Ramsey's Theorem for a Class of Categories

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1. INTRODUCTION AND BASIC TERMINOLOGY

In this paper we present a Ramsey theorem for certain categories which is sufficiently general to include as special cases the finite vector space analog to Ramsey's theorem (conjectured by Gian-Carlo Rota), the Ramsey theorem for n-parameter sets [2], as well as Ramsey's theorem itself [4, 6]. The Ramsey theorem for finite affine spaces is obtained here simultaneously with that for vector spaces. That these two are equivalent was already known [5, 1], and the arguments previously used to show that the affine theorem implies the projective theorem are also special cases of the results of this paper.

The argument used here to establish the main result is essentially the same as that used for n-parameter sets [2]. What we do here is to abstract the properties of n-parameter sets which suffice to allow the induction argument. In particular, the properties described for n-parameter sets in Remarks 1-3 of [2] are essential.

In order to state the Ramsey property for a category C we must have a notion of rank with which to index the objects and subobjects of the category. To this end, it is convenient to consider henceforth only categories C with the following property:

(a) The objects of C are the nonnegative integers $0, 1, 2, \ldots,$ and if $l > k$, $C(l, k) = \varnothing$, where $C(l, k)$ is the set of all morphisms from l to k in C.

Using this property, we define a rank on subobjects of an object l in C. Namely, if $k \xrightarrow{f} l$ and $k' \xrightarrow{f'} l$ are representatives of the same subobject of l, then there must be isomorphisms $k \xrightarrow{\alpha} k'$ and $k' \xrightarrow{\beta} k$. But by (a), this means that $k = k'$. We define the rank of this subobject to be k, and we refer to it as a *k-subobject* of l. We denote by $C\begin{bmatrix} l \\ k \end{bmatrix}$ the class of subobjects of l in C of rank k. We make the convention that for $k < 0$, or $l < 0$,

$C\begin{bmatrix} l \\ k \end{bmatrix} = \varnothing$. In order to make our induction argument work, we need a finiteness condition. We assume in addition to (a) that all categories considered here satisfy:

(b) For each pair of integers there is an integer $y_{k,l}$ such that $C\begin{bmatrix} l \\ k \end{bmatrix}$ is a finite set with $y_{k,l}$ elements. In particular, $y_{0,0} = 1$.

For convenience, all categories we consider are assumed to satisfy

(c) All morphisms of C are monomorphisms.

If $k \xrightarrow{f} l$ is a morphism of C, we let $\bar{f}$ denote the induced mapping on subjects of l. That is, if $s \xrightarrow{g} k$ represents a subobject of k, then $\bar{f}$ takes this subobject into the subobject of l represented by the composition fg. This is clearly well defined, and $\bar{f}\colon C\begin{bmatrix} k \\ s \end{bmatrix} \to C\begin{bmatrix} l \\ s \end{bmatrix}$. An *$r$-coloring* of $C\begin{bmatrix} l \\ s \end{bmatrix}$ is a function $c\colon C\begin{bmatrix} l \\ s \end{bmatrix} \to \{1, \ldots, r\}$. We say that a subobject has color i if its image under c is i. An r-coloring c of $C\begin{bmatrix} l \\ s \end{bmatrix}$ induces an r-coloring on $C\begin{bmatrix} k \\ s \end{bmatrix}$ by the composition $c\bar{f}$, where $k \xrightarrow{f} l$ is in C. If the image of $c\bar{f}$ is only a single element, we say that c *has a monochromatic k-subobject*, namely, the k-subobject represented by $k \xrightarrow{f} l$.

We can now state the Ramsey property for a category C satisfying (a)-(c):

Given integers k, l, r, there exists a number n, depending only on k, l, r, so that for all $m \geqslant n$, every r-coloring of $C\begin{bmatrix} m \\ k \end{bmatrix}$ has a monochromatic l-subobject.

When C has morphisms $k \xrightarrow{f} l$ which are all the monomorphic functions from $\{1, \ldots, k\}$ to $\{1, \ldots, l\}$, then this is just the statement of Ramsey's Theorem. If C has morphisms $k \xrightarrow{f} l$ which are the linear monomorphisms from $V_k = \langle v_1, \ldots, v_k \rangle$ to $V_l = \langle v_1, \ldots, v_l \rangle$, where $v_1, v_2, \ldots$ form a basis for a vector space V over $GF(q)$, then this is the statement of Rota's conjecture. In this case, the k-subobjects of l correspond to the subspaces of V_l of dimension k. Other examples of special cases of the Ramsey property will be given later.

2. STATEMENT OF THE MAIN RESULT

In order to establish the Ramsey property for certain categories C, we consider a somewhat stronger version of it which makes the induction argument easier.

$C(k; l_1, \ldots, l_r)$: There is a number $N = N_c(k; r; l_1, \ldots, l_r)$ depending only on k, r, $l_1, \ldots, l_r$, such that for any $m \geqslant N$ and any r-coloring c of $C\begin{bmatrix} m \\ k \end{bmatrix}$, there is an i, $1 \leqslant i \leqslant r$, and a morphism $l_i \xrightarrow{f} m$ such that

$$\begin{tikzcd} C\begin{bmatrix} l_i \\ k \end{bmatrix} \arrow[r, "\bar{f}"] \arrow[dr] & C\begin{bmatrix} m \\ k \end{bmatrix} \arrow[r, "c"] & \{1, \ldots, r\} \\ & \{i\} \arrow[ur, "\text{incl}"'] & \end{tikzcd}$$

commutes, where $\mathrm{incl}(i) = i$.

This statement always holds for $k < 0$, since $C\begin{bmatrix} l_i \\ k \end{bmatrix} = \varnothing$, by convention. If all the l_i are equal, this becomes the Ramsey property stated above.

Theorem 1 below provides the induction step in establishing $C(k; l_1, \ldots, l_r)$ for certain categories. It establishes $B(k+1; l_1, \ldots, l_r)$ if we know $A(k; l_1, \ldots, l_r)$ for all r and l_i provided the categories A and B are related in a special way. This relation is given by the conditions below. For a functor M from A to B with $M(x) = y$ for integers x and y, we denote by $\bar{M}$ the induced function from subobjects of x to subobjects of y. This is given by letting $\bar{M}$ take the subobject represented by $s \xrightarrow{f} x$ in A into the subobject represented by $M(s) \xrightarrow{M(f)} y$ in B.

Conditions on Categories A and B

There is a functor M from A to B with $M(l) = l+1$, $l = 0, 1, \ldots$, a functor P from B to A with $P(l) = l$, $l = 0, 1, \ldots$, an integer $t \geqslant 0$, and for each $l = 0, 1, \ldots t$ morphisms, $l \xrightarrow{\phi_{lj}} l+1$, $1 \leqslant j \leqslant t$, satisfying the following:

I. For each $k+1 = 0, 1, 2, \ldots$, the diagonal d in the following diagram is epic, where $\amalg$ (together with the indicated injections) is coproduct, and d is the unique map determined by the coproduct to make the diagram commute:

$$\begin{tikzcd}
B\begin{bmatrix} l \\ k+1 \end{bmatrix} \arrow[ddd, "\bar{\phi}_{l1}"'] \arrow[drr, "i_1"] & & \\
\ddots & B\begin{bmatrix} l \\ k+1 \end{bmatrix} \arrow[r, "i_t"] \arrow[ddl, "\bar{\phi}_{lt}"] & \amalg \arrow[ddll, "d"'] \\
& & \\
B\begin{bmatrix} l+1 \\ k+1 \end{bmatrix} & & A\begin{bmatrix} l \\ k \end{bmatrix} \arrow[ll, "\bar{M}"] \arrow[uu, "i_0"']
\end{tikzcd}$$

II. For each $s \xrightarrow{g} l$ in B and each $j = 1, \ldots, t$ the following diagram commutes:

$$\begin{tikzcd}
l \arrow[r, "\phi_{lj}"] & l+1 \\
s \arrow[u, "g"] \arrow[r, "\phi_{sj}"] & s+1 \arrow[u, "M(P(g))"']
\end{tikzcd}$$

III. For some $l \xrightarrow{e} l+1$ in A, the following diagram commutes for all $j = 1, \ldots, t$:

$$
\begin{tikzcd}
& l+1 \arrow[dr, "\phi_{l+1,j}"] & \\
l \arrow[ur, "\phi_{lj}"] \arrow[dr, "\phi_{lj}"'] & & l+2 \\
& l+1 \arrow[ur, "M(e)"'] &
\end{tikzcd}
$$

Remark. Let $s + l \xrightarrow{h} l$ in B. Then by III there is some $s \xrightarrow{e} s + 1$ in A such that

$$
\begin{tikzcd}
& s+1 \arrow[dr, "\phi_{s+1,j}"] & \\
s \arrow[ur, "\phi_{sj}"] \arrow[dr, "\phi_{sj}"'] & & s+2 \\
& s+1 \arrow[ur, "M(e)"'] &
\end{tikzcd}
$$

commutes in B for each j. By II, the diagram

$$
\begin{tikzcd}
l \arrow[r, "\phi_{lj}"] & l+1 \\
s+1 \arrow[u, "h"] \arrow[r, "\phi_{s+1,j}"] & s+2 \arrow[u, "M(P(h))"']
\end{tikzcd}
$$

commutes for each j. Thus

$$
\begin{tikzcd}
& s+1 \arrow[r, "h"] & l \arrow[dr, "\phi_{lj}"] & \\
s \arrow[ur, "\phi_{sj}"] \arrow[dr, "\phi_{sj}"'] & & & l+1 \\
& s+1 \arrow[urr, "M(P(h)e)"'] & &
\end{tikzcd}
$$

commutes for each j.

THEOREM 1. *Let A and B be two categories satisfying the conditions above. Assume* $A(k; l_1, \ldots, l_r)$ *holds for all* $l_1, \ldots, l_r$ *and* $r > 0$.

Then $B(k+1; l_1, \ldots, l_r)$ *holds for all* $l_1, \ldots, l_r$, *and* $r > 0$.

3. PROOF OF MAIN RESULT

We will eventually need a lemma about n-dimensional arrays of points. We state it now without proof. Proofs can be found in [3] and [2]. (It is a special case of Corollary 4 below, in fact.) We denote by A^n the set of n-tuples $(x_1, \ldots, x_n)$ of elements of x_i of a set A.

LEMMA 1. *Given integers* $r > 0$, $t \geqslant 0$, *there exists an integer* $N = N(r,t)$, *depending only on* r *and* t, *such that if* $n \geqslant N$, A *is a set of* t *elements, and* A^n *is* r*-colored in any way, then there exists a set of* t n*-tuples* $(x_1(j), \ldots, x_n(j))$, $1 \leqslant j \leqslant t$, *all the same color with the property that for each* i, $1 \leqslant i \leqslant n$, *either* $x_i(j) = j$ *for all* j, *or* $x_i(j) = a_i$ *for all* j *and some* $a_i \in A$.

Proof of Theorem 1. We use induction on $L = l_1 + \ldots + l_r$. $B(k+1; l_1, \ldots, l_r)$ holds vacuously if $l_i < k+1$ for any i or if $k+1 < 0$ and trivially if $t = 0$. So we assume $l_i \geqslant k+1 \geqslant 0$ and $t > 0$. If any $l_i = 0$, then $k+1 = 0$, and $B(k+1; l_1, \ldots, l_r)$ holds trivially, since $y_{0,0} = 1$. So we may assume all $l_i > 0$, and, in particular, that $L > 0$. Assume, then, that $B(k+1; l_1, \ldots, l_r)$ holds for $L-1$, and let $l_1 + \ldots + l_r = L$, $l_i > 0$.

DEFINITION. For $1 \leqslant h \leqslant m$, suppose $k+1 \xrightarrow{f} l+h$ is in B, and $f = M(f')$ for some $k \xrightarrow{f'} l+h-1$ in A. For any fixed choice of $j_h, j_{h+1}, \ldots, j_m - 1$, $1 \leqslant j_i \leqslant t$, let $\phi_i = \phi_{l+i,j}$. Then the $(k+1)$-subobject of $l+m$ represented by the composition

$$k+1 \xrightarrow{f} l+h \xrightarrow{\phi_h} l+h+1 \longrightarrow \cdots \longrightarrow l+m-1 \xrightarrow{\phi_{m-1}} l+m$$

is said to have *signature* $(h; j_m - 1, \ldots, j_h)$ with respect to l and m. (The signature need not be unique for a given subobject, nor must every subobject have a signature.) An r-coloring of $B\begin{bmatrix} l+m \\ k+1 \end{bmatrix}$ such that all $(k+1)$-subobjects with the same signature have the same color is called an (l,m)*-coloring.*

For integers l and m we define recursively some numbers needed to prove Lemma 2 below.

$$v_1 = N_A(k; r^{t^{m-1}}; l, \ldots, l)$$

$$v_2 = N_A(k; r^{t^{m-2}}; v_1 + 1, \ldots, v_1 + 1)$$

$$\vdots$$

$$v_m = N_A(k; r^{t^0}; v_m - 1 + 1, \ldots, v_m - 1 + 1)\,.$$

The existence of these numbers is guaranteed by the hypothesis of Theorem 1.

LEMMA 2. *With the same assumptions as in Theorem 1, let* $l \geqslant 0$, $m \geqslant 1$ *be integers; let* $x \geqslant v_m + 1$; *and let* $B\begin{bmatrix} x \\ k+1 \end{bmatrix} \xrightarrow{c} \{1, \ldots, r\}$ *be an* r*-coloring. Then there exists* $l+m \xrightarrow{g} x$ *in* B *such that* $c\bar{g}$ *is an* (l,m)*-coloring of* $B\begin{bmatrix} l+m \\ k+1 \end{bmatrix}$.

Proof. We use induction on m. For $m = 1$ the lemma is trivially true. Assume for some $m \geqslant 2$ that it holds for $m - 1$. Then by induction, and by the choice of the v_i, there is some $v_1 + m \xrightarrow{g} x$ in B such that $B\begin{bmatrix} v_1+m \\ k+1 \end{bmatrix}$ is $(v_1 + 1, m - 1)$-colored by $c\bar{g}$.

We now color $B\begin{bmatrix} v_1+1 \\ k+1 \end{bmatrix}$ as follows: Two subobjects, represented by $k + 1 \xrightarrow{f} v_1 + 1$ and $k + 1 \xrightarrow{f'} v_1 + 1$ have the same color if and only if for each choice of $j_m - 1, \ldots, j_1$, $1 \leqslant j_i \leqslant t$, the subobjects represented by the compositions

$$k + 1 \xrightarrow{f} v_1 + 1 \xrightarrow{\phi_1} v_1 + 2 \longrightarrow \cdots \longrightarrow v_1 + m - 1 \xrightarrow{\phi_{m-1}} v_1 + m$$

and

$$k + 1 \xrightarrow{f'} v_1 + 1 \xrightarrow{\phi_1} v_1 + 1 \longrightarrow \cdots \longrightarrow v_1 + m - 1 \xrightarrow{\phi_{m-1}} v_1 + m$$

have the same color, where $\phi_i = \phi_{v_1+i, j_i}$, $1 \leqslant i \leqslant m - 1$. This is an $r^{t^{m-1}}$-coloring of $B\begin{bmatrix} v_1+1 \\ k+1 \end{bmatrix}$; call it c'.

Next, we color $A\begin{bmatrix} v_1 \\ k \end{bmatrix}$ by the coloring induced by M. That is, a subobject in $A\begin{bmatrix} v_1 \\ k \end{bmatrix}$ is assigned the same color as its image under $\overline{M}$ in $B\begin{bmatrix} v_1+1 \\ k+1 \end{bmatrix}$. In other words, $c'\overline{M}$ is the coloring we use. By the choice of v_1, there is some i, $1 \leqslant i \leqslant r^{t^{m-1}}$, and some $l \xrightarrow{w} v_1$ in A such that the following diagram commutes:

$$\begin{tikzcd}
A\begin{bmatrix} l \\ k \end{bmatrix} \arrow[r, "\overline{w}"] \arrow[dr] & A\begin{bmatrix} v_1 \\ k \end{bmatrix} \arrow[r, "c'\overline{M}"] & \{1, \ldots, r^{t^{m-1}}\} \\
& \{i\} \arrow[ur, "\text{incl}"'] &
\end{tikzcd}$$

Thus all the subobjects in $\overline{M}(A\begin{bmatrix} l \\ k \end{bmatrix})$ have the same color in $B\begin{bmatrix} l+1 \\ k+1 \end{bmatrix}$ colored by $c'\overline{M(w)}$.

Suppose $k + 1 \xrightarrow{f} l + h$ is in B, $1 \leqslant h \leqslant m$, with $f = M(f')$ for some $k \xrightarrow{f'} l + h - 1$ in A. Consider the following diagram:

$$
\begin{tikzcd}
 & v_1+h \arrow[r, "\phi_{v_1+h,j_h}"] & v_1+h+1 \arrow[r] & \cdots \\
k+1 \arrow[ur, "M(w_h f')"] \arrow[r, "f"'] & l+h \arrow[u, "u_h"'] \arrow[r, "\phi_{l+h,j_h}"] & l+h+1 \arrow[u, "u_{h+1}"'] \arrow[r] & \cdots
\end{tikzcd}
$$

$$
\begin{tikzcd}
\cdots \arrow[r] & v_1+m-1 \arrow[r, "\phi_{v_1+m-1,j_{m-1}}"] & v_1+m \\
\cdots \arrow[r] & l+m-1 \arrow[u, "u_{m-1}"'] \arrow[r, "\phi_{l+m-1,j_{m-1}}"] & l+m \arrow[u, "u_m"']
\end{tikzcd}
$$

where $u_1 = M(w)$, $u_i = M(P(u_{i-1}))$, $i = 2, 3, \ldots, m$, and $w_1 = w$, $w_i = P(u_{i-1})$, $i = 2, 3, \ldots, m$. By condition II this commutes for each choice of $j_h, j_{h+1}, \ldots, j_m - 1$. Consider any subobject of $l + m$ with signature $(1; j_m - 1, \ldots, j_1)$ with respect to l and m. Let it be represented by $k + 1 \xrightarrow{e} l + m$, where e is the bottom row of the diagram above with $h = 1$. Then $u_m e$ represents a subobject of $v_1 + m$. By the definition of c' and the choice of w, all such subobjects with the same signature $(1; j_m - 1, \ldots, j_1)$ have the same color in $B\begin{bmatrix} v_1+m \\ k+1 \end{bmatrix}$, since the diagram above commutes. On the other hand, consider a subobject of $l+m$ with signature $(h; j_m - 1, \ldots, j_h)$, $h \geqslant 2$, and let it be represented by $k + 1 \xrightarrow{e} l + m$, where e is the bottom row of the diagram. By the commutativity of the diagram, $u_m e = bM(w_h f')$, where $v_1 + h \xrightarrow{b} v_1 + m$ is the top row of the diagram. This means that $u_m e$ has signature $(h - 1; j_m - 1, \ldots, j_h)$ with respect to $v_1 + 1$ and $m - 1$. Since $c\bar{g}$ was a $(v_1 + 1, m - 1)$-coloring of $B\begin{bmatrix} v_1+m \\ k+1 \end{bmatrix}$, the color of this subobject is determined only by the j_i. Thus the color of any subobject with signature $(h; j_m - 1, \ldots, j_h)$ with respect to l and m, $h \geqslant 1$, has its color under the coloring $c\bar{g}\bar{u}_m$ determined only by the j_i. So $c\bar{g}\bar{u}_m$ is an (l, m)-coloring, and the lemma is proved.

We may now proceed with the proof of Theorem 1. Let

$$l = \max_{1 \leqslant i \leqslant r} N_B(k + 1; r; l_1, \ldots, l_{i-1}, l_i - 1, l_{i+1}, \ldots, l_r),$$

a number which must exist by the induction hypothesis. Let $y = r^{y_{l,k+1}}$, where $y_{l,k+1}$ is the number given by property (b). Let $m = N(y, t)$, where $N(y, t)$ is the number given by Lemma 1. Let v_m be the number used in the hypothesis of Lemma 2 (depending on l and m), and let $x \geqslant v_m + 1$. Finally, let $B\begin{bmatrix} x \\ k+1 \end{bmatrix} \xrightarrow{c} \{1, \ldots, r\}$ be an r-coloring. By Lemma 2 there is some $l + m \xrightarrow{g} x$ in B such that $c\bar{g}$ is an (l, m)-coloring of $B\begin{bmatrix} l+m \\ k+1 \end{bmatrix}$. We now color the m-tuples $(j_1, \ldots, j_m)$, $1 \leqslant j_i \leqslant t$, by letting $(j_1, \ldots, j_m)$ and $(k_1, \ldots, k_m)$ have the same color if and only if for each $k + 1 \xrightarrow{h} l$ in B the subobjects represented by the compositions

$$k+1 \xrightarrow{h} l \xrightarrow{\phi_{l,j_1}} l+1 \longrightarrow \cdots \longrightarrow l+m-1 \xrightarrow{\phi_{l+m-1,j_m}} l+m$$

and

$$k+1 \xrightarrow{h} l \xrightarrow{\phi_{l,k}} l+1 \longrightarrow \cdots \longrightarrow l+m-1 \xrightarrow{\phi_{l+m-1,k_m}} l+m$$

both have the same color in $B\begin{bmatrix} l+m \\ k+1 \end{bmatrix}$. This is a y-coloring of the m-tuples.

By Lemma 1 and the choice of m, we can find t m-tuples $(j_1(z), \ldots, j_m(z))$, $1 \leqslant z \leqslant t$, all having the same color such that for each i either $j_i(z) = z$ for all z or $j_i(z) = j_i$ for all z and some fixed j_i. Let $i_1, \ldots, i_d$ be the i for which $j_i(z) = z$ (there must be at least one of these since there are t m-tuples here). For $0 \leqslant a \leqslant d$, let h_a denote the composition

$$l+i_a \xrightarrow{\phi_{l+i_a, j_{i_a+1}}} l+i_a+1 \longrightarrow \cdots$$
$$\cdots \longrightarrow l+i_{a+1}-2 \xrightarrow{\phi_{l+i_{a+1}-2, j_{i_{a+1}-1}}} l+i_{a+1}-1,$$

where we let $i_0 = 0$ and $i_{d+1} = m+1$. Consider the following diagram:

$$\begin{tikzcd}
l \arrow[r, "h_0"] & l+i_1-1 \arrow[r, "\phi_{l+i_1-1,j}"] \arrow[d, "\phi_{l+i_1-1,j}"] & l+i_1 \arrow[r, "h_1"] & l+i_2-1 \arrow[r] \arrow[d, "\phi_{l+i_2-1,j}"] & \cdots \\
 & l+i_1 \arrow[rr, "M(e_1)"] & & l+i_2 \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & l+i_{d-2}-1 \arrow[r, "\phi_{l+i_{d-2}-1,j}"] \arrow[d, "\phi_{l+i_{d-2}-1,j}"] & l+i_{d-2} \arrow[r, dash, "h_{d-2}"] & {} \\
\cdots \arrow[r] & l+i_{d-2} \arrow[rr, dash, "M(e_{d-2})"] & & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r, "h_{d-2}"] & l+i_{d-1}-1 \arrow[r, "\phi_{l+i_{d-1}-1,j}"] \arrow[d, "\phi_{l+i_{d-1}-1,j}"] & l+i_{d-1} \arrow[r, "h_{d-1}"] & l+i_d-1 \arrow[r, "\phi_{l+i_d-1,j}"] & l+i_d \arrow[r, "h_d"] & l+m \\
{} \arrow[r, "M(e_{d-2})"] & l+i_{d-1} \arrow[rrru, "M(e_{d-1})"'] & & & &
\end{tikzcd}$$

where the $l+i_{d-s}-1 \xrightarrow{e_{d-s}} l+i_{d-s+1}-1$ in A are those guaranteed by the Remark (following Condition III) to make this diagram commute for each $j = 1, 2, \ldots, t$.

By the choice of the h_a we have for any $k+1 \xrightarrow{h} l$ that the t subobjects represented by

$$k+1 \xrightarrow{h} l \xrightarrow{h_0} l+i_1-1 \xrightarrow{\phi_{l+i_1-1,j}} l+i_1 \longrightarrow \cdots$$

$$\cdots \xrightarrow{M(e_{d-1}e_{d-2}\cdots e_2e_1)} l+i_d \xrightarrow{h_d} l+m \text{ in } B\begin{bmatrix} l+m \\ k+1 \end{bmatrix}, \quad 1 \leqslant j \leqslant t,$$

all have the same color. By Condition II, the following diagram commutes for all j:

$$\begin{tikzcd} l+i_1-1 \arrow[r, "\phi_{l+i_1-1,j}"] & l+i_1 \\ l \arrow[u, "h_0"] \arrow[r, "\phi_{l,j}"] & l+1 \arrow[u, "M(P(h_0))"'] \end{tikzcd}$$

Then letting $\alpha = h_d M(e_{d-1} \cdots e_1 P(h_0))$ we see that for $k+1 \xrightarrow{h} l$ in B the subobjects represented by the t compositions

$$k+1 \xrightarrow{h} l \xrightarrow{\phi_{l,j}} l+1 \xrightarrow{\alpha} l+m, \quad 1 \leqslant j \leqslant t,$$

all have the same color. Thus $\overline{cg\alpha\phi_{l,j}}$ are equal for all $j = 1, 2, \ldots, t$, on $B\begin{bmatrix} l \\ k+1 \end{bmatrix}$.

Now consider any subobject of $\overline{M}(A\begin{bmatrix} l \\ k \end{bmatrix})$ in $B\begin{bmatrix} l+1 \\ k+1 \end{bmatrix}$. Let it be represented by $k+l \xrightarrow{f} l+1$ in B, where $f = M(f')$, $k \xrightarrow{f'} l$ in A. Then the subobject represented by αf has signature $(i_d; j_m, \ldots, j_{i_d+1})$ with respect to l and m, since αf is just $h_d M(e_{d-1} \cdots e_1 P(h_0) f')$. Since $l+m$ is (l, m)-colored by $c\bar{g}$, all subobjects of $l+m$ with this signature have the same color. Thus $c\bar{g}\alpha$ gives the same color to any subobject of $\overline{M}(A\begin{bmatrix} l \\ k \end{bmatrix})$, since the signature was independent of the choice of f. That is, $\overline{cg\alpha M}(A\begin{bmatrix} l \\ k \end{bmatrix} = \{q\}$ for some q, $1 \leqslant q \leqslant r$.

Consider the coloring $\overline{cg\alpha\phi_{l,1}}$ on $B\begin{bmatrix} l \\ k+1 \end{bmatrix}$. By the choice of l, either there is some $l_p \xrightarrow{f_p} l$ in B such that

$$B\begin{bmatrix} l_p \\ k+1 \end{bmatrix} \xrightarrow{\overline{cg\alpha\phi_{l,1}f_p}} \{p\}, \quad p \neq q,$$

or there is some $l_q - 1 \xrightarrow{f_q} l$ in B such that

$$B\begin{bmatrix} l_q-1 \\ k+1 \end{bmatrix} \xrightarrow{\overline{cg\alpha\phi_{l,1}f_q}} \{q\}.$$

In the former case, we have the desired monochromatic subobject, and the theorem is proved. Hence we may assume that

$$B\begin{bmatrix} l_q-1 \\ k+1 \end{bmatrix} \xrightarrow{\overline{cg\alpha\phi_{l,1}f_q}} \{q\}\,.$$

We recall that $\overline{cg\alpha\phi_{l,1}} = \overline{cg\alpha\phi_{l,j}}$ on $B\begin{bmatrix} l \\ k+1 \end{bmatrix}$ for all j. In particular,

$$\overline{cg\alpha\phi_{l,j}f_q}\left(B\begin{bmatrix} l_q-1 \\ k+1 \end{bmatrix}\right) = \{q\} \quad \text{for all } j\,.$$

By Condition II, $\phi_{l,j}f_q = M(P(f_q))\,\phi_{l_q-1,j}$, $j = 1,\ldots,t$. Thus

$$\overline{cg\alpha M(P(f_q))\,\phi_{l_q-1,j}}\left(B\begin{bmatrix} l_q-1 \\ k+1 \end{bmatrix}\right) = \{q\}, \quad j = 1,\ldots,t\,.$$

Now consider any subobject in $\overline{M}\left(A\begin{bmatrix} l_q-1 \\ k \end{bmatrix}\right)$, and let it be represented by $k+1 \xrightarrow{f} l_q$ in B, where $f = M(f')$, $k \xrightarrow{f'} l_q - 1$ in A. The subobject represented by $M(P(f_q))f = M(P(f_q)f')$ is in $\overline{M}\left(A\begin{bmatrix} l \\ k \end{bmatrix}\right)$, and thus has color q by the coloring $\overline{cg\alpha}$. So $\overline{cg\alpha M(P(f_q))}$ colors all subobjects in $M\left(A\begin{bmatrix} l_q-1 \\ k \end{bmatrix}\right)$ color q. We also saw above that $\overline{cg\alpha M(P(f_q))}$ colors all subobjects in $\overline{\phi_{l_q-1,j}}\left(B\begin{bmatrix} l_q-1 \\ k+1 \end{bmatrix}\right)$ color q. But by Condition I, this accounts for all of $B\begin{bmatrix} l_q \\ k+1 \end{bmatrix}$, and hence $l_q \xrightarrow{g\alpha M(P(f_q))} x$ is the desired morphism, and the theorem is proved.

4. CONSEQUENCES

PROPOSITION 1. *Let $\mathscr{C}$ be a class of categories such that for each category B in $\mathscr{C}$ there is a category A in $\mathscr{C}$ such that A and B satisfy the conditions of Theorem 1. Then $B(k;l_1,\ldots,l_r)$ holds for all k, $l_1,\ldots,l_r$ and all B in $\mathscr{C}$.*

Proof. $B(-1;l_1,\ldots,l_r)$ holds vacuously for all $l_1,\ldots,l_r$, as observed at the beginning of the proof of Theorem 1. This holds for all B in $\mathscr{C}$. Thus for each B we can find a suitable A and apply Theorem 1 to obtain $B(0;l_1,\ldots,l_r)$ for all $l_1,\ldots,l_r$. Proceeding in this fashion from 0 to 1 to 2, etc., we obtain $B(k;l_1,\ldots,l_r)$ for all k, $l_1,\ldots,l_r$ and B in $\mathscr{C}$.

COROLLARY 1 (Ramsey). *Let C be the category with objects the nonnegative integers and morphisms $k \xrightarrow{f} l$ all the monomorphic functions from $\{1,\ldots,k\}$ into $\{1,\ldots,l\}$, where composition is just composition of functions. Then $C(k;l_1,\ldots,l_r)$ holds in general.*

Proof. We must find a class $\mathscr{C}$ containing C which satisfies the conditions of Proposition 1. For $\mathscr{C}$ choose the single category C itself. This clearly satisfies (a)-(c). So for A and B both equal to C, we must show that they satisfy the conditions of Theorem 1.

Let P be the identity functor on C. For any $k \xrightarrow{f} l$ in C, let $M(f)$ be the function $k+1 \xrightarrow{f'} l+1$ in C given by letting $f'(x) = f(x)$, $x \leqslant k$, and $f'(k+1) = l+1$. Let ϕ_l be the function from $\{1,\ldots,l\}$ to $\{1,\ldots,l+1\}$ which acts identically on $\{1,\ldots,l\}$. That is, $\phi_l(x) = x$ for $x \leqslant l$. Then we claim these choices, together with choosing $t = 1$ satisfy I-III.

Consider a subobject in $C\begin{bmatrix} l+1 \\ k+1 \end{bmatrix}$ represented by some $k+1 \xrightarrow{f} l+1$. First suppose $f(s) = l+1$ for some s. Then f represents the same subobject as $f_{\pi_{s,k+1}}$, where $\pi_{s,k+1}$ is the permutation of $\{1,\ldots,k+1\}$ fixing everything except s and $k+1$, which it interchanges. $\pi_{s,k+1}$ is an isomorphism and is its own inverse. Let $k \xrightarrow{f'} l$ be defined by letting $f'(x) = f\pi_{s,k+1}(x)$, $1 \leqslant x \leqslant k$. Then clearly $M(f') = f\pi_{s,k+1}$. Thus the subobject we chose is in $\overline{M}(C\begin{bmatrix} l \\ k \end{bmatrix})$. The only other subobjects are represented by some $k+1 \xrightarrow{f} l+1$ where $f(\{1,\ldots,k+1\}) \subset \{1,\ldots,l\}$. Then letting $k+1 \xrightarrow{f'} l$ be defined by $f'(x) = f(x)$, $1 \leqslant x \leqslant k+1$, we have $f = \phi_l f'$, and the subobject is in $\overline{\phi_l}(C\begin{bmatrix} l \\ k+1 \end{bmatrix})$. This establishes I. II is clear from the definitions. III follows by taking e to be ϕ_l, since $M(\phi_l)(x) = x$ for $1 \leqslant x \leqslant l$. This establishes Corollary 1. We note that if one examines the argument used in the proof of Theorem 1 for this special case, the usual proof of Ramsey's Theorem emerges.

Let V be an infinite-dimensional vector space over $GF(q)$ with basis $v_1, v_2, \ldots$. For each $k = 0, 1, \ldots$, let $V_k = \langle v_1, \ldots, v_k \rangle$, $V_0 = \langle 0 \rangle$. Let C be the category which has objects $0, 1, \ldots$, and morphisms $k \xrightarrow{\phi} l$, where ϕ is a linear monomorphism from V_k to V_l. Composition is ordinary composition of mappings. C clearly satisfies (a)-(c).

COROLLARY 2 (Vector Space Analog). *For the category C described above, $C(k; l_1, \ldots, l_r)$ holds in general.*

Proof. We apply Proposition 1 to a class containing C. Let A be an infinite-dimensional vector space over $GF(q)$ with basis $a_1, a_2, \ldots$, and let $A_m = \langle a_1, \ldots, a_m \rangle$, $A_0 = \langle 0 \rangle$. For $m = 0, 1, 2, \ldots$, the category C_m is defined as follows: The objects of C_m are $0, 1, 2, \ldots$, and the morphisms $k \xrightarrow{(w,\phi)} l$ are all pairs (w, ϕ) where $w \in A_m \otimes V_l$ and ϕ is a linear monomorphism from V_k to V_l. Let $k \xrightarrow{(w,\phi)} l$, where $w = \sum_{i=1}^m a_i \otimes w_i$, $w_i \in V_l$, and $l \xrightarrow{(x,\psi)} n$ be morphisms in C_m. Then their composition is defined to be $k \xrightarrow{(y,\psi\phi)} n$, where $y = x + \sum_{i=1}^m a_i \otimes \psi(w_i)$. Thus we can think of these morphisms as certain special affine transformations from $A_m \otimes V_k$ into $A_m \otimes V_l$. (a)-(c) are satisfied for the C_m. We choose for our class $\mathscr{C}$ all the C_m. When $m = 0$, we get the category C of Corollary 2.

For each m, let $B = C_m$ and $A = C_{m+1}$. We show that these satisfy Theorem 1. To define M, consider a morphism $k \xrightarrow{(w,\phi)} l$ in C_{m+1}. Then $w \in A_{m+1} \otimes V_l$ can be written uniquely as $w = w' + a_{m+1} \otimes w_{m+1}$, where $w' \in A_m \otimes V_l$. Let $\phi': V_{k+1} \to V_{l+1}$ be determined by letting $\phi'(v_{k+1}) = v_{l+1} + w_{m+1}$, and $\phi' = \phi$ on V_k. Then define $M((w, \phi)) = (w', \phi')$, where $k+1 \xrightarrow{(w',\phi')} l+1$ is in C_m. One can verify by a direct check that M preserves composition. We next define P. Let $k \xrightarrow{(w,\phi)} l$ be in C_m. Then $P((w, \phi)) = (w'', \phi'')$, where $w'' = w + a_{m+1} \otimes 0$, and $\phi'' = \phi$. Clearly P preserves composition. Also, since the identity morphism for k in C_m is $(0, 1_k)$, where 1_k is the identity transformation on V_k, and similarly for C_{m+1}, we see that $M(l) = l+1$ and $P(l) = l$ for each l. Finally, let $t = |A_m| = q^m$, and for each element $a \in A_m$ and each l let $\phi_{la} = (a \otimes v_{l+1}, e_l)$ in C_m, where e_l is the map from V_l to V_{l+1} acting identically on V_l. Then these choices are sufficient to satisfy I-III.

To check I, let $k+1 \xrightarrow{(w',\phi')} l+1$ represent a $(k+1)$-subobject of $l+1$ in C_m. First suppose $\phi'(V_{k+1}) \not\subset V_l$. Then we can choose some isomorphism $\psi: V_{k+1} \to V_{k+1}$ such that $\phi'\psi(V_k) \subset V_l$ and $\phi'\psi(v_{k+1}) = v_{l+1} + v'$ for some

$v' \in V_l$. Furthermore, for a suitable choice of $v \in A_m \otimes V_{k+1}$ we have $(w', \phi')(v, \psi) = (\hat{w}', \phi'\psi)$, with $\hat{w}' \in A_m \otimes V_l$. Of course (w', ϕ') and $(\hat{w}', \phi'\psi)$ represent the same subobject since (v, ψ) is an isomorphism. Now let $k \xrightarrow{(w,\phi)} l$ be in C_{m+1}, where $\phi = \phi\psi'$ on V_k, and $w = \hat{w}' + a_{m+1} \otimes v'$. Then we have $M((w, \phi)) = (\hat{w}', \phi'\psi)$. Thus all subobjects represented by a (w', ϕ') with $\phi'(V_{k+1}) \not\subset V_l$ are in $\overline{M}(C_{m+1}\left[\begin{smallmatrix} l \\ k \end{smallmatrix}\right])$. On the other hand, if $\phi'(V_{k+1}) \subset V_l$, then $(w', \phi') = (w'' + a \otimes v_{l+1}, \phi')$ for some $a \in A_m$ and some $w'' \in A_m \otimes V_l$. But

$$(w'' + a \otimes v_{l+1}, \phi') = (a \otimes v_{l+1}, e_l)(w'', \phi'') = \phi_{l,a}(w'', \phi''),$$

where $\phi'' = \phi'$ on V_{k+1}, $\phi'': V_{k+1} \to V_l$. Thus the subobject is in $\overline{\phi_{la}}(C_m\left[\begin{smallmatrix} l \\ k+1 \end{smallmatrix}\right])$. This establishes I.

To check II, let $s \xrightarrow{(w,\phi)} l$ in C_m. Then $M(P((w, \phi))) = (w', \phi')$, $s + 1 \xrightarrow{(w',\phi')} l + 1$, where $w' = w$ and ϕ' is the mapping determined by letting $\phi' = \phi$ on V_s, and $\phi'(v_{k+1}) = v_{l+1}$. Clearly

$$(a \otimes v_{l+1}, e_l)(w, \phi) = (w', \phi')(a \otimes v_{s+1}, e_s).$$

This establishes II.

Finally, for III, consider in C_{m+1} the morphism

$$l \xrightarrow{(a_{m+1} \otimes v_{l+1}, e_l)} l + 1.$$

$M((a_{m+1} \otimes v_{l+1}, e_l)) = (0, \psi')$, where ψ' acts identically on V_l, and $\psi'(v_{l+1}) = v_{l+2} + v_{l+1}$. Now we have for each $a \in A_m$,

$$(a \otimes v_{l+2}, e_{l+1})(a \otimes v_{l+1}, e_l) = (0, \psi')(a \otimes v_{l+1}, e_l).$$

This establishes III.

Thus $C_m(k; l_1, \ldots, l_r)$ holds in general for all m by Proposition I. In particular, as noted above, if $m = 0$, this establishes Corollary 2. We note also that for $m = 1$ the subobjects of an object l can be considered to be affine subspaces of V_l. Thus we have also proved the affine version of Ramsey's Theorem, which we state below.

COROLLARY 3 (Affine Analog). *For $C = C_1$ as described above, $C(k; l_1, \ldots, l_r)$ is true in general.*

The application of Theorem 1 to the case $A = C_1$, $B = C_0$ is just the statement that the affine analog for k and all $l_1, \ldots, l_r$ implies the vector space analog for $k + 1$ and all $l_1, \ldots, l_r$. This result was already known [1, 5], and the previous proof is the same as the proof of Theorem 1 specialized to this case. There was another way given in [5] to show that Corollary 3 implies Corollary 2. Namely, it shows that $C_1(k; l_1, \ldots, l_r)$ implies $C_0(k; l_1, \ldots, l_r)$. This argument is also a special case of Theorem 1, and we can describe it here.

Actually, we replace C_0 with the equivalent C_0', defined by letting $k \xrightarrow{f} l$ in C_0' if and only if $k - 1 \xrightarrow{f} l - 1$ is in C_0. We also must adjoin an identity 1_0 to C_0'. If $k \xrightarrow{(w,\phi)} l$ is in C_1, then $M((w, \phi)) = (0, \phi)$ in C_0', where we recall that $k + 1 \xrightarrow{(0,\phi)} l + 1$ in C_0'. We let $t = 0$, thus making the choices of P and ϕ_{lj} unnecessary. Clearly $C_0'\left[\begin{smallmatrix} l+1 \\ k+1 \end{smallmatrix}\right] = \overline{M}(C_1\left[\begin{smallmatrix} l \\ k \end{smallmatrix}\right])$, and I is satisfied. II is vacuously true as is III, since $t = 0$. Hence by Theorem 1, if $C_1'(k;l_1,\ldots,l_r)$ holds for all $l_1,\ldots,l_r$, then $C_0'(k+1;l_1,\ldots,l_r)$ holds and this is just $C_0(k;l_1,\ldots,l_r)$, as desired.

Finally we obtain the Ramsey theorem for n-parameter sets. We refer the reader to [2] to see that the definitions used there are essentially the same as those we will use here. In particular, the categories corresponding to the notions in [2] are the quotient categories described in the last paragraph in this paper. That is, the partially ordered sets of subobjects are isomorphic.

Let G be a finite group, and let $A = \{a_1,\ldots,a_{t_0}\}$ be a finite set. Let $C(A,G)$ be the category with objects $0, 1, 2, \ldots,$ and morphisms described as follows:

For each k and l, the morphisms $k \xrightarrow{(f,s)} l$ are diagrams

$$G \xleftarrow{s} \{1,\ldots,l\} \cup A \xrightarrow{f} \{1,\ldots,k\} \cup A,$$

where f is any epimorphic function which acts identically on A, and s is any function such that $s(a) = 1 \in G$ for $a \in A$. Composition of the morphisms $k \xrightarrow{(f,s)} l$ and $l \xrightarrow{(g,t)} m$ is given by $k \xrightarrow{(fg,\, sg\cdot t)} m$, where fg is ordinary composition of functions, and $sg \cdot t$ is defined by $s(g(x)) \cdot t(x) = (sg\cdot t)(x)$ in G for $x \in \{1,\ldots,m\} \cup A$.

We note several things about this choice for $C(A,G)$. First there is no mention of the relationship of G to A. G need not be a permutation group on A, nor even act on it at all. This was a necessary assumption for part of the proof in [2]. Second, we allow $|A| < 2$ here, where in [2], $|A| \geqslant 2$ was required. Actually, in the situation in [2] where the n-parameter sets under consideration had constant set $B \subset A$, we did not need $|B| \geqslant 2$. But this took a separate argument. What we have there is the general result for n-parameter sets for arbitrary sets of constants B.

COROLLARY 4 (n-Parameter Sets). *If* $C = C(A,G)$, *then* $C(k;l_1,\ldots,l_r)$ *holds in general.*

Proof. Again, we consider a class $\mathscr{C}$ containing $C(A,G)$ for which Proposition 1 holds. There is more than one possibility here. We will give the proof in detail for one class $\mathscr{C}$. Then we will describe another class but omit the detailed verification of I-III. It is this second class $\mathscr{C}$ which provides a more direct translation of the proof in [2]. The first $\mathscr{C}$ we describe now is somewhat different.

Let $\{a_, a_2, a_3,\ldots\}$ be an infinite set. For each $t = 1, 2, 3, \ldots,$ let $A_t = \{a_1,\ldots,a_t\}$, and let $C_t = C(A_t, G)$. Thus $C(A,G)$ above is C_{t_0} here. We claim that $A = C_{m+1}$ and $B = C_m$ satisfy Theorem 1, for all $m \geqslant 1$.

To see this we first define M. Let $k \xrightarrow{(f,s)} l$ be in C_{m+1}. Then $M((f,s)) = (f',s')$, where $k = 1 \xrightarrow{(f',s')} l+1$ in C_m is defined as follows. For $x \in A_m \cup \{1,\ldots,l\}$, $f'(x) = f(x)$ if $f(x) \in A_m \cup \{1,\ldots,k\}$, $f'(x) = k+1$ if $f(x) = a_{m+1}$, and $f'(l+1) = k+1$. For $x \in A_m \cup \{1,\ldots,l\}$, $s'(x) = s(x)$, and $s'(l+1) = 1$. One can check that M does preserve composition. For the identity map $(e_l, 1)$, l in C_{m+1}, where e_l acts identically on l and $1(x) = 1 \in G$, $x \in A_{m+1} \cup \{1,\ldots,l\}$, we see that $M((e_l,1)) = (e_{l+1}, 1)$ in C_m, so $M(l) = l+1$.

Next we define P. Let $k \xrightarrow{(h,r)} l$ be in C_m. Then $P((h,r)) = (h'',r'')$, where $k \xrightarrow{(h'',r'')} l$ in C_{m+1} is defined by letting $h''(x) = h(x)$ and $u''(x) = u(x)$

for $x \in A_m \cup \{1, \ldots, l\}$, and $h''(a_{m+1}) = a_{m+1}$, $r''(a_{m+1}) = 1 \in G$. P clearly preserves composition, and $P(l) = l$ for all l.

Finally, for each l and any $g \in G$ and any j, $1 \leqslant j \leqslant m$, let $\phi_{l,(j,g)} = (d_{jl}, 1_{gl})$, or just $(j,g)_l$ for short, where $d_{jl}(x) = x$ for $x \in \{1, \ldots, l\} \cup A_m$, $d_{jl}(l+1) = a_j$, and $1_{gl}(x) = 1 \in G$ for $x \in \{1, \ldots, l\} \cup A_m$, $1_{gl}(l+1) = g$. These ϕ's are indexed by the pairs (j,g). We let $t = |A_m|\ |G| = m\ |G|$, and for the choices above we verify I-III.

Let $k+1 \xrightarrow{(f,s)} l+1$ represent a subobject in $C_m \begin{bmatrix} k+1 \\ l+1 \end{bmatrix}$. Suppose first that $f(l+1) \notin A_m$. Let π be a permutation on $\{1, \ldots, k+1\} \cup A_m$ fixing all $a \in A_m$ and such that πf takes $l+1$ onto $k+1$. Let $u = (s(l+1))^{-1}$, and let $(f', s') = (f, s)(\pi, 1_{uk}\pi)$, where as above, 1_{uk} maps $\{1, \ldots, k\} \cup A_m$ onto $1 \in G$, and $k+1$ onto u. Then $f' = \pi f$ and $s' = 1_{uk} f \cdot s$. In particular, since $(\pi, 1_{uk}\pi)$ is an isomorphism in C_m (its inverse is $(\pi^{-1}, 1_{u^{-1}k})$), we see that (f, s) and (f', s') represent the same subobject of $l+1$. Now let $k \xrightarrow{(f'', s'')} l$ be defined in C_{m+1} as follows. For $x \in A_m \cup \{1, \ldots, l\}$, we let $f''(x) = f'(x)$ if $f'(x) \neq k+1$, and $f''(x) = a_{m+1}$ if $f'(x) = k+1$. We let $f''(a_{m+1}) = a_{m+1}$. For $x \in A_m \cup \{1, \ldots, l\}$, we let $s''(x) = s'(x)$, and $s''(a_{m+1}) = 1$. Then $M((f'', s'')) = (f', s')$. So the subobject represented by (f, s) is in $\overline{M}(C_{m+1} \begin{bmatrix} l \\ k \end{bmatrix})$. This is the case, then, for any (f, s) with $f(l+1) \notin A_m$. On the other hand, suppose $f(l+1) = a_j \in A_m$. Let $k+1 \xrightarrow{(f', s')} l$ in C_m be defined by $f'(x) = f(x)$ and $s'(x) = s(x)$ for $x \in \{1, \ldots, l\} \cup A_m$. Then $(f, s) = (j, s(l+1))_l (f', s')$ and (f, s) represents a subobject in $\overline{(j, s(l+1))_l}(C_m \begin{bmatrix} l \\ k+1 \end{bmatrix})$. This establishes I.

For II, we note that for $k \xrightarrow{(f,s)} l$ in C_m, $M(P((f,s)))$ is the morphism $k+1 \xrightarrow{(f', s')} l+1$ in C_m, where $f'(x) = f(x)$ and $s'(x) = s(x)$ for $x \in \{1, \ldots, l\} \cup A_m$, and $f'(l+1) = k+1$, $s'(l+1) = 1$. Then for each j and g we see that $(j,g)_l (f,s) = (f', s')(j,g)_k$, establishing II.

To verify III, we consider $(m+1, 1)_l$ in C_{m+1}. Then $M((m+1, 1)_l)$ is the morphism $l+1 \xrightarrow{(e', 1)} l+2$ in C_m where $1(x) = 1$ for all x in $\{1, \ldots, l+2\} \cup A_m$ and $e'(x) = x$ for $x \in \{1, \ldots, l\} \cup A_m$, and $e'(l+1) = l+1$, $e'(l+2) = l+1$. Then clearly $(j,g)_{l+1}(j,g)_l = (e', 1)(j,g)_l = M((m+1,1)_l)(j,g)_l$. This establishes III and completes the proof of Corollary 4.

The alternate choice for the class $\mathscr{C}$ to prove Corollary 4 is as follows. For each $m = 0, 1, 2, \ldots$, let $A'_m = A \cup (\{1, \ldots, m\} \times G)$, and let $C'_m = C(A'_m, G)$. Then $C'_0 = C$. Let $\mathscr{C}$ be the class of all C'_m. For each m, C'_{m+1} and C'_m satisfy Theorem 1.

For $k \xrightarrow{(f,s)} l$ in C'_{m+1}, we let $M((f,s)) = (f', s')$, where for $x \in A'_m \cup \{1, \ldots, l\}$ we let

$$f'(x) = f(x) \text{ and } s'(x) = s(x) \text{ if } f(x) \in A'_m \cup \{1, \ldots, k\};$$

for $f(x) = (m+1, g)$, we let $f'(x) = k+1$, $s'(x) = g \cdot s(x)$; and $f'(l+1) = k+1$, $s'(l+1) = 1$. For $k \xrightarrow{(f,s)} l$ in C'_m we define $P((f,s)) = (f', s')$ in C'_{m+1} by letting $f'(x) = f(x)$, $s'(x) = s(x)$ if $x \in A'_m \cup \{1, \ldots, l\}$, and $f'((m+1, g)) = (m+1, g)$, $s'((m+1, g)) = 1$. For $a \in A'_m$ and $g \in G$, as before, we let $\phi_{l,(a,g)} = (d_{al}, 1_{gl})$. Then I, II and III can be verified, with $t = |A'_m|\ |G|$.

Now we still do not have an exact translation of the proof in [2]. In particular, we have taken no account of any action of G on A. To handle this we consider a set A and a group G acting on A, $a \to a^g \in A$ for $g \in G$. We consider the category $C(A, G)$ and obtain from it the category $\overline{C(A,G)}$ by identifying any two morphisms $k \to^{(f,s)} l$ and $k \to^{(g,u)} l$ for which $f(x) = g(x)$ and $s(x) = u(x)$ if $f(x) \in \{1, \ldots, k\}$, and $f(x)^{s(x)} = g(x)^{u(x)}$ otherwise. By considering G to act on $(\{1, \ldots, m\} \times G)$ by $(i, g)^h = (i, gh)$ for all $h \in G$, we obtain the categories $\overline{C'_m} = \overline{C(A'_m, G)}$. The categories $\overline{C'_{m+1}}$ and $\overline{C'_m}$ satisfy Theorem 1, where we take for M and P the functors determined by the M and P for C'_{m+1} and C'_m above by their action on classes of identified morphisms. For the ϕ's we use classes of identified $\phi_{l,(a,g)}$ from above. There are $|A'_m|$ of these, represented by the $\phi_{l,(a,1)}$. Thus we let $t = |A'_m|$ here. Letting $\mathscr{C}$ be the class consisting of all $\overline{C'_m}$, we can apply Proposition 1. This is the exact translation of the proof in [2].

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