Pursuit–Evasion Games on Graphs

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ABSTRACT

Two players, Red and Blue, each independently choose a vertex of a connected graph G. Red must then pay Blue an amount equal to the distance between the vertices chosen. In this note, we investigate the value $v(G)$ of this pursuit–evasion game for various classes of graphs G, as well as those optimal mixed strategies for achieving $v(G)$. It is shown that some rather counterintuitive behavior can occur. For example, there exist graphs G in which, for any optimal mixed strategy, Red never selects a vertex in the center of G.

1. PRELIMINARIES

Suppose G is a finite connected graph* with p vertices, and with no loops or multiple edges. By the *distance* $d(u, v)$ between two vertices u and v of G, we mean the minimum number of edges in any path between u and v. We define on G a two-person (Blue, Red) zero-sum game† with payoff matrix $D = (d(u, v): u, v \in G)$ as follows: Blue and Red each independently select a vertex of G, with no prior knowledge of the other player. If Blue selects u and Red selects v, then Red must pay Blue the amount $d(u, v)$. We imagine this process being carried out repeatedly for a large number of times. Of course, Blue's

*For undefined concepts in graph theory, see [1] or [4].
†For undefined concepts in game theory, see [3] or [5].

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objective is to collect as much as possible from Red while, conversely, Red tries to minimize his total payment to Blue.

A *strategy* for this game is defined as a probability vector $x = (x(v): v \in G)$ where $x(v)$ denotes the probability that the player chooses vertex v. It is said to be *completely mixed* if $x(v) > 0$ for all $v \in G$. Similarly, a real payoff matrix for a two-person zero-sum game is said to be *completely mixed* if all optimal strategies for both players are completely mixed. Strategies x and y for Blue and Red, respectively, *solve* the game specified by the payoff matrix A if

$$\max_i (Ay)_i = \min_j (x^T A)_j ,$$

in which case the *value* $v(A)$ of the game is given by the quantity on either side. It was shown by von Neumann (see [8]) that, for any A,

$$\max_x \min_y x^T Ay = \min_y \max_x x^T Ay = v(A) . \tag{1}$$

Finally, we will let rad(G), diam(G), and cen(G) denote the radius, diameter, and center of a graph G, respectively, and for $v \in G$, we let $e(v)$ denote the eccentricity of G, i.e., $e(v) = \max_u d(u, v)$.

2. THE PURSUIT-EVASION GAME ON *G*

By $v(G)$ we will mean $v(D)$ where $D = D(G)$ is the distance matrix of G.

Fact 1. If G is vertex-transitive then the (uniform) strategy $\overline{p} = (1/p, \ldots, 1/p)$ is optimal for both players.

Proof. Since the set of optimal strategies for each player is convex, and $\overline{p}$ is a convex combination of optimal strategies if G is vertex-transitive, then the result follows. Note that in this case

$$v(G) = \overline{p}^T D \overline{p} . \quad ∎$$

As special cases of Fact 1, we have

$$v(K_p) = (p - 1)/p$$

and

$$v(C_p) = (p^2 - \varepsilon)/(4p) \quad \text{with} \quad \varepsilon = \begin{cases} 0 & \text{if } p \text{ is even} \\ 1 & \text{if } p \text{ is odd} , \end{cases}$$

where K_p and C_p denote the *complete graph* and *cycle* on p vertices, respectively. Note that, when p is even, another optimal strategy (for either player) on C_p is to assign probability $1/2$ to two diametrically opposite vertices.

Fact 2. If G is a graph with p vertices then

$$\begin{aligned} &\text{(i) } v(G) = 1 - 1/p, && \text{if } G = K_p, \\ &\text{(ii) } 1 \le v(G) \le \text{rad}(G) && \text{if } G \ne K_p. \end{aligned} \tag{2}$$

Proof. The right-hand side of (ii) follows by noting that Red can always choose a vertex in cen(G). If $G = K_p$ then (i) follows by an earlier remark. Suppose $G \ne K_p$. Let a and b be nonadjacent vertices in G. One strategy x_0 for Blue is to assign probability $1/2$ to both a and b. Suppose the best counter-strategy for Red assigns probability y_v to vertex v. Then the value of this pair of strategies is at least

$$\frac{1}{2} \sum_v (d(v,a) + d(v,b))y_v \ge 1 .$$

Thus,

$$v(G) = \max_x \min_y x^T D y \ge \min_y x_0 D y \ge 1 .$$

This proves (2). ∎

Fact 3. $v(G) < \text{diam}(G)$.

Proof. Let x and y be optimal strategies for Blue and Red, respectively. Thus,

$$v(G) = \max_i (Dy)_i = \min_j (x^T D)_j .$$

At least one component of x, say x_1, must be positive. Therefore,

$$\begin{aligned} (x^T D)_1 &= x_1 d_{11} + \sum_{i>1} x_i d_{i1} \\ &\le (1 - x_1)\, \text{diam}(G) \\ &< \text{diam}(G) = \max_{i,j} d_{ij} . \end{aligned}$$

Hence,

$$v(G) \le (x^T D)_1 < \text{diam}(G) . \quad ∎$$

Fact 4. If G is a graph with p vertices then

$$\begin{aligned} &\text{(i) } v(G) = (p-1)/2, && \text{if } G = P_p, \text{ a path with } p \text{ vertices}, \\ &\text{(ii) } v(G) \le (p-2)/2, && \text{if } G \ne P_p . \end{aligned} \tag{3}$$

Proof. It is easy to see that $\text{rad}(G) \leq (p - 1)/2$ [and thus, $v(G) \leq (p - 1)/2$] unless $G = P_p$, for p even. In the latter case, Red can assign probability $1/2$ to each of the central vertices of P_p to get $v(P_p) \leq (p - 1)/2$ as well. Now, suppose G is not a path. Then, G contains a spanning tree T with a vertex w of degree at least 3. Let T' be the tree obtained by deleting all the endpoints of T. Thus, T' has $p' \leq p - 3$ vertices, and

$$v(G) \leq \text{rad}(G) \leq \text{rad}(T) = \text{rad}(T') + 1 \leq (p' - 1)/2 + 1 \leq (p - 2)/2$$

unless p' is even and $T' = P_{p'}$. In the latter case, either $p' \leq p - 4$ and $v(G) \leq \text{rad}(T') + 1 = p'/2 + 1 \leq (p - 2)/2$, or else $p' = p - 3$ is even and $T' = P_{p'}$. In this last case, T consists of a path with an even number $p - 1$ of vertices together with an additional edge incident to the internal vertex w. For any vertex x of G, the sum of the distances to the two central vertices of the path is at most

$$(p - 1)/2 + ((p - 1)/2 - 1) = p - 2.$$

Thus, if Red assigns probability $1/2$ to each of the central vertices of the path, then Red will pay at most $(p - z)/2$. This shows that

$$v(G) \leq \frac{1}{2}(p - 2)$$

and the proof is complete. ∎

3. COUNTEREXAMPLES

The *perimeter per* (G) of G is the set of vertices v such that, for some u, $d(u, v) = \text{diam}(G)$. Evidently *per* (G) contains at least two distinct vertices. In paths of order $p > 2$, the center and the perimeter are disjoint. In circuits of order $p > 2$, the perimeter and the center both consist of all vertices.

It is tempting to conjecture that, if x is an optimal strategy for Blue, then the support of x lies in *per* (G). To see that this conjecture is false, consider Fig. 1.

If $\bar{x}$ is any strategy with its support in per $(G) = \{1, 2\}$, then by choosing y with $y_6 = y_{10} = 1/2$, Red pays no more than $5/2$.

However, let x have $x_1 = x_2 = x_3 = 1/3$. For any vertex v, $d(v, 1) + d(v, 2) + d(v, 3) \geq 8$. Therefore Red has to pay at least $8/3 > 5/2$. Thus $\bar{x}$ is not optimal.

It also seems plausible to conjecture that for every v in $\text{cen}(G)$, there exists an optimal strategy y for Red such that the support of y contains v. This too is false. Consider Fig. 2. Choose $\bar{y}$ with probability $1/2$ on vertices 4 and 5 for Red. For any vertex v, $d(v, 4) + d(v, 5) \leq 3$. Therefore Red pays at most $3/2$.

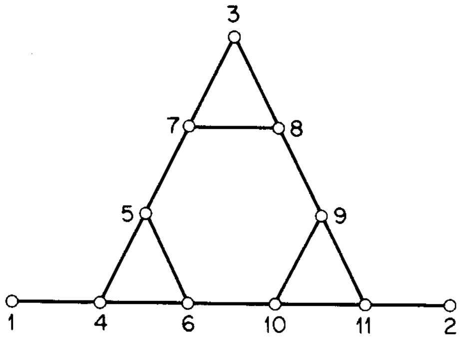

FIGURE 1

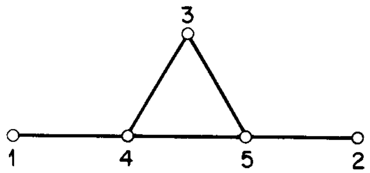

FIGURE 2

We will show that any strategy y for Red putting positive probability $y_3 > 0$ on vertex 3 pays more than 3/2 and therefore is not optimal. Choose the strategy x for Blue given by $x_1 = x_2 = 1/2$. For any vertex v, if $v \neq 3$, then $d(v, 1) + d(v, 2) \geq 3$, and $d(3, 1) + d(3, 2) = 4$. Thus,

$$\left(\frac{1}{2}\right) \sum_v y_v(d(v, 1) + d(v, 2)) \geq \left(\frac{3}{2}\right)(1 - y_3) + 2y_3 > \frac{3}{2}.$$

Therefore, y cannot be optimal. ∎

Even the weaker conjecture that some optimal strategy for Red contains a vertex in cen(G) is false. In fact, for Fig. 3, any optimal strategy for Red avoids the center.

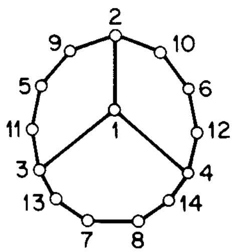

FIGURE 3

The center of G is $\{1\}$. Consider a strategy $\bar{y}$ for Red given by $\bar{y}_2 = 1/5$, $\bar{y}_3 = \bar{y}_4 = 2/5$. It can be easily checked that for any vertex v,

$$\left(\frac{1}{5}\right) d(2, v) + \left(\frac{2}{5}\right) d(3, v) + \left(\frac{2}{5}\right) d(4, v) \leq \frac{14}{5}.$$

Thus $v(G) \leq 14/5$ and Red has to pay no more than $14/5$.

Now suppose Red has a strategy y with $y_1 > 0$ (putting positive probability on the center vertex 1). If there is a strategy x for Blue that forces Red to pay more than $14/5$, then y is not optimal.

Let us choose x so that $x_3 = x_6 = 3/10$, $x_7 = x_8 = 2/10$. Then

$$\left(\frac{3}{10}\right) d(1,5) + \left(\frac{3}{10}\right) d(1,6) + \left(\frac{2}{10}\right) d(1,7) + \left(\frac{2}{10}\right) d(1,8) = 3$$

and for $i \neq 1$,

$$f(i) := \left(\frac{3}{10}\right) d(i,5) + \left(\frac{3}{10}\right) d(i,6) + \left(\frac{2}{10}\right) d(i,7) + \left(\frac{2}{10}\right) d(i,8) \geq \frac{14}{5}.$$

(It is enough to compute $f(2) = f(3) = f(4) = 14/5, f(9) = f(10) = 16/5,$

$$f(5) = f(6) = 3, f(11) = f(12) > 3, f(13) = f(14) = 3.)$$

Therefore

$$x^T D y = \sum_{i=1}^{14} y_i f(i) \geq y_1 \cdot 3 + (1 - y_1) \cdot \frac{14}{5} > \frac{14}{5}.$$

This means Red has to pay more than $14/5$ and so y is not optimal.

4. GAMES ON DIGRAPHS

The pursuit–evasion game on graphs extends naturally to strongly connected digraphs, the distance matrix of which need not constitute a metric space. The digraphs considered from now on are assumed to be strongly connected (e.g., see [9]).

Suppose Blue and Red each independently pick a vertex of a digraph G, with no prior knowledge of the choice of the other player. If Blue picks v and Red picks w, which may or may not be distinct from v, then Red must pay Blue one of two amounts. If Red and Blue are playing a game of "Seek," then Red pays Blue $d(w, v)$, because that is the distance Red must travel to arrive at Blue's vertex. If Red and Blue are playing a game of "Fetch," then Red pays Blue $d(v, w)$, because Blue must be transported that distance to be brought to Red's vertex. In general, of course, on a digraph $G, d(w, v)$ need not equal $d(v, w)$.

For a digraph G, define the *inradius* by $ir(G) = \min_w \max_v d(v, w)$ and the *outradius* by $or(G) = \min_w \max_v d(w, v)$. Then in a game of Fetch, Red has a pure strategy that limits his payments to Blue to no more than $ir(G)$. In a game of Seek, Red has a pure strategy that limits his payments to Blue to no more than $or(G)$. However, as in the pursuit–evasion game on graphs, there are no pure strategies x and y such that (x, y) is optimal for (Blue, Red), because no element of the distance matrix is simultaneously the minimum of its row and the maximum of its column. When Blue and Red use mixed strategies, the value of Fetch is $v(D)$ and the value of Seek is $v(D^T)$. It is not true in general that $v(D) = v(D^T)$. For example, for Fig. 4, the value of Fetch is $v(D) = 7/6$, with optimal strategies $x = (0, 1/2, 1/6, 1/3)$, $y = (0, 1/4, 1/3, 5/12)$, while the value of Seek is $v(D^T) = 1$, with optimal strategies $x = (0, 1/2, 0, 1/2)$, $y = (1, 0, 0, 0)$.

When the distance matrix D is completely mixed, $v(D) = v(D^T)$ because both equal the reciprocal of the sum of the elements of the inverse of D. Thus a graph-theoretic characterization of strongly connected digraphs with a completely mixed distance matrix would provide a sufficient condition for $v(D) = v(D^T)$. Such a characterization would provide a partial answer to a more general open question: When does the value of Seek equal the value of Fetch? The following analogue to Fact 2 is not difficult to prove:

Fact 5. If G is a strongly connected digraph with p vertices then

$$\begin{array}{lll} \text{(i)} & v(G) = 1 - 1/p, & \text{if } G = \overline{K}_p, \\ \text{(ii)} & v(G) \geq 1 - 1/(p+1) & \text{if } G \neq \overline{K}_p, \end{array}$$

where $\overline{K}_p$ denotes a directed complete graph on p vertices.

5. FAIR GAMES

A game is defined to be "fair" if its value is 0, because such a game has, on average, no net payments between players. Suppose Blue and Red play the pursuit–evasion game on a (connected undirected) graph with distance matrix D.

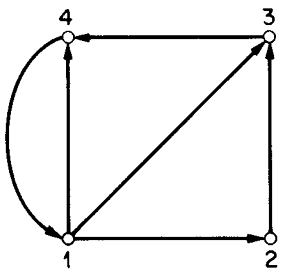

FIGURE 4

Whenever Blue chooses vertex v and Red chooses $w \neq v$, Red pays Blue $d(w, v)$, as before. However, now suppose that, whenever Blue and Red land on the same vertex, Blue must pay Red a constant penalty λ, which is independent of the particular vertex they both happen to land on. In this modified game, the payoff matrix is $D - \lambda I$. How large should λ be for the game to be fair?

Because $D(G)$ is an irreducible nonnegative matrix, Blackwell's theorem [2] implies that $\lambda = \rho(D(G))$, where $\rho(D)$ is the *spectral radius* or *Perron–Frobenius root* of D, that this fair game is completely mixed, and that the unique optimal strategies for Blue and Red are the unique positive left and right eigenvectors, respectively, of $D(G)$, normalized so that each eigenvector sums to 1. Because $D(G)$ is symmetric, the normalized positive left and right eigenvectors are identical, i.e., the optimal (mixed) strategies of Blue and Red are the same. By contrast, in the unfair pursuit–evasion game without penalty, defined above, optimal strategies for Blue and for Red differ, in general. This game-theoretic point of view provides an interpretation of the Perron–Frobenius root and eigenvectors of the distance matrix of a connected graph.

By obvious analogy, Fair Seek and Fair Fetch can be defined on strongly connected digraphs, with the payoff matrices $D^T - \lambda I$ and $D - \lambda I$, respectively. Because D is not symmetric in general for a strong digraph, Blackwell's theorem implies that the unique optimal strategy of Blue will not in general equal the unique optimal strategy of Red.

CONCLUDING REMARKS

There are numerous other aspects of this topic that space limitations do not allow us to discuss here, such as the relationship between $v(G)$, the minimum column sum of $D = D(G)$ and the spectral radius $\rho(D)$ of D, the fact that the only *trees* T possessing completely mixed strategies are paths (because of an explicit form of $D^{-1}(T)$ available from [6]), and the connection to the related results of Gross [7] for pursuit–evasion games on m-connected metric spaces.

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