the denominator is even and $\sum_{i=0}^{r} a_i$ is an integer. This proves

THEOREM 2. *C_{16t+13} is not graceful for $t \geq 0$.*

We conclude this section with several doubly infinite classes of configurations that are not strictly graceful.

Define $C_n^{(m)}$ to be the configuration formed by starting with the "extended edges of a regular n-gon" configuration C_n and replacing each of the n lines with very closely spaced m parallel lines. We now modify $C_n^{(m)}$ by moving the m lines in each parallel class c_i so as to go through a single point p_i. The points p_i are chosen symmetrically around the center of the configuration, and very far away from it. The resulting configuration of mn lines, having rotational symmetry of $2\pi/n$ radians, we denote by $\overline{C}_n^{(m)}$. We show a portion of $\overline{C}_n^{(m)}$ in FIGURE 6.

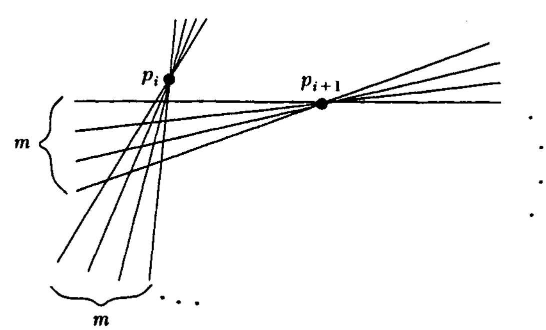


FIGURE 6
A portion of $\overline{C}_n^m$.

In FIGURE 7 we show a portion of $\overline{C}_n^{(m)}$ with values assigned to the regions, normalized from a strictly graceful labeling so that the central region has value 0.

We assume from now on that $n = 2t + 1$ is odd. It is not hard to check that the total number $\overline{R}$ of regions in $\overline{C}_{2t+1}^{(m)}$ is

$$\overline{R} = 1 + (2t+1)(tm^2 + 2m - 1). \qquad (6)$$

Thus, the assumption that $\overline{C}_{2t+1}^{(m)}$ is (strictly) gracefully labeled, implies that for some x, the region values are $x, x+1, x+2, \ldots, x+\overline{R}-1$.

As before, we now compute the sum $\overline{S}$ of all the region values in two ways. On one hand,

$$\overline{S} = \sum_{j=0}^{\overline{R}-1} (x+j) = \overline{R}x + \binom{\overline{R}}{2}. \tag{7}$$

On the other hand, each $a_i(j)$, $0 \le i < 2t+1$ occurs equally often, say $R(j)$ times, in the region values (by symmetry), where

$$R(j) = \frac{t(t+3)}{2}m^2 + (t+1)m - t - (j-1)2tm$$

by a straightforward (but perilous) computation. Thus,

$$\overline{S} = \sum_{i=0}^{2t} \sum_{j=1}^{m} R(j)a_i(j). \tag{8}$$

Our final step is to make certain modular assumptions on m and $n = 2t+1$ to obtain a contradiction, thereby showing that for these m and n, no strictly graceful labeling of $\overline{C}_n^{(m)}$ exists.

For the first choice, we take:

$$t = 4u, \qquad m = 4v,$$

with u and v odd, and $u - v \equiv 4 \pmod 8$. Then $\overline{R}$ is even, and an easy computation shows that $\overline{S} \not\equiv 0 \pmod{32}$ but that $R(j) \equiv 0 \pmod{32}$ for all j. This clearly contradicts (8).

For the second choice, we take

$$t = 4u + 2, \qquad m \equiv 2u + 3 \pmod 4.$$

Then a similar calculation now shows that

$$\overline{S} \not\equiv 0 \pmod 4$$

but that $R(j) \equiv 0 \pmod 4$ for all j, again contradicting (8). Thus we have the

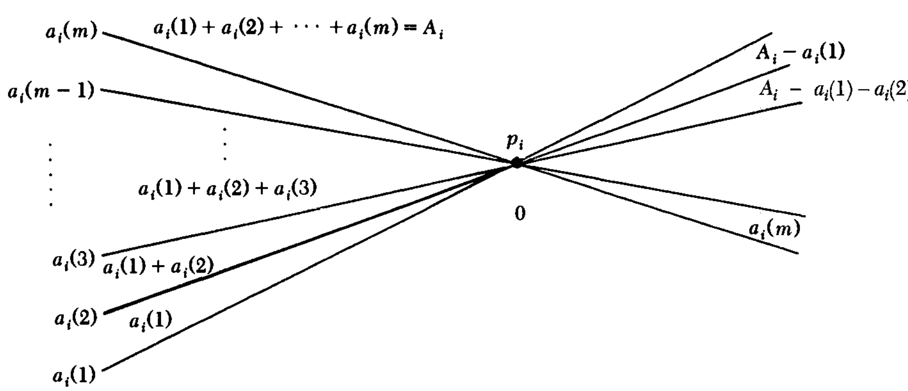


FIGURE 7
Generic labels assigned to regions of $\overline{C}_n^{(m)}$.

following theorem.

THEOREM 3. *If*

$$n = 8u + 5, \qquad m \equiv 2u + 3 \pmod 4$$

or

$$n = 16w + 9, \qquad m \equiv 8w + 20 \pmod{32},$$

for nonnegative integers u and w, then $\overline{C}_n^{(m)}$ *is not strictly graceful.*

Since for $m \geq 4$, $\overline{C}_n^{(m)}$ has points lying on more than three lines, we cannot rule out the possibility of a twisted graceful labeling.

6. Concluding remarks A number of challenging open questions remain unanswered. We list several of these below.

(i) Is there *any* graceful configuration consisting of five or more lines in general position (i.e., no two parallel and no three concurrent)? We suspect that there are not. (The 5-line configuration shown in FIGURE 1(j) is a "near miss".) For example, are the extended edge configurations C_{2k+1} for regular $(2k+1)$-gons all nongraceful?

(ii) An easier exercise would be to show that almost all configurations are not graceful (something we definitely believe). If there is a simple counting argument showing this, it has eluded us.

(iii) Are there configurations that have only *twisted* graceful labelings, or does the existence of a twisted labeling imply the existence of a nontwisted one? The configurations $\overline{C}_n^{(m)}$ in Section 5 with $m \geq 4$ might be good candidates for such configurations.

(iv) What are the analogous results (and questions!) in three or more dimensions?

REFERENCES

1. J. Gallian, A survey: recent results, conjectures and open problems in labeling graphs, *J. Graph Th.* 13 (1989), 491–504.
2. S. W. Golomb, How to number a graph, *Graph Th. and Computing*, R. C. Read, editor, Academic Press, New York, 1972.
3. D. E. Knuth (personal communication).