NUMBERS IN RAMSEY THEORY

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1. Introduction

Ramsey theory can be loosely described as the study of structure which is preserved under finite decomposition. Its underlying philosophy is captured succinctly by the statement "Complete disorder is impossible". Since the publication of the seminal paper of Ramsey [R30] in 1930, the subject has grown with increasing vitality, and is currently among the most active areas in combinatorics. However, most of the recent work has focussed on far-ranging generalizations of the original concepts, dealing for example, with extensions to n-dimensional vector spaces, (and their cominatorial analogues, n-parameter sets), lattices, groups, various transfinite sets, induced and restricted variations, etc. (cf. [GRS80], [G83], [NR78], [NR79]). Progress on determining the basic numbers themselves, the so-called Ramsey numbers $r(k, \ell)$, has been painfully slow. We recall that for positive integers k and ℓ, $r(k, \ell)$ denotes the *least* integer n such that if the edges of the complete graph K_n are arbitrarily 2-colored, say with colors red and blue, then there is always formed either a K_k with all edges red, or a K_ℓ with all edges blue. (The existence of $r(k, \ell)$ is guaranteed by Ramsey's Theorem.) For example, the first lower bound for $r(k, k)$, due to Erdös [E47], was published in 1947:

$$r(k, k) > (1 + o(1)) \frac{1}{e\sqrt{2}} \cdot k \cdot 2^{k/2}. \tag{1.1}$$

In the 40 years since its proof, this bound has only been improved by a factor of 2!

However, some breakthroughs have occurred recently, such as the first significant improvement on the upper bound

$$r(k, k) \leqslant \binom{2k-2}{k-1},$$

originally given by Erdös and Szekeres [ES35] in 1935. It is now known (cf.[R]) that for suitable positive constants c_1 and c_2,

$$r(k, k) < c_1 \binom{2k-1}{k-1} / (\log k)^{c_2}, \tag{1.2}$$

(where all logarithms in this paper are to the base e). We will give a proof (Theorem 2.14) that

$$r(k, k) < 6 \binom{2k-2}{k-1} / \log\log k \tag{1.3}$$

for k sufficiently large.

In this paper, we will attempt to describe the current state of knowledge in this and some related areas. We will usually not give the arguments which lead to the sharpest bounds known, but rather we will concentrate on outlining some of the basic methods needed for the various improvements discussed. We will, however, try to provide appropriate references to the sharper results when we can.

2. Asymptotic bounds on classical problems

In this section we will review a few basic bounds on the numbers $r(k, \ell)$ and then outline some very recent improvements.

2.1 Lower bounds

Theorem 2.1 [E47]

$$r(k, k) > (1 + o(1)) \frac{1}{e\sqrt{2}} k \cdot 2^{k/2}.$$

Proof: We show that if

$$\binom{n}{k} 2^{1-\binom{k}{2}} < 1 \tag{2.1}$$

then in fact $r(k, k) > n$, i.e., there exists a 2-coloring of K_n containing no monochromatic K_k. (This is clearly sufficient since a simple calculation shows that if (2.1) fails then

$$n > \left[\frac{1}{e\sqrt{2}} + o(1)\right] k \cdot 2^{k/2}.$$

Consider a random 2-coloring of K_n, where the color of each edge is chosen independently with probability 1/2. Let T be a subset of the vertex set of size $|T| = k$. The probability of the event A_T that the set $[T]^2$ of pairs of T is monochromatic is

$$Pr(A_T) = 2^{1-\binom{k}{2}}.$$

Thus, the probability of the event that for some k-element set T of vertices, the set $[T]^2$ is monochromatic, is bounded above by

$$\binom{n}{k} 2^{1-\binom{k}{2}}.$$

Hence, if (2.1) holds, then there must exist a 2-coloring of the edges of K_n having no monochromatic K_k. ■

Note that this proof, which was one of the earliest uses of the "probabilistic method", gives no information about how such a 2-coloring might actually be constructed. The best constructive lower bounds currently known (see (2.12)) are much weaker.

The bound in Theorem 2.1 has been improved slightly by Spencer [Sp75] to

$$r(k, k) > (1 + o(1)) \frac{\sqrt{2}}{e} k \cdot 2^{k/2}, \tag{2.2}$$

which is at present the sharpest known.

The proof of (2.2) is based on an ingenious probabilistic inequality of Lovász [EL75] known as the "local lemma". The setting for this result is the following. Let Ω be a probability space and let $A_1, A_2, \ldots, A_n$ be events in Ω. We say that the graph Γ with vertex set $\{1, 2, \ldots, n\}$ is a *dependence graph* of $\{A_1, A_2, \ldots, A_n\}$ if:

$$\{i\} \text{ not joined to } J_1, J_2, \ldots, J_s \Rightarrow A_i \text{ and } A_{J_1} \cap A_{J_2} \cap \cdots \cap A_{J_s} \text{ are independent}.$$

Theorem 2.2 (Lovász local lemma). Let $A_1, A_2, \ldots, A_n$ be events in a probability space Ω with dependence graph Γ. Suppose there exist $x_1, x_2, \ldots, x_n$ with $0 < x_i \leqslant 1$ such that

$$Pr(A_i) \leqslant (1 - x_i) \prod_{\{i,j\} \in \Gamma} x_j, \quad 1 \leqslant i \leqslant n.$$

Then

$$Pr(\bigcap_i \bar{A}_i) > 0\,.$$

The proof of this version can be found in [Sp77]. A slightly more convenient form of the local lemma results from the following observation. Set

$$y_i = \frac{1 - x_i}{x_i Pr(A_i)}$$

so that

$$x_i = \frac{1}{1 + y_i Pr(A_i)}\,.$$

Since $1 + z \leqslant \exp(z)$, we have

Corollary 2.3.

Suppose $A_1, A_2, \ldots, A_n$ are events in a probability space having dependence graph Γ, and there exist positive $y_1, y_2, \ldots, y_n$ satisfying

$$\log y_i > \sum_{\{i,j\} \in \Gamma} y_j Pr(A_j) + y_i Pr(A_i)$$

for $1 \leqslant i \leqslant n$. Then

$$Pr(\bigcap_i \bar{A}_i) > 0\,.$$

We next illustrate the use of this result in deriving a lower bound for $r(k, 3)$. The first bound on $r(k, 3)$, again due to Erdös [E61], was that $r(k, 3) > ck^2/(\log k)^2$ for a suitable $c > 0$. The following result gives the sharpest known bound currently known.

Theorem 2.4 [Sp77]

$$r(k, 3) \geqslant \left(\frac{1}{27} + o(1)\right) k^2/(\log k)^2\,. \tag{2.3}$$

Proof: The proof is a modification of that of Spencer [Sp77]. Let the edges of K_n be independently 2-colored red and blue with the probability that an edge is colored red always being p. To each 3-element subset of vertices S associate the event A_S that all the edges spanned by S have been colored red. Similarly, to each k-element subset K associate the event B_K that all the edges spanned by K have been colored blue. Observe that

$$r(k, 3) > n \;\text{ if }\; Pr(\bigcap_S \bar{A}_S \cap \bigcap_K \bar{B}_K) > 0\,.$$

Let Γ denote the graph with $\binom{n}{3} + \binom{n}{k}$ vertices corresponding to all possible A_S and B_K, where $\{A_S, B_K\}$ is an edge of Γ if and only if $|S \cap K| \geqslant 2$ (i.e., the events A_S and B_K are dependent), and the same applies to pairs of the form $\{A_S, A_{S'}\}$ and $\{B_K, B_{K'}\}$. Let N_{AA} denote the number of vertices of the form A_S for some S joined to some other vertex of this form (so that $N_{AA} = 3(n-2)$), and let N_{AB}, N_{BA} and N_{BB} be defined analogously. In this case, Corollary 2.3 implies:

If there exist positive p, y, z such that:

$$p < 1 ,$$

(2.4)
$$\log y > yPr(A_S)(N_{AA} + 1) + zPr(B_K)N_{AB} ,$$
$$\log z > yPr(A_S)N_{BA} + zPr(B_K)(N_{BB} + 1)$$

then

$$r(k, 3) > n .$$

Now,

$$Pr(A_S) = p^3 , Pr(B_K) = (1 - p)^{\binom{k}{2}} \leq \exp\left\{-p\binom{k}{2}\right\} .$$

Also, we have the bounds

$$N_{AB} \leq \binom{n}{k} < \left(\frac{ne}{k}\right)^k , N_{BB} + 1 < \left(\frac{ne}{k}\right)^k ,$$

$$N_{AA} + 1 < 3n , \quad N_{BA} = \binom{k}{2}(n - k) + \binom{k}{3} < \frac{1}{2} k^2 n .$$

Set

$$p = c_1 n^{-1/2} , \qquad k = c_2 n^{1/2} \log n$$

$$z = \exp(c_3 n^{1/2}(\log n)^2) , \quad y = 1 + \epsilon$$

where c_1, c_2, c_3, ϵ are positive constants to be specified later. We now verify (2.4).

First, observe that

$$zPr(B_K) \max\{N_{AB}, N_{BA}\} < zPr(B_K)\left(\frac{ne}{k}\right)^k$$

$$< \exp\left\{n^{1/2}(\log n)^2\left[c_3 + c_2\left(\frac{1 - c_1 c_2}{2}\right) + o(1)\right]\right\} .$$

Thus, if

(2.5)
$$c_3 + c_2\left(\frac{1 - c_1 c_2}{2}\right) < 0$$

then the second line of (2.4) will hold for k large.

In a similar way, if in addition

(2.6)
$$(1 + \epsilon)c_1^3 c_2^2 < 2c_3$$

then the last line of (2.4) will hold. Finally, by choosing

$$c_1 = \frac{1}{\sqrt{3}} + o(1) , c_2 = 3\sqrt{3}/2 + o(1) , c_3 = \frac{3\sqrt{3}}{8} + o(1) , \epsilon = o(1)$$

appropriately, an easy calculation shows that (2.5) and (2.6) hold. This implies

$$k = \left(\frac{3\sqrt{3}}{2} + o(1)\right) n^{1/2} \log n$$

which in turn implies (2.3). ■

We remark that in the same way the following more general result can be established.

Theorem 2.5 [Sp77]

Let $\ell \geqslant 3$ be fixed. Then for a suitable constant $c = c(\ell) > 0$,

$$r(k, \ell) > c(k/\log k)^{(\ell+1)/2} \tag{2.7}$$

The best available lower bound for general k and ℓ comes from the following result (by using the probability method).

Theorem 2.6 [Sp75]

If for some $p \in (0,1)$ we have

$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{\ell} (1-p)^{\binom{\ell}{2}} < 1 \tag{2.8}$$

then $r(k,\ell) > n$.

In particular, it follows from Theorem 2.6 that

$$r(k,\ell) > c\binom{k+\ell-2}{k-1}^{1/4} \tag{2.9}$$

for an absolute constant $c > 0$. Erdös conjectures that for fixed ℓ,

$$r(k, \ell) > k^{\ell-1}/(\log k)^{c_\ell}$$

for a suitable constant $c_\ell > 0$ and k sufficiently large.

2.2 Constructive lower bounds

In the preceding sections, all of the bounds given were based on the use of the probability method. As a consequence, the proofs do not produce any explicit colorings but rather, they only prove that such colorings exist. To remedy this not entirely satisfactory state of affairs, attempts have been made over the years to construct good colorings, unfortunately without much success. For the case of $r(k, 3)$, Erdös [E66] has given a construction which shows that

$$r(k, 3) > k^{\alpha+o(1)}$$

where

$$\alpha = \frac{2}{3}\left(\frac{\log 2}{\log 3 - \log 2}\right) = 1.139\ldots,$$

improving an earlier construction of his which gave a somewhat weaker result (cf. [E57]). A breakthrough for $r(k, k)$ finally occurred several years ago, however, with the result of Frankl [F77], who gave the first Ramsey construction which grew faster than any polynomial. In this section, we will outline a more recent theorem of Frankl and Wilson [FW81] on set intersections which yields the best constructive bound for $r(k, k)$ currently available.

Theorem 2.7 [FW81]

Suppose $\mathscr{F}$ is a family of k-sets of $\{1, 2, \ldots, n\}$ such that for some prime power q,

$$F, F' \in \mathscr{F}, F \neq F' \Rightarrow |F \cap F'| \not\equiv k \pmod{q}.$$

Then

$$|\mathscr{F}| \leqslant \binom{n}{q-1}. \tag{2.10}$$

Proof: Let $A_1, A_2, \ldots, A_{\binom{n}{j}}$ be all the j-element subsets and $B_1, B_2, \ldots, B_{\binom{n}{i}}$ be all the i-element subsets of $\{1, 2, \ldots, n\}$, where $i < j$. Define the $\binom{n}{i}$ by $\binom{n}{j}$ matrix $N(i, j)$ as follows: the (u, v)-entry of $N(i, j)$ is 1 if $B_u \subset A_v$, and 0 if $B_u \not\subset A_v$, for $1 \leqslant u \leqslant \binom{n}{i}$, $1 \leqslant v \leqslant \binom{n}{j}$. Let V denote the vector space (over $\mathbf{R}$) generated by the row vectors $V_1, V_2, \ldots, V_{\binom{n}{q-1}}$ of $N(q-1, k)$. Of course

$$\dim V \leqslant \binom{n}{q-1}. \tag{2.11}$$

It is easy to check that

$$N(i, q-1)N(q-1, k) = \binom{k-i}{q-1-i} N(i, k).$$

Thus, for $0 \leqslant i < q-1$, the row vectors of $N(i, k)$ belong to V.

Consider the product

$$M(i, k) := N(i, k)^t N(i, k).$$

The (u, v)-entry of the $\binom{n}{k}$ by $\binom{n}{k}$ matrix $M(i, k)$ is $\binom{|A_u \cap A_v|}{i}$ for $1 \leqslant u, v \leqslant \binom{n}{k}$. Moreover, the row vectors of $M(i, k)$ are linear combinations of the rows of $N(i, k)$, and consequently, belong to V.

Now, choose real numbers a_i, $0 \leqslant i < q$, so that

$$\sum_{i=0}^{q-1} a_i \binom{x}{i} = \binom{x-k-1}{q-1}.$$

Set $M := \sum_{i=0}^{q-1} a_i M(i, k)$ where addition is performed componentwise, i.e., the (u, v)-entry of M is

$$m(u, v) = \sum_{i=0}^{q-1} a_1 \binom{|A_u \cap A_v|}{i}.$$

It follows from the definition that the row vectors of M are in V, and consequently

$$\operatorname{rank} M \leqslant \dim V.$$

Let $M(\mathscr{F})$ denote the minor of M spanned by the elements $m(u, v)$ for which $A_u, A_v \in \mathscr{F}$. Since for $q = p^\alpha$, p prime and $\alpha \geqslant 1$, we have

$$\binom{a}{q-1} \equiv 0 \pmod{p} \text{ iff } a \not\equiv -1 \pmod{q}$$

then it follows that

$$m(u, v) \equiv 0 \pmod{p} \quad \text{for } u \neq v, \text{ and}$$

$$m(u, u) \not\equiv 0 \pmod{p}.$$

Thus, $\det M(\mathscr{F}) \not\equiv 0 \pmod{p}$ and so,

$$|\mathscr{F}| = \text{rank } M(\mathscr{F}) \leqslant \text{rank } M \leqslant \binom{n}{q-1}$$

as required. ■

Corollary 2.8.

Let q be a prime power and form the graph $G = G(n, q)$ with vertex set

$$V(G) = \{F \subseteq \{1, 2, \dots, n\} : |F| = q^2 - 1\}$$

and edge set

$$E(G) = \{\{F, F'\} : |F \cap F'| \not\equiv -1 \pmod q\} .$$

Then neither G nor its complement G' contains a complete subgraph on more than $\binom{n}{q-1}$ vertices.

Proof: If $F_1, F_2, \dots, F_m$ forms a complete subgraph in $\overline{G}$ then $|F_i \cap F_j| \not\equiv -1 \pmod q$ for every $i \neq j$, and so, by Th. 2.7, we have the desired conclusion. If $F_1, F_2, \dots, F_m$ forms a complete subgraph in $\overline{G}$ then

$$|F_i \cap F_j| \in \{q-1, 2q-1, \dots, q^2 - q - 1\}$$

for $i \neq j$. However, we can now apply the following result of Ray-Chaudhuri and Wilson [RW75] (which can be proved along the lines of Theorem 2.7):

If $\mathscr{F}$ is a family of k-sets of $\{1, 2, \dots, n\}$ and

$$s := |\{|F \cap F'| : F, F' \in \mathscr{F}, F \neq F'\}|$$

then

$$|\mathscr{F}| \leqslant \binom{n}{s} .$$

Thus,

$$m \leqslant \binom{m}{q-1}$$

in this case as well, and the Corollary is proved. ■

Choosing $n = p^3$, $q = p$, we obtain

$$r(k, k) \geqslant \exp((1 + o(1)) \log^2 k / 4 \log\log k) \tag{2.12}$$

which is the best constructive lower bound on $r(k, k)$ currently known.

A graph which has often been suggested as a natural candidate for giving an exponentially large (constructive) Ramsey bound is the following. Let p be a prime which is congruent to 1 modulo 4. Form the graph G_p with vertex set $\boldsymbol{Z}_p$ (the integers modulo p) and edge set $\{\{i, j\} : 0 \neq i - j$ is a quadratic residue modulo $p\}$. Since $p \equiv 1 \pmod 4$, $\{i, j\}$ is an edge *iff* $\{j, i\}$ is. Recent results of Shearer [Sh86] show that for small p, bounds obtained for $r(k, k)$ by these techniques are not too bad (see also Mathon [Ma87]). It is known that G_p shares many properties with a random graph of the same size (e.g., see [GS71], [BT81]). However, it can be shown that infinitely often G_p will contain a complete subgraph of size $c \log p \log\log p$.

2.3 Upper bounds

The earliest upper bounds on $r(k, \ell)$, due to Erdös and Szekeres [ES35], follows from the (immediate) recurrence

$$r(k,\ell) \leqslant r(k-1,\ell) + r(k,\ell-1)\,. \tag{2.13}$$

This leads at once to the upper bound

$$r(k,\ell) \leqslant \binom{k+\ell-2}{k-1} \tag{2.14}$$

and, in the special case that $k = \ell$,

$$r(k,k) \leqslant \binom{2k-2}{k-1} \sim c \cdot 4^k/\sqrt{k}\,. \tag{2.15}$$

For fixed values of ℓ, these bounds have been improved by Graver and Yackel [GY68] to

$$r(k,\ell) \leqslant c\,\frac{k^{\ell-1}\log\log k}{\log k}\,,$$

although a widely quoted upper bound for large values of k and ℓ, published by Yackel [Y72] now appears to be flawed (cf. [LSW]). In this section we present several recent improved estimates on $r(k,\ell)$. The bound on $r(k,3)$ (Theorems 2.9, 2.10) and its striking proof is due to Shearer [Sh83], and is based on an earlier argument of Ajtai, Komlós and Szemerédi [AKS80]. We point out that the ideas introduced in [AKS80] have also been applied very effectively by these authors to several other problems as well (cf. [AKS81a], [AKS81b]). The general bound on $r(k,\ell)$ in Theorem 2.11 is due to the second author and represents the first significant improvement on (2.14) in the 50 years since it appeared. In particular, it implies that

$$r(k,\ell) = o\left(\binom{k+\ell-2}{k-1}\right)$$

as $k+\ell \longrightarrow \infty$. Note that (2.13) does not imply $r(k,k) = o\left(\binom{2k}{k}\right)$, even assuming the (unrealistic) boundary conditions $r(j,3) = 1$ for all j.

Theorem 2. 9 [Sh83]

Let G be a triangle-free graph on n vertices with average degree d. Let $\alpha = \alpha(G)$ be the independence number of G and define

$$f(d) := (d\log d - d + 1)/(d-1)^2\,,\, f(0) = 1\,,\, f(1) = 1/2\,.$$

Then

$$\alpha \geqslant nf(d)\,. \tag{2.16}$$

Proof: First note that f is continuous for $0 \leqslant d < \infty$ and that $0 < f(d) < 1$, $f'(d) < 0$, $f''(d) > 0$. Further, f satisfies

$$(d+1)f(d) = 1 + (d-d^2)f'(d)\,. \tag{2.17}$$

We will prove (2.16) by induction on n. For $n \leqslant d/f(d)$, the theorem clearly holds since the neighbors of any vertex of G must form an independent set, and consequently

$$\alpha \geqslant d \geqslant nf(d)\,.$$

For the vertex v in G, let $d_1 = d_1(v)$ denote the degree of v. Also, let $d_2 = d_2(v)$ denote the average degree of the neighbors of v. We claim that we can always find a vertex v so that

$$(d_1+1)f(d) \leqslant 1 + (dd_1 + d - 2d_1d_2)f'(d) \tag{2.18}$$

holds. To see this, note that the average value of d_1d_2 is the same as the average value of d_1^2, and is at least d^2. Thus, the average value of the RHS of (2.18) is at least as large as the average value of the RHS of (2.17). Since, the average value of the LHS of (2.18) is equal to the LHS of (2.17), then (2.18) holds on the average and consequently, the desired vertex v must exist.

Now, let G^* be formed from G by deleting v and all its neighbors. Of course, G^* is also triangle-free, and has $n - d_1 - 1$ vertices and $\frac{1}{2} nd - d_1 d_2$ edges. Let $d^* = (nd - 2d_1 d_2)/(n - d_1 - 1)$ be the average degree in G^*. By the induction hypothesis, G^* contains an independent set of size $(n - d_1 - 1)f(d^*)$. By adding v back to this set, we obtain an independent set in G of size

$$1 + (n - d_1 - 1)f(d^*) .$$

Since $f''(d) \geqslant 0$ for $0 < d < \infty$ we have

$$f'(d) \geqslant f(d) + (d^* - d)f'(d) .$$

Therefore,

$$\begin{aligned} 1 + (n - d_1 - 1)f(d^*) &\geqslant 1 + (n - d_1 - 1)f(d) + (n - d_1 - 1)(d^* - d)f'(d) \\ &\geqslant 1 + (n - d_1 - 1)f(d) + (dd_1 + d - 2d_1 d_2)f'(d) \\ &\geqslant (n - d_1 - 1)f(d) + (d_1 + 1)f(d) \qquad \text{by (2.18)} \\ &= nf(d) . \end{aligned}$$

Thus, $\alpha \geqslant nf(d)$ as required, and the theorem is proved. ∎

As an immediate consequence, we have:

Theorem 2.10

(2.19) $$r(k, 3) < k^2/\log(k/e) .$$

Proof: Let G be a triangle-free graph on n vertices and suppose $\alpha(G) \leqslant k$. Since the neighbors of any vertex v form an independent set then we must have degree $(v) \leqslant k$. Thus, the *average* degree d in G is at most k. Therefore, by (2.16)

(2.20) $$k \geqslant \alpha(G) \geqslant nf(d) \geqslant nf(k) .$$

Hence, if $n > k/f(k)$ then G must contain an independent set of size $k + 1$. This implies

(2.21) $$R(k + 1, 3) \leqslant 1 + k/f(k)$$

which in turn implies (2.19). ∎

Theorem 2.9 is a slight extension of an earlier result of Ajtai, Komlós and Szemerédi [AKS80] who proved

$$\alpha(G) \geqslant n/(100d \log d)$$

for any triangle-free graph G with n vertices and average degree d. If 3 is replaced by an arbitrary but fixed value ℓ, then the best bound on $r(k, \ell)$ is given by the following result of Ajtai, Komlós and Szemerédi:

Theorem 2.11 [AKS80]

(2.22) $$r(k, \ell) \leqslant (5000)^{\ell} k^{\ell - 1}/(\log k)^{\ell - 2}$$

for k sufficiently large (depending on ℓ).

It will be convenient for the next result to define the related quantity $r^*(k, \ell)$, the largest value of n such that there is a red-blue coloring of the edges of K_n having no red K_k and no blue K_ℓ. Thus,

$$r^*(k, \ell) = r(k+1, \ell+1) - 1 .$$

For arbitrary k and ℓ (not satisfying $k >> \ell$ required by Theorem 2.11), the best current upper bound is given by a recent result of the second author:

Theorem 2.12 [R]

$$r^*(k, \ell) \leq c_1 \binom{k+\ell}{k} \Big/ (\log(k+\ell))^{c_2} \tag{2.23}$$

for suitable positive constants c_i. The proof of Theorem 2.12 is based on the following result, which is of independent interest.

Theorem 2.13

If $\ell = \log k$ then

$$r^*(k, \ell) \leq \binom{k+\ell}{k} \Big/ (k+\ell)^c \tag{2.24}$$

for a suitable constant $c > 0$.

Because of space limitations, we will not give the proof of (2.23). Rather, we will illustrate the method used in proving it by proving the following weaker result.

Theorem 2.14

$$r^*(k, \ell) < 6 \binom{k+\ell}{k} \Big/ \log\log(k+\ell) \tag{2.25}$$

for $k + \ell$ sufficiently large.

Proof: The proof of (2.25) will require the use of several auxiliary lemmas. We first establish these.

Lemma 2.15 [Go59] Let G be a graph with n vertices and $\beta \binom{n}{2}$ edges, and let $T(X)$ denote the number of triangles in a graph X. Then

$$T(G) + T(\bar{G}) \geq (\beta^3 + (1-\beta)^3) \binom{n}{3} - \beta(1-\beta)\binom{n}{2} . \tag{2.26}$$

Proof: Let $d_1, d_2, \ldots, d_n$ be the degrees of the vertices of G. Then

$$\begin{aligned} T(G) + T(\bar{G}) &= \binom{n}{3} - \frac{1}{2}\sum_i (n - d_i - 1)d_i \\ &= \binom{n}{3} - \frac{1}{2}\sum_i ((n-1)d_i - d_i^2) \\ &\geq \frac{1}{6}(n(n-1)(n-2) - 3\beta n(n-1)^2 + 3\beta^2 n(n-1)^2) \\ &= (\beta^3 + (1-\beta)^3 \binom{n}{3} - \beta(1-\beta)\binom{n}{2} . \quad \blacksquare \end{aligned}$$

Lemma 2.16 Suppose m and n satisfy $m > e^2 n > 0$.

Then

(2.27) $$r^*(m, n) \leqslant 5 \binom{m+n}{n} \Big/ \log(m/e^2 n) .$$

Proof: By applying (2.13) repeatedly, we obtain

(2.28) $$r(m+1, n+1) \leqslant \sum_{j=0}^{n} \binom{m+n-j-3}{n-3} r(j+1, 3)$$

Let $s := m + n$ and $t := \lfloor (s/n)^{1/2} \rfloor - 1$. We break the sum on the RHS of (2.28) into two parts.

$$S_1 := \sum_{j \leqslant t} \binom{s-j-3}{n-3} r(j+1, 3) \leqslant \binom{s-3}{n-3} \sum_{j \leqslant t} \binom{j+2}{2}$$

(2.29) $$\leqslant \binom{s}{n} \left(\frac{n}{s}\right)^3 \binom{t+3}{3} .$$

Also, by (2.19)

$$S_2 := \sum_{j=t+1}^{s} \binom{s-j-3}{n-3} r(j+1, 3) \leqslant \sum_{j=t+1}^{s} \binom{s-j-3}{n-3} (j+1)^2 \Big/ \log\left(\frac{j+1}{e}\right)$$

$$\leqslant \frac{4}{\log(s/e^2 n)} \sum_{j \geqslant 0} \binom{s-j-3}{n-3} \binom{j+2}{2}$$

$$= \frac{4}{\log(s/e^2 n)} \left\{ \binom{n-1}{2} \binom{s}{n} - (s-2)(n-2) \binom{s-1}{n-1} + \binom{s-2}{2} \binom{s-2}{n-2} \right\}$$

$$= \frac{4}{\log(s/e^2 n)} \left(1 - \frac{n(2s-n-1)}{s(s-1)}\right) \binom{s}{n}$$

$$< \frac{4}{\log(s/e^2 n)} \cdot \binom{s}{n} .$$

Thus, using the fact that

$$\left(\frac{n}{s}\right)^3 \binom{t+3}{3} < 1/\log\,(s/e^2 n) ,$$

we obtain

$$r^*(m, n) < r(m+1, n+1) < S_1 + S_2 < 5 \binom{m+n}{n} \Big/ \log(m/e^2 n)$$

for $m > e^2 n$. ∎

Lemma 2.17 Suppose for some positive m and n,

(2.30) $$r^*(m, n) \geqslant 16(m+n)^2 .$$

Let $s := m + n$ and $\epsilon := \frac{1}{8s} \min(m, n)$. Then at least one of the following holds:

(i) $$r^*(m-1,n) \geqslant \frac{m}{s}\left(1+\frac{\epsilon}{s}\right) r^*(m,n);$$

(ii) $$r^*(m,n-1) \geqslant \frac{n}{s}\left(1+\frac{\epsilon}{s}\right) r^*(m,n);$$

(iii) $$r^*(m-2,n) \geqslant \frac{m(m-1)}{s(s-1)}\left(1+\frac{\epsilon}{s}\right)\left(1+\frac{\epsilon}{s-1}\right) r^*(m,n);$$

(iv) $$r^*(m,n-2) \geqslant \frac{n(n-1)}{s(s-1)}\left(1+\frac{\epsilon}{s}\right)\left(1+\frac{\epsilon}{s-1}\right) r^*(m,n).$$

Proof: Set $N := r^*(m,n)$ and let K_N be 2-colored so that no red K_{m+1} and no blue K_{n+1} is formed. Let $d_i^{(r)}$ denote the number of red edges incident to the ith vertex of K_N, with $d_i^{(b)}$ denoting the analogous quantity for the blue edges. If $d_i^{(r)} \geqslant \frac{m}{s}\left(1+\frac{\epsilon}{s}\right)N$ for some i, then by considering the "red" neighbors of this vertex we get (i). Similarly, if $d_i^{(b)} \geqslant \frac{n}{s}\left(1+\frac{\epsilon}{s}\right)N$ then (ii) must hold. Hence, we can assume that

(2.31) $$d_i^{(r)} < \frac{m}{s}\left(1+\frac{\epsilon}{s}\right)N\,, \quad d_i^{(b)} < \frac{n}{s}\left(1+\frac{\epsilon}{s}\right)N$$

for $1 \leqslant i \leqslant N$. Since

$$d_i^{(r)} + d_i^{(b)} = N-1$$

then

$$d_i^{(r)} > N-1-\frac{n}{s}\left(1+\frac{\epsilon}{s}\right)N$$

(2.32) $$= N\left(1-\frac{n}{s}-\frac{\epsilon n}{s^2}\right)-1$$

$$= N\left(\frac{m}{s}-\frac{\epsilon n}{s^2}\right)-1\,.$$

On the other hand, if β is defined by

$$\frac{1}{2}\sum_i d_i^{(r)} = \beta\binom{N}{2}\,,$$

then Lemma 2.15 implies that there are either at least

$$A := \beta^3\binom{N}{3} - \frac{1}{2}\beta(1-\beta)\binom{N}{2}$$

red triangles, or at least

$$B := (1-\beta)^3\binom{N}{3} - \frac{1}{2}\beta(1-\beta)\binom{N}{2}$$

blue triangles in K_N. Consider the first possibility. Then some edge, say e_0, is contained in at least

(2.33) $$\frac{3A}{\beta\binom{N}{2}} = \beta^2 N - \frac{3}{2}(1 - \beta) \geqslant \beta^2 N - 2$$

red triangles. Thus,(iii) would follow (by considering the set of vertices of these red triangles, excluding the endpoints of e_o) if we could show

(2.34) $$\beta^2 N - 2 \geqslant \frac{m(m-1)}{s(s-1)}\left\{1 + \frac{\varepsilon}{s}\right\}\left\{1 + \frac{\varepsilon}{s-1}\right\} N$$

This we now indicate how to do. First, observe that by (2.32)

$$\sum_i d_i^{(r)} = 2\beta\binom{N}{2} > N^2\left\{\frac{m}{s} - \frac{\varepsilon n}{s^2}\right\} - N$$

so that

(2.35) $$\beta > \frac{\left\{N\frac{m}{s} - \frac{\varepsilon n}{s^2}\right\} - 1}{N - 1}$$

Thus it will be enough to show that

(2.36) $$\frac{m}{s} - \frac{\varepsilon n}{s^2} + \left(\frac{m}{s} - \frac{\varepsilon n}{s^2} - 1\right)/ (N-1) \geqslant$$
$$\frac{m(m-1)}{s(s-1)}\left\{1 + \frac{\varepsilon}{s}\right\}\left\{1 + \frac{\varepsilon}{s-1}\right\}^2 N + 2$$

for $N \geqslant 16s^2$

We will not carry out all the details of this computation, which are relatively straight forward (but unenlightening).The basic point is that the important term of (2.36) is

(2.37) $$\left\{\frac{m}{s} - \frac{\varepsilon n}{s^2}\right\}^2 - \frac{m(m-1)}{s(s-1)}\left\{1 + \frac{\varepsilon}{s}\right\}\left\{1 + \frac{\varepsilon}{s-1}\right\}$$

which can be rewritten as

(2.38) $$\left[\frac{m^2}{s^2} - \frac{m(m-1)}{s(s-1)} - \varepsilon\left\{\frac{2mn}{s^3} + \left(\frac{1}{s} + \frac{1}{s-1}\right)\frac{m(m-1)}{s(3-1)}\right\}\right.$$
$$\left. + \varepsilon^2\left\{\frac{n^2}{s^4} - \frac{m(m-1)}{s^2(s-1)^2}\right\}\right\} N$$

In turn the main contribution to (2.38) turns out to be

(2.39) $$\frac{m^2}{s^2} - \frac{m(m-1)}{s(s-1)} = \frac{mn}{s^2(s-1)}$$

by the definition of s.

The value assigned to ε now guarantees (2.36), and therefore (2.34), holds.

2. If $\max\{m/n, n/m\} \geqslant x$ then *halt*; otherwise go to 3.
3. If $r^*(m, n) < 16(m + n)^2$ then *halt*; otherwise go to 4.
4. Select a pair (m^*, n^*) for which one of the possibilities of Lemma 2.17 occurs. (Thus, $m^* + n^* = m + n - 1$ or $m + n - 2$.) Let G^* be a graph with $r^*(m^*, n^*)$ vertices. Set $m = m^*$, $n = n^*$, $G = G^*$ and go to 2.

Suppose now that the algorithm halts at some graph G' of size $r^*(k', \ell')$. Let $t' := k' + \ell'$ and $p = t - t'$. Note that if $y := \dfrac{\log \log t}{\log t}$ then at each pass through step 4 of the algorithm, the value of $\epsilon = \dfrac{\min(m, n)}{8(m + n)}$ satisfies

$$\epsilon \geqslant \frac{n}{8(m+n)} \geqslant \frac{1}{8(1 + m/n)} \geqslant \frac{1}{8(1+x)} = y . \tag{2.41}$$

Hence, by the time we have reached G', we have (by Lemma 2.17) accumulated the "gain factors" to obtain the estimate

$$r^*(k', \ell') = \frac{(k'+1)(k'+2)\cdots(k)(\ell'+1)(\ell'+2)...(scrL)}{(t'+1)(t'+2)\cdots(t)}\left(1 + \frac{y}{t'+1}\right)\cdots\left(1 + \frac{y}{t}\right) r^*(k, \ell)$$

$$= \frac{\binom{t'}{k'}}{\binom{t}{k}} \prod_{i=1}^{p}\left(1 + \frac{y}{t'+i}\right) r^*(k, \ell) . \tag{2.42}$$

We now consider several cases, depending on how soon the algorithm halts, and why.

Case 1. $t' > \sqrt{t} + 2$.

Subcase (a). $r^*(k', \ell') < 16\, t'^2$.

Since t is large then so is t'. Thus, by (2.7) we must have $\min(k', \ell') = 2$, say (by symmetry) $\ell' = 2$. Thus,

$$\begin{aligned} r^*(k', \ell') = r^*(k', 2) &\leqslant (k'+1)^2/\log(k'+1)/e) \\ &< (k'+1)^2/\log(\sqrt{t}/e) \\ &= 2(k'+1)^2/\log(t/e^2) . \end{aligned} \tag{2.43}$$

Therefore, by (2.42)

$$\begin{aligned} r^*(k, \ell) &\leqslant \frac{\binom{t}{k}}{\binom{k'+2}{2}} r^*(k', \ell') \\ &\leqslant \frac{\binom{t}{k}}{\binom{k'+2}{2}} \cdot \frac{2(k+1)^2}{\log(t/e^2)} < 5\binom{k+\ell}{k}/\log(k+\ell) \end{aligned} \tag{2.44}$$

for $t = k + \ell$ large.

The case that there are B blue trinagles follows in exactly the same way (using the symmetry of β and $1 - \beta$ in the expressions involved) to yield (iv). This completes the proof of Lemma 2.17.

We need one final observation before proceeding to the proof of (2.25).

Fact.

$$\sum_{i=r+1}^{s} 1/i > \log \frac{s + i}{r + 1} \tag{2.40}$$

Proof: The sequence

$$a_n = \sum_{i=i}^{n} 1/i - \log(n + 1)$$

is monotone increasing. Consequently

$$\begin{aligned} \sum_{i=r+1}^{s} 1/i &= \sum_{i=i}^{s} 1/i - \sum_{i=i}^{r} 1/i \\ &= \log(s + 1) + a_s - \log(r + 1) - a_r \\ &> \log \frac{s + 1}{r + 1} \end{aligned}$$

Proof of Theorem 2.14. Let G be a graph with $r^*(k,\ell)$ vertices. We will consider the following algorithm.

1. Set $m = k$, $n = \ell$, $t = k + \ell$ and

$$x = \frac{1}{8} \frac{\log t}{\log \log t} - 1 .$$

Subcase (b). $k'/\ell' \geqslant x = \frac{1}{8}\,\frac{\log t}{\log\log t} - 1.$

Thus, by Lemma 2.16,

$$r^*(k', \ell') \leqslant 5\binom{k'+\ell'}{k'} \Big/ \log(k'/e^2\ell')$$

so that

$$r^*(k, \ell) \leqslant \frac{5\binom{k+\ell}{k}}{\binom{k'+\ell'}{k'}} \cdot \frac{\binom{k'+\ell'}{k'}}{\log(k'/e^2\ell')}$$

$$\leqslant 5\binom{k+\ell}{k} \Big/ \log\left[\frac{1}{e^2}\,\frac{\log t}{\log\log t} - 1\right] < 6\binom{k+\ell}{k} \Big/ \log\log(k+\ell) \tag{2.45}$$

for $k + \ell$ sufficiently large.

Case 2. $t' \leqslant \sqrt{t} + 2$

Thus,

$$p = t - t' \geqslant t - \sqrt{t} - 2$$

so that the factor in (2.42) we gain is

$$\prod_{i=1}^{p}\left(1 + \frac{y}{t'+i}\right) \geqslant \prod_{i=0}^{t-\sqrt{t}-3}\left(1 + \frac{y}{t-i}\right)$$

$$> y \sum_{i=0}^{t-\sqrt{t}-3} \frac{1}{t-i} + 1$$

$$> y \log \frac{t+1}{\sqrt{t}} \qquad \text{by (2.40)}$$

$$> \frac{1}{2}\, y \log t = \frac{1}{2} \log\log t\,.$$

Hence,

$$r^*(k, \ell) \leqslant \frac{\binom{k+\ell}{k}}{\binom{k'+\ell'}{k'}} \cdot \frac{1}{\frac{1}{2}\log\log t} \cdot r^*(k', \ell') < 2\binom{k+\ell}{k} \Big/ \log\log(k+\ell) \tag{2.46}$$

for $k + \ell$ large.

Thus in all cases (2.25) holds and the theorem is proved. ∎

2.4 Many colors

Up this point, we have investigated the problem of estimating the size of the largest monochromatic clique formed whenever the edges of K_n are 2-colored. In this section we will discuss the case in which t colors are permitted.

Define $r(k_1, k_2, \ldots, k_t)$ to be the least integer n with the property that for every t-coloring of the edges of K_n, there exists an i, $1 \leqslant i \leqslant t$, and a complete subgraph K_{k_i} of K_n having all edges colored by the ith color. As before, Ramsey's Theorem guarantees the existence of $r(k_1, \ldots, k_t)$.

The same argument as that used for $r(k, \ell)$ by Erdös and Szekeres gives a general (recursive) upper bound:

Proposition 2.18

$$r(k_1, k_2, \ldots, k_t) \leqslant r(k_1 - 1, k_2, \ldots, k_t) + r(k_1, k_2 - 1, \ldots, k_t) + \cdots$$

$$\ldots + r(k_1, k_2, \ldots, k_t - 1) - (t - 2) .$$

Proof: Set n equal to the RHS of (2.47) and let the edges of K_n be t-colored. Let C_i denote the set of all vertices joined to a given vertex v by an edge having the ith color. Then

$$\sum_{i=1}^{t} |C_i| = n - 1$$

and therefore there exists j, $1 \leqslant j \leqslant t$, with

$$|C_j| \geqslant r(k_1, \ldots, k_j - 1, \ldots, k_t) .$$

Suppose now that for no i, $1 \leqslant i \leqslant t$, does there exist a subset of C_j of size k_i which spans edges having only the ith color. Then by the definition of $r(k_1, \ldots, k_j - 1, \ldots, k_t)$, C_j must contain a K_{k_j-1} with all its edges colored by the jth color. Adding the vertex v to this K_{k_j-1}, we obtain the desired copy of K_{k_j} having all edges with the jth color. ■

It follows now by induction on $k_1, \ldots, k_t$ and t that, for example,

$$r(k_1, \ldots, k_t) \leqslant \frac{(k_1 + \cdots + k_t - t)!}{(k_1 - 1)! \cdots (k_t - 1)!} . \tag{2.48}$$

For $t > 2$, (2.48) can easily be improved by a factor which tends to 0 as $t \longrightarrow \infty$. For a discussion of general lower bounds for $r(k_1, \ldots, k_t)$, the reader is referred to [A72]. For the remainder of the section, we will restrict ourselves to the interesting special case that

$k_1 = \ldots = k_t = 3$. Denote $r(\overbrace{3, 3, \ldots, 3}^{t})$ by $r(3; t)$.

Theorem 2.19

For a suitable constant $c > 0$,

$$c(315)^{t/5} \leqslant r(3; t) \leqslant t!(e - 1/24) , \; t \geqslant 4 .$$

The lower bound is due to Frederickson [Fr79] (cf. [CG83]). For a statement of the upper bound, the reader is referred to the survey paper of Chung and Grinstead [CG83]. Here we will prove only the slightly weaker:

Theorem 2.20

$$\frac{3^t + 3}{2} \leqslant r(3; t) \leqslant \lfloor t!\, e \rfloor , \; t \geqslant 4 . \tag{2.49}$$

Proof: To prove the upper bound, we first show

$$r(3; t) \leqslant t(r(3; t - 1) - 1) + 2 \tag{2.50}$$

Fix a vertex v of an r-colored complete graph K_n with $n = t(r(3; t - 1) - 1) + 2$ vertices. Let C_i denote the set of all vertices x with the property that $\{x, v\}$ is colored by the ith color. By the pigeon-hole principle, there exists j with

$$|C_j| \geqslant r(3; t-1) . \tag{2.51}$$

We can assume that no pair in C_j is colored by the jth color (since otherwise x together with such a pair forms a triangle in the jth color). However, it follows from (2.51) and the definition of $r(3; t-1)$ that C_j contains a monochromatic triangle. Iterating (2.50) now yields the upper bound on (2.49).

To prove the lower bound, let us call a set A of integers *sum-free* if $a, b \in A$ implies $a + b \notin A$. Let s_t denote the largest integer such that the set $\{1, 2, \ldots, s_t\}$ can be partitioned into t sets $A_1, \ldots, A_t$, each of which is sum-free. (A theorem of Schur [Sc16] guarantees that s_t is finite; see Theorem 5.1.)

Claim. For any $t > 0$,

$$s_{t+1} \geqslant 3s_t + 1 . \tag{2.52}$$

Proof: Let

$$\{1, 2, \ldots, s_t\} = A_1 \cup \cdots \cup A_t$$

be a partition of $\{1, 2, \ldots, s_t\}$ into t sum-free sets. Then the sets

$$B_i = \{3a : a \in A_i\} \cup \{3a - 1 : a \in A_i\} , \quad 1 \leqslant i \leqslant t ,$$

$$B_{t+1} = \{3\ell + 1 : 0 \leqslant \ell \leqslant s_t\} ,$$

are sum-free and moreover, form a partition of $\{1, 2, \ldots, 3s_t + 1\}$. ■

Since $s_1 = 1$, it follows by iterating (2.52) that

$$s_t \geqslant \frac{1}{2}(3^t - 1) . \tag{2.53}$$

Now, set $m = s_t + 1$ and let $C_1 \cup \cdots \cup C_t$ be the t-coloring of the edges of K_m with vertex set $\{1, 2, \ldots, m\}$ defined by

$$\{u, v\} \in C_k \text{ iff } |u - v| \in A_k .$$

Suppose this graph contains a monochromatic triangle $\{u, v, w\}$, with $u < v < w$. This implies $v - u, w - v, w - u \in A_k$ for some k, contradicting the fact that A_k is sum-free. Thus,

$$m = s_t + 1 \leqslant r(3; t) - 1$$

i.e.,

$$r(3; t) \geqslant \frac{3^t - 1}{2} + 2 = \frac{3^t + 3}{2}$$

as required. ■

The outstanding open problem concerning $r(3; t)$ is to decide whether or not $\lim_{t\to\infty} r(3; t)^{1/t}$ is finite. (The limit is known to exist [Ch73].) Erdős currently is offering \$100 for the answer (with a proof!)

3. Ramsey numbers for other graphs

During the past 10 years, an impressive number of papers have appeared which investigate analogues of Ramsey's Theorem for other graphs besides complete graphs. An initial motivation for these studies was the hope that progress here might lead to a deeper understanding of the classical (complete graph) problems. While this hope has not yet materialized (for example, the Ramsey numbers are known for all graphs on 5 vertices *except* K_5), the subject has developed into a lively and interesting area in its own right. In this section we will describe several of the results (including some very recent ones) which we find particularly attractive.

We begin with a definition. For graphs G and H, we let $r(G, H)$ denote the least integer n so that in any coloring of the edges of K_n by red and blue (say), there must always be formed either a red copy of G or a blue copy of H. In particular, if $G = K_k$ and $H = K_\ell$ then $r(K_k, K_\ell) = r(k, \ell)$ from the preceding section. When $G = H$, we abbreviate $r(G, G)$ by $r(G)$.

One of the simplest and most general results in this topic is the following theorem of Chvátal and Harary [CH72].

Theorem 3.1

$$r(G, H) \geqslant (\chi(G) - 1)(c(H) -) + 1 \tag{3.1}$$

where $\chi(G)$ denotes the chromatic number of G and $c(H)$ denotes the size of the largest connected component of H.

Proof: Let $m = (\chi(G) - 1)(c(H) - 1)$ and think of K_m as being $\chi(G) - 1$ copies of $K_{c(H)-1}$ with edges interconnecting all pairs of vertices in different copies of $K_{c(H)-1}$. Color all edges within each copy of $K_{c(H)-1}$ blue, and all remaining edges red. Certainly, there is no red copy of G since this would imply that the chromatic number of G is at most $\chi(G) - 1$. On the other hand, there can be no blue copy of H since the largest blue component of K_n has size $c(H) - 1$. Therefore, $r(G, H) > m$. ■

Corollary 3.2 [C77] For any tree T_m with m vertices,

$$r(T_m, K_n) = (m - 1)(n - 1) + 1 . \tag{3.2}$$

Proof: The lower bound follows from (3.1). For $m = 2$ or $n = 2$, (3.2) is immediate. Assume that (3.2) holds for all values of m' and n' with $m' + n' < m + n$. Let $s = (m - 1)(n - 1) + 1$ and consider a 2-colored K_s. Let T' be a tree formed from T by the removal of some endpoint x (where x is connected to, say y, in T_m). By the induction hypothesis, the K_s contains either a *blue* K_n and we are done, or a *red* T'. Hence, we may assume the latter holds. Now, remove the $m - 1$ vertices of this red T', leaving a 2-colored $K_{s-(m-1)} = K_{(m-1)(n-2)+1}$. Again, by the induction hypothesis, this graph contains either a red T_m or a blue K_{n-1}; we may assume the latter. Thus, in the original K_s we have a red T' and a blue T_{n-1} disjoint from it. Finally, we examine the edges emanating from y to the blue K_{n-1}. If any of these edges is red, then we have found a red T_m. If none of the edges is red, then $K_{n-1} \cup \{y\}$ forms a blue K_{n-1}. ■

Just as in the case of $r(k, \ell)$, it is possible to consider using more than two colors. We let $r(G; t)$ denote the least integer n such that if the edges of K_n are arbitrarily t-colored, then a monochromatic copy of G must always be formed. One of the sharper bounds for a number of this type is given by the following result of Chung [Ch74] and Irving [I74] (see also [CG75]). Let C_4 denote the cycle on four vertices.

Theorem 3.3 [I74].

For all t,

$$r(C_4; t) \leqslant t^2 + t + 1 \tag{3.3}$$

If $t - 1$ is a prime power, then

$$r(C_4; t) \geqslant t^2 - t + 2 . \tag{3.4}$$

Note the dramatic difference between the size of $r(C_4; t)$ and that of $r(C_3; t) = r(3; t)$ given in (2.49) (which grows exponentially in t), giving one more example of the difference between the behavior of odd and even cycles in graphs.

Returning again to the case of two colors, one might ask how *slowly* $r(G)$ can grow, as a function of the number of vertices $v(G)$ of G. This is answered by the following result of Burr and Erdös.

Theorem 3.4 [BE76]

If G is a connected graph with n vertices then

$$r(G) \geqslant \lfloor (4n-1)/3 \rfloor . \tag{3.8}$$

Furthermore, for each $n \geqslant 3$, there exist graphs for which equality in (3.8) is achieved.

If G is allowed to be disconnected (but, as always, having no isolated vertices) then $r(G)$ can be much smaller.

Theorem 3.5 [BE76]

There exist positive constants c, c' such that

$$n + \frac{\log n}{\log 2} - c \log\log n \leqslant \min\{r(G) \colon v(G) = n\} \leqslant n + c'\sqrt{n} . \tag{3.9}$$

It is conjectured that the lower bound in (3.9) is closer to the correct answer.

If G is restricted in various ways, then the Ramsey number can also be restricted. A beautiful example of this behavior is given by a recent result of Chvátal et al. (which was conjectured by Burr and Erdös [BE76]).

Theorem 3.6 [CRST83]

For each positive integer d there exists a constant $c(d)$ such that for any graph G with n vertices and maximum degree d,

$$r(G) \leqslant c(d)n . \tag{3.10}$$

Space limitations prevent us from giving the proof, which uses the powerful regularity lemma of Szemerédi (cf. [Sz76] and Lemma 6.8).

A related result of Beck [**Bec83b**] asserts the following, where we now assume that not only is the maximum degree of G bounded by d, but also the chromatic number of G is bounded by χ.

Theorem 3.7

If G has n vertices then

$$r(G) < (2n)^{2d^{2\chi-3}} .$$

Burr and Erdös have conjectured a stronger form of Theorem 3.6, which is still unresolved.

Conjecture 3.8. For each d there exists a constant $c'(d)$ so that if G is any graph with n vertices and, for any subgraph G' of G, the average degree of a vertex in G' is at most d, then

$$r(G) \leqslant c'(d)n .$$

An interesting variation of $r(G)$ has been studied by a number of authors recently. This is the *size* Ramsey number, denoted by $r_e(G)$, and defined to be the least integer m for which there exists a graph H with m edges, so that in any 2-coloring of the edges of H, a monochromatic copy of G must always be formed.

A beautiful and unexpected result of Beck (settling a $100 problem of Erdös) shows that r_e can be quite small.

Theorem 3.9 [Bec83a]

If n is sufficiently large then for P_n, the path on n vertices,

$$r_e(P_n) < 900n .$$

Beck actually proves a stronger density theorem which implies Theorem 3.9 at once.

Theorem 3.10 [Bec83a]

If n is sufficiently large, there exists a graph G with fewer than 900 n edges so that any subgraph H of G containing at least half the edges of G must contain a copy of P_n.

Beck also considered the corresponding problem for trees T with maximum degree d. For this case, he proves:

Theorem 3.11 [Bec83a]

If a tree T_n has n vertices and maximum degree d then for n sufficiently large,

$$r_e(T_n) < dn(\log n)^{12} . \tag{3.11}$$

We will not give the proofs of Beck's results here (which employ, among other things, the use of the probabilistic method and the Lovász local lemma). The reader is referred to the original papers for the details.

A question left open by Beck was whether the logarithmic term in (3.11) could be replaced by a constant. This was very recently resolved in the affirmative by a striking result of Friedman and Pippenger.

Theorem 3.12 [FP]

Let $0 < \delta < 1$ and let d be fixed. For every n there is a graph G with $e = O(n)$ edges such that, even after the deletion of all but δe edges, G continues to contain every tree with n vertices and maximum degree at most d.

The ingenious proof of Theorem 3.12 given in [FP] is definitely non-trivial, and uses earlier results of Beck, Alon and Chung [AC], Lubotzky, Phillips and Sarnak [LPS] and a powerful result of their own implying the universality of expanding graphs with respect to small trees. More precisely, this last result is:

Theorem 3.13 [FP]

If H is a nonempty graph so that every subset of size $x \leqslant 2n - 2$ vertices has at least $(d + 1)x$ neighbors then H contains every tree with n vertices and maximum degree at most d.

The restriction here on the maximum degree is necessary, as Beck [Bec83a] observes, by showing that there are trees T_n with n vertices for which $r_e(T_n) \geqslant n^2/8$. By contrast, any tree T_n' with n vertices satisfies $r(T_n) \leqslant 4n + 1$ (see [EG73]).

For general graphs G with n vertices and maximum degree d, it has been shown by Rödl and Szemerédi [RS] that

$$r_e(G) = o(n^2) .$$

On the other hand, there exist graphs G with maximum degree 3 having

$$r_e(G) > n(\log n)^c$$

for an absolute constant $c > 0$.

We conclude this section with several rather nice open problems in this area.

Conjecture 3.14. (Erdös) For some $\epsilon > 0$,

$$r(C_4, K_n) = o(n^{2-\epsilon}) .$$

Conjecture 3.15. (Erdös) If G has $\binom{n}{2}$ edges then

$$r(G) \leqslant r(K_n) .$$

More generally, if G has $\binom{n}{2} + t$ edges, $0 \leqslant t \leqslant n$, then

$$r(G) \leqslant r(K_n(t))$$

where $K_n(t)$ denotes the graph formed by connecting a new vertex to t of the vertices of a K_n.

Conjecture 3.16. (Erdös-Graham [EG73])

$$r(C_5;t)/r(C_3;t) \longrightarrow 0$$

as $t \longrightarrow \infty$.

Define the *induced* Ramsey number $r^*(G)$ of a graph G to be the least integer m for which there exists a graph H with m vertices so that in any 2-coloring of the edges of H, there is always an *induced* monochromatic copy of G in H. The existence of $r^*(G)$ was shown independently by Deuber [D75], Erdös, Hajnal and Pósa [EHP75], and Rödl [R73].

Problem 3.17. If G has n vertices, is it true that

$$r^*(G) < c^n$$

for some absolute constant c?

This can be shown to hold when G is bipartite (using techniques from [R73]). It is also known that for general graphs G,

$$r^*(G) < 2^{2^{n^{1+o(1)}}} .$$

Problem 3.18. Is it true that

$$r^*(P_n) < cn \ ?$$

Problem 3.19. (Trotter) Is it true that for each d there is a $c(d)$ such that if G_n has n vertices and maximum degree d then $r^*(G_n) \leqslant n^{c(d)}$?

Problem 3.20. (Erdös) Is it true that if G has e edges then

$$r(G) < 2^{ce^{1/2}} \tag{3.12}$$

for some absolute constant c? If true, then (3.12) would be, apart from the value of the constant, best possible.

4. Hypergraphs

In this section we consider extensions of the preceding questions to the general setting of hypergraphs. More precisely, set $[n] = \{1, 2, \ldots, n\}$ and

$$[n]^p = \{A: A \subseteq [n], |A| = p\} .$$

We will consider colorings of $[n]^p$ where p is an arbitrary integer. This extends the case $p = 2$ considered earlier.

Definition. The symbol $n \longrightarrow (\ell_1, \ell_2, \ldots, \ell_t)^p$ will denote the validity of the following statement: For every t-coloring of $[n]^p$ there exists i, $1 \leqslant i \leqslant t$, and a set T, $|T| = \ell_i$ so that $[T]^p$ is colored only by color i.

If $\ell_1 = \ell_2 = \ldots = \ell_t = \ell$, then this will be abbreviated by writing $n \longrightarrow (\ell)_t^p$. The general Ramsey numbers are defined as follows:

$$r_p(\ell_1, \ell_2, \ldots, \ell_t) = \min\{n_0: n \longrightarrow (\ell_1, \ell_2, \ldots, \ell_t)^p \text{ for } n \geqslant n_0\} ,$$

$$r_p(\ell, t) = \min\{n_0: n \longrightarrow (\ell)_t^p \text{ for } n \geqslant n_0\}$$

$$r_p(\ell) = r_p(\ell, 2) .$$

Recall that in Section 1 we gave a proof that

$$(\sqrt{2}+o(1))^\ell \leqslant r_2(\ell) \leqslant (4+o(1))^\ell \tag{4.1}$$

so that the growth of the function $r_2(\ell)$ is exponential in ℓ. For $p \geqslant 3$, much less is known about the order of magnitude of the function $r_p(\ell)$. An upper bound follows from the proof of Ramsey's theorem.

Theorem 4.1 [ER52]

$$\mathrm{Log}_{p-1}(r_p(\ell)) \leqslant c_p \ell \tag{4.2}$$

where Log_{p-1} denotes the $(p-1)$-fold iterated logarithm.

Proof: The proof proceeds by induction on p. For $p = 2$, (4.2) follows from (4.1), while for $p = 1$, (4.2) is just the pigeon-hole principle. Suppose (4.2) holds for $p - 1$, so that $r_{p-1}(\ell)$ exists and satisfies (4.2). Set

$$u = r_{p-1}(\ell - 1) + 1$$

and define the numbers $x_{u-1}, \ldots, x_{p-2}$ by

$$x_{u-1} = 1, \text{ and for } p - 2 \leqslant i < u - 1 ,$$

$$x_i = x_{i+1} 2^{\binom{i+1}{p-1}} + 1 .$$

Let $n = x_{p-2} + (p - 2)$. We prove

$$r_p(\ell) \leqslant n . \tag{4.3}$$

Suppose now that $[n]^p$ is 2-colored, say by the coloring χ. Select distinct points $v_1, v_2, \ldots, v_{p-2}$ in $[n]$ and define

$$V_{p-2} = [n] - \{v_1, v_2, \ldots, v_{p-2}\} .$$

In general, now, suppose $v_1, v_2, \ldots, v_i$ and V_i have been defined. We proceed as follows:

(α) Select $v_{i+1} \in V_i$ arbitrarily;

(β) Partition $V_i - \{v_{i+1}\}$ into equivalence classes by defining

$$v \equiv v' \textit{ iff}$$

for every choice of $v_{j_1}, v_{j_2}, \ldots, v_{j_{p-1}} \in \{V_i, \ldots, V_{i+1}\}$ the sets $\{v_{j_1}, v_{j_2}, \ldots, v_{j_{p-1}}, v\}$ and $\{v_{j_1}, v_{j_2}, \ldots, v_{j_{p-1}}, v'\}$ have the same color;

(γ) Define V_{i+1} to be the set of those v belonging to the largest equivalence class in (β).
Thus,

$$|V_{i+1}| \geqslant (|V_i| - 1)2^{-\binom{i+1}{p-1}} \geqslant x_{i+1} .$$

We continue until $v_1, v_2, \ldots, v_u$ are constructed. This is possible since V_i is nonempty for $i = p-2, \ldots, u-1$ (and $|V_i| \geqslant x_i \geqslant 1$). The sequence $v_1, v_2, \ldots, v_u$ therefore has the following property: The color of $\{v_{i_1}, v_{i_2}, \ldots, v_{i_{p-1}}, v_{i_p}\}$, $i_1 < i_2 < \ldots < i_p$, is not changed if v_{i_p} is replaced by any v_j where $j > i_{p-1}$. In other words, the color of any (ordered) p-set depends only its first $p-1$ elements. Let χ^* be the 2-coloring of the $(p-1)$-element subsets of $\{v_1, v_2, \ldots, v_{u-1}\}$ defined by

$$\chi^*\{v_{i_1}, v_{i_2}, \ldots, v_{i_{p-1}}\} = \chi\{v_{i_1}, v_{i_2}, \ldots, v_{i_{p-1}}, v_u\} .$$

By the choice of $u = r_{p-1}(\ell - 1) + 1$ we obtain a χ^*-monochromatic ℓ-set $\{v_{i_1}, v_{i_2}, \ldots, v_{i_{\ell-1}}, v_u\}$ (which, of course, is also χ-monochromatic). This proves (4.3). Since

$$u = r_{p-1}(\ell - 1) + 1 \leqslant r_{p-1}(\ell)$$

then by induction

$$\mathrm{Log}_{p-2}(u) \leqslant \mathrm{Log}_{p-2}(r_{p-1}(\ell)) \leqslant c_{p-1}\ell .$$

On the other hand,

$$n < \prod_{i=p-2}^{u-1} \left(2^{\binom{i+1}{p-1}} + 1\right) < 2^{2\binom{u}{p}} .$$

Therefore,

$$\mathrm{Log}_{p-1}(r_p(\ell)) \leqslant \mathrm{Log}_{p-2}\left[2\binom{u}{p}\right] \leqslant c_p\ell$$

as required. ■

We next discuss lower bounds on $r_p(\ell)$.

Theorem 4.2

$$r_3(\ell) \geqslant \frac{\ell}{e} \cdot 2^{\ell^2/6}(1 + o(1)) . \tag{4.4}$$

Proof: The proof of this result is quite similar to that of Theorem 2.1, so we will only give a sketch. Consider a random 2-coloring of the set $[n]^3$. The probability that a particular ℓ-set is monochromatic is $2^{1-\binom{\ell}{3}}$. Thus, if

$$\binom{n}{\ell} 2^{1-\binom{\ell}{3}} < 1$$

then $n < r_3(\ell)$. A simple computation now yields (4.4). ■

Although the next theorem is a major tool in establishing lower bounds for $r_p(\ell)$, $p \geqslant 4$, we will not give a proof. Instead, we will present a similar, but much less technical proof, of a related result (Theorem 4.5).

Theorem 4.3 [EHR65] (Stepping-Up Lemma)

If $n \not\rightarrow (\ell)_2^p$ for $p \geqslant 3$ then

$$2^n \not\rightarrow (2\ell + p - 4)_2^{p+1}$$

(where $m \not\rightarrow (k)$ indicates that $m \longrightarrow (k)$ does not hold). This together with Theorem 4.2 implies

Theorem 4.4

For $p \geqslant 3$,

$$\text{(4.5)} \qquad \operatorname{Log}_{p-2}(r_p(\ell)) \geqslant c_p' \ell^2 .$$

The asymptotic behavior of $r_p(\ell)$ is not known for $p \geqslant 3$. However, because of the Stepping-Up Lemma, any improvement on the lower bound of $r_3(\ell)$ would yield a corresponding improvement on the lower bounds for $r_p(\ell)$, $p > 3$. The major open problem here (and indeed, one of the main unsolved problems in Ramsey theory) is the determination of the order of growth of $r_3(\ell)$. P. Erdös is currently offering $500 for an answer to the following problem.

Problem 4.5. Is there an absolute constant $c > 0$ such that

$$\text{(4.6)} \qquad \log\log r_3(\ell) \geqslant c\ell \ ?$$

It is interesting to note that if *four* colors are allowed (rather than two), then the analogue to (4.6) is valid, i.e.,

$$\text{(4.7)} \qquad \log\log r_3(\ell,4) \geqslant c\ell .$$

This is a consequence of the next result.

Theorem 4.5 (Hajnal; cf. [EHMR84], Th. 26.3)

If $n \not\rightarrow (\ell)_2^2$ then $2^n \not\rightarrow (\ell + 1)_4^3$.

Proof: Let $[n]^2 = C_1 \cup C_2$ be a 2-coloring with no monochromatic ℓ-set, and let $<$ be the lexicographic order on $\mathscr{P}(n)$, the power set of $[n]$, given by:

$$a_1 < a_2 \textit{ iff } \max(a_1 - (a_1 \cap a_2)) < \max(a_2 - (a_1 \cap a_2)) .$$

For $a_1 \neq a_2 \in \mathscr{P}(n)$, we also define

$$\delta(a_1,a_2) = \max\{i:\ i \in (a_1 - a_2) \cup (a_2 - a_1)\} .$$

For $a_1,a_2,a_3 \in \mathscr{P}(n)$ with $a_1 < a_2 < a_3$, let

$$\delta_1 = \delta(a_1,a_2),\ \delta_2 = \delta(a_2,a_3) .$$

Finally, set

$$\{a_1,a_2,a_3\} \in S_1 \text{ iff } \{\delta_1,\delta_2\} \in C_1 \text{ and } \delta_1 < \delta_2,$$

$$\{a_1,a_2,a_3\} \in S_2 \text{ iff } \{\delta_1,\delta_2\} \in C_1 \text{ and } \delta_1 > \delta_2,$$

$$\{a_1,a_2,a_3\} \in S_3 \text{ iff } \{\delta_1,\delta_2\} \in C_2 \text{ and } \delta_1 < \delta_2,$$

$$\{a_1,a_2,a_3\} \in S_4 \text{ iff } \{\delta_1,\delta_2\} \in C_2 \text{ and } \delta_1 > \delta_2.$$

Suppose now that there is a family $X \subseteq \mathscr{P}(n)$, $|X| = \ell + 1$, which is monochromatic. Assume that $[X]^3 \subseteq S_1$ (the other three cases are similar). Write

$$X = \{a_1, a_2, \ldots, a_{\ell+1}\}, \quad a_1 < a_2 < \ldots < a_{\ell+1}$$

and let $\delta_i := \delta(a_i, a_{i+1})$, $1 \leqslant i \leqslant \ell$. For $i \leqslant \ell - 1$, $\{a_i, a_{i+1}, a_{i+2}\} \in S_1$ so that

$$\delta_i = \delta(a_i, a_{i+1}) < \delta(a_{i+1}, a_{i+2}) = \delta_{i+1}.$$

Thus,

$$\delta_1 < \delta_2 < \ldots < \delta_\ell.$$

However, for arbitrary $1 \leqslant i < j \leqslant \ell$, $\{a_i, a_{i+1}, a_{j+1}\} \in S_1$ and consequently,

$$\{\delta(a_i, a_{i+1}), \delta(a_{i+1}, a_{j+1})\} = \{\delta_i, \delta_j\} \in C_1$$

(where $\delta(a_{i+1}, a_{j+1}) = \delta(a_j, a_{j+1}) = \delta_j$ follows from the monotonicity of the a_k's). This implies that $\{\delta_1, \delta_2, \ldots, \delta_\ell\}$ is monochromatic, contradicting our hypothesis on the coloring of $[n]^2$. ∎

Applying Theorem 4.5 with the estimate of Theorem 2.1, we obtain:

Corollary 4.6. For ℓ sufficiently large,

$$\log\log r_3(\ell, 4) > \frac{1}{2}\ell \log 2.$$

We conclude this section with several remarks. Theorem 4.1 is a quantitative form of Ramsey's theorem. The upper bounds on $r_p(\ell)$ are due to Erdös and Rado [ER52]. The lower bounds on $r_p(\ell)$ are due to Erdös and Hajnal [EH72] (cf. [GRS80]).

In summary, the order of growth of the function $r_3(\ell)$ is not known. However, if we allow $t \geqslant 4$ colors, then we do know that $r_3(\ell; t)$ is doubly exponential in ℓ. If just *three* colors are allowed, there is a modest improvement of (4.4) due to Erdös and Hajnal [EH]:

(4.8) $$r_3(\ell; 3) > \exp(c\ell^2 \log^2 \ell).$$

Finally, we state one more related problem which was considered in [EH72].

For fixed n, ℓ, u, v and p, the notation

(4.9) $$n \longrightarrow \left(\ell, \begin{bmatrix} u \\ v \end{bmatrix}\right)^p$$

will denote the truth of the following statement: For any red-blue coloring of $[n]^p$, either there is an ℓ-set $X \subseteq [n]$ with all elements of $[X]^p$ red, or there is a u-set $Y \subseteq [n]$ with at least v elements of $[Y]^p$ blue. Let $f_p(n, u, v)$ denote the largest value of ℓ for which (4.9) holds. The conjecture of Erdös and Hajnal concerns the behavior of $f_p(n, u, v)$ as v increases from 1 to $\binom{u}{p}$. It asserts that there exist

$$1 < v_1 < v_2 < \ldots < v_{p-2} \leqslant \binom{u}{p}, \quad \text{where } v_i = v_i(u)$$

(but is independent of n) such that the function $f_p(n, u, v)$ grows like a power of n for $v \in [1, v_1 - 1]$, like a power of $\log n$ for $v \in [v_1, v_2 - 1]$, like a power of $\log \log n$ for $v \in [v_2, v_3 - 1]$, etc., and finally, like a power of $\mathrm{Log}_{p-2} n$ for $v \in [v_{p-2}, \binom{u}{p}]$.

5. The Theorems of Schur and Rado-Folkman-Sanders

Set $m = \lfloor t!e \rfloor$ and suppose that $[1, m] = C_1 \cup C_2 \cup \ldots \cup C_t$ is a t-coloring of the integers in the interval $[1, m]$. Consider the induced t-coloring of the edges of the complete graph K_{m+1} with vertex set $\{1, 2, \ldots, m+1\}$ where the edge $\{u, v\}$ is colored by the ith color *iff* $|u - v| \in C_i$. By Theorem 2.20, we must find a monochromatic triangle $\{u, v, w\}$ with $u < v < w$. However, this means that the integers $x = v - u$, $y = w - v$ and $z = w - u = x + y$ all belong to the same C_j for some j.

This argument yields the quantitative form of a theorem which was proved some 70 years ago by I. Schur:

Theorem 5.1 [Sc16]

Suppose

$$\bar{m} \geqslant r(3; t) - 1 \geqslant \lfloor t!e \rfloor$$

and the set of integers $\{1, 2, \ldots, \bar{m}\}$ is t-colored. Then there exist integers $x, y, z \in \{1, 2, \ldots, \bar{m}\}$ having the same color such that $x + y = z$.

Recall (cf. the proof of Theorem 2.20) that the minimum $\bar{m}$ for which Theorem 5.1 holds was denoted by $s_t + 1$. By Theorem 2.19, $s_t \geqslant c(315)^{t/5}$ for a suitable positive c. By replacing $(315)^{1/5} = 3.16\ldots$ by a slightly larger value, it is possible to guarantee that x and y are distinct. We shall now deal with this problem in a somewhat more general form:

Let $X = \{x_1, x_2, \ldots, x_k\}$ be a set of integers. Denote by ΣX the set

$$\{\sum_{i \in I} x_i \colon \varnothing \neq I \subseteq \{1, 2, \ldots, k\}\}$$

of all $2^k - 1$ sums of nonempty subsets of the x_i. Of course, these sums do not all have to be distinct but, in general, $|\Sigma X| \leqslant 2^k - 1$ for any k-element set X.

Our main concern here will be to discuss the following two theorems:

Theorem 5.2 (Non-Repeated Sums Theorem).

For any choice of positive integers k and t there is a least integer $S(k, t)$ so that if $n \geqslant S(k, t)$ and $\{1, 2, \ldots, n\}$ is t-colored, say $\{1, 2, \ldots, n\} = C_1 \cup \ldots \cup C_t$, then there exists a set X with $|X| = k$ so that $|\Sigma X| = 2^k - 1$ and $\Sigma X \subseteq C_i$ for some i.

Theorem 5.3 (Disjoint Unions Theorem).

For any choice of positive integers k and t there is a least integer $\cup(k, t)$ so that if $n \geqslant \cup(k, t)$ and $\{1, 2, \ldots, n\}$ is t-colored, say $\{1, 2, \ldots, n\} = C_1 \cup \ldots \cup C_t$, then there exists a collection of k pairwise disjoint nonempty sets $Z_1, Z_2, \ldots, Z_k \subseteq \{1, 2, \ldots, n\}$ so that all nonempty unions of the form $\bigcup_{i \in I} Z_i$, $\varnothing \neq I \subseteq \{1, 2, \ldots, k\}$, are in a single C_j for some j.

The non-repeated sums theorem was proved by Rado [Ra33], Folkman (unpublished; cf. [GR71b]) and Sanders [Sa68]; the disjoint unions theorem was derived by Graham and Rothschild [GR71a] as a consequence of a general Ramsey theorem for n-parameter sets.

The following proposition is part of the folklore.

Proposition 5.4

$$\log_2 S(k,t) \leqslant \cup(k,t) < \binom{S(k,t)+1}{2}. \tag{5.1}$$

Proof: First we prove the upper bound. Partition a set X of cardinality $\binom{S(k,t)+1}{2}$ into blocks X_j, $1 \leqslant j \leqslant S(k,t)$, satisfying $|X_j| = j$. Suppose that all the subsets of X are t-colored (and therefore, so are the sets X_j). Assign to each integer $j \leqslant S(k,t)$ the color that X_j has, and apply Theorem 5.2. This immediately yields the required sets $Z_1, Z_2, \ldots, Z_k$.

Now we prove the lower bound. Suppose that $\{1, 2, \ldots, 2^{\cup(k,t)}\}$ is t-colored. This induces a t-coloring of the nonempty subsets of $\{1, 2, \ldots, \cup(k,t)\}$ by assigning to the set $J \subseteq \{1, 2, \ldots, \cup(k,t)\}$ the same color as that assigned to the integer $\sum_{j \in J} 2^j$. By Theorem 5.3 we must have k pairwise disjoint sets $Z_1, Z_2, \ldots, Z_k$ with all unions monochromatic. Set $x_i = \sum_{z \in Z_i} 2^z$, $1 \leqslant i \leqslant k$, and $X = \{x_1, x_2, \ldots, x_k\}$. Clearly the set ΣX is monochromatic and $|\Sigma X| = 2^k - 1$, which implies $S(k,t) \leqslant 2^{\cup(k,t)}$, as required. ∎

A. Taylor [T81] has given upper bounds for $S(k,t)$ and $\cup(k,t)$ which are of the form of a many times iterated exponential. More precisely, for integers p and q, define:

$$T_p(1) = p,$$

$$T_p(q+1) = p^{T_p(q)}.$$

Thus, $T_p(q)$ is a tower of p's of height q.

Theorem 5.5

$$S(k,2) \leqslant T_3(4k-1), \quad \cup(k,2) \leqslant T_3(4k-2). \tag{5.2}$$

In [T81], Taylor also gives similar upper bounds for the general values $S(k,t)$ and $\cup(k,t)$ in which both the terms and the heights of the towers now depend on k and t.

Because of the lower bound in Proposition 5.4, the first inequality of (5.2) is a consequence of the second one. We will give a proof of the second inequality here which is based in part on ideas from [R82] and [Fu85], and is slightly different from the proof of Taylor.

The following result is a special case of a more general theorem of Erdös [E65].

Lemma 5.6. Let $A_1, A_2, \ldots, A_r$ be sets with cardinalities $a_1, a_2, \ldots, a_r$, respectively, which satisfy:

$$a_1 = t + 1$$

and

$$a_j = t \prod_{i=1}^{j-1} \binom{a_i}{2} + 1, \quad j = 2, \ldots, r. \tag{5.3}$$

Suppose that the set $A_1 \times A_2 \times \ldots \times A_r$ is t-colored. Then there exist sets $B_i \subseteq A_i$ with $|B_i| = 2$ for $1 \leqslant i \leqslant r$, such that the set $B_1 \times B_2 \times \ldots \times B_r$ is monochromatic.

Proof: We will proceed by induction on r. For $r = 1$, the statement is immediate. Suppose the statement has been proved for $t = j - 1$ and assume that $A_1 \times A_2 \times \ldots \times A_j$ has been t-colored. By induction, for each $a \in A_j$ there exist sets $B_i(a) \subseteq A_i$, $|B_i(a)| = 2$, such that $\prod_{i=1}^{j-1} B_i(a) \times \{a\}$ is monochromatic. Since

$$a_j = t \prod_{i=1}^{j-1} \binom{a_i}{2} + 1$$

must exist distinct $a, a' \in A_j$ having $B_i(a) = B_i(a') := B_i$, and moreover, so that the set

$$\prod_{i=1}^{j-1} B_i \times \{a\} \cup \prod_{i=1}^{j-1} B_i \times \{a'\} = (\prod_{i=1}^{j-1} B_i) \times \{a, a'\}$$

is monochromatic. ∎

Remark: Note that for $t = 2$, (5.3) gives

$$a_1 = 3 \,, \ a_2 = 7 \text{ and for } j \geqslant 3 \,,$$

$$a_j \leqslant 2 \prod_{i=1}^{j-1} \left[\frac{a_i^2}{2} \right] \leqslant \frac{1}{2} \cdot 3^{\sum_{i=1}^{j-1} 2 \cdot 3^i} < \frac{1}{2} \cdot 3^{3^j} .$$

Thus,

$$\sum_{j=1}^{r} a_j < 3^{3^{r-1}} + \frac{1}{2} \cdot 3^{3^r} < 3^{3^r} . \tag{5.4}$$

Definition. Let $\mathscr{X} = \{X_1, X_2, \ldots, X_n\}$ be a collection of pairwise disjoint sets. By $\cup(\mathscr{X})$ we mean the set of all unions of the X_i, i.e.,

$$\cup(\mathscr{X}) := \{\bigcup_{i \in I} X_i : I \subseteq \{1, 2, \ldots, n\}\} .$$

Lemma 5.7. Let r and t be fixed integers and let $a_1, a_2, \ldots, a_r$ be defined by (5.3). Further, let $\mathscr{X}$ be a collection of $m := \sum_{j=1}^{r} a_j$ disjoint sets $X_1, X_2, \ldots, X_m$, and suppose that the set $\cup(\mathscr{X})$ is t-colored. Then there exist pairwise disjoint nonempty sets $Y_0, Y_1, \ldots, Y_r \in \cup(\mathscr{X})$ so that the set

$$\{Y_0 \cup Y : Y \in \cup(\{Y_1, Y_2, \ldots, Y_r\})\} \tag{5.5}$$

is monochromatic.

Proof: For the given values of r and t, let $A_1, A_2, \ldots, A_r$ be the sets, and $a_1, a_2, \ldots, a_r$ be the numbers, from Lemma 5.6. Now, let $\cup(\mathscr{X}) = C_1 \cup C_2 \cup \ldots \cup C_t$ be a t-coloring. We will find the desired monochromatic set of the (5.5) as a consequence of Lemma 5.6.

Let ϕ: $A_1 \cup A_2 \cup \ldots \cup A_r \longrightarrow \mathscr{A}$ be a bijection. Suppose that the elements of each of the sets A_i are linearly ordered by $\leqslant_i$ for $i = 1, 2, \ldots, r$. To each r-tuple $(v_1, v_2, \ldots, v_r) \in A_1 \times A_2 \times \ldots \times A_r$ we assign the set $\bigcup_{i=1}^{r} \bigcup_{u_i \leqslant_i v_i} \phi(u_i)$. Consider the t-coloring

$$A_1 \times A_2 \times \ldots \times A_r = C_1' \cup C_2' \cup \ldots \cup C_t'$$

defined by

$$(v_1, v_2, \ldots, v_r) \in C_j' \text{ iff } \bigcup_{j=1}^{r} \bigcup_{u_i \leqslant_i v_i} \phi(u_i) \in C_j .$$

Since the sets of the form (5.5) correspond to sets $B_1 \times B_2 \times \ldots \times B_r$, $B_i \subseteq A_i$, $|B_i| = 2$, then Lemma 5.7 follows at once from Lemma 5.6. ∎

Note that when $t = 2$ then by (5.4) we have $m < 3^{3^r}$.

Proof of Theorem 5.5. We will apply Lemma 5.7 iteratively. Let $m = T_3(4k-2)$ and suppose that the power set $\mathscr{P}(Z)$ of some m-element set Z is 2-colored. By Lemma 5.7 we can find pairwise disjoint nonempty sets $Y_0^{(1)}, Y_1^{(1)}, \ldots, Y_{m_1}^{(1)}$ with $m_1 = T_3(4k-4)$ such that the set

$$\{Y_0^{(1)} \cup Y^{(1)} \colon Y^{(1)} \in \cup(\{Y_1^{(1)}, Y_2^{(1)}, \ldots, Y_{m_1}^{(1)}\})$$

is monochromatic. Now apply Lemma 5.7 again to obtain nonempty pairwise disjoint sets

$$Y_0^{(2)}, Y_1^{(2)}, \ldots, Y_{m_2}^{(2)} \in \cup(\{Y_1^{(1)}, Y_2^{(1)}, \ldots, Y_{m_1}^{(1)}\})$$

with $m_2 = T_3(4k-6)$ so that the set

$$\{Y_0^{(2)} \cup Y^{(2)} \colon Y^{(2)} \in \cup(\{Y_1^{(2)}, Y_2^{(2)}, \ldots, Y_{m_2}^{(2)}\})\}$$

is monochromatic, etc.

This process is continued until we finally obtain nonempty pairwise disjoint sets

$$Y_0^{(2k-1)}, Y_1^{(2k-1)} \in \cup(\{Y_1^{(2k-2)}, Y_1^{(2k-2)}, \ldots, Y_{m_{2k-2}}^{(2k-2)}\})$$

(since $m_{2k-1} = T_3(0) = 1$) so that the set

$$\{Y_0^{(2k-1)} \cup Y^{(2k-1)} \colon Y^{(2k-1)} \in \cup(\{Y_1^{(2k-1)}\})$$

is monochromatic.

Now, since each of the sets $Y_0^{(i)}$, $i = 1, 2, \ldots, 2k-1$, has one of two colors, then there must exist indices $i_1 < i_2 < \ldots < i_k$ such that all the sets $Y_0^{(i_j)}$, $1 \leqslant j \leqslant k$, have the same color. However, by the construction of the $Y_0^{(i)}$, the color of any (nonempty) set $Y = Y_0^{(s_1)} \cup Y_0^{(s_2)} \cup \ldots \cup Y_0^{(s_\ell)}$ with $s_1 < s_2 < \ldots < s_\ell$ depends only on the order of $Y_0^{(s_1)}$. Thus, $\cup(\{Y_0^{(i_1)}, Y_0^{(i_2)}, \ldots, Y_0^{(i_k)}\})$ is monochromatic and the theorem is proved. ■

If our state of knowledge in the hypergraph case of Ramsey's theorem is considered unsatisfactory (we do not even know the order of growth of the Ramsey function $r_3(\ell)$), here the situation is much worse. While the upper bounds we have derived are expressed in terms of multiply-iterated exponential functions, the best lower bounds currently known are incomparably smaller.

Theorem 5.8

$$S(k, 2) \geqslant \frac{k^2}{e^2} \cdot 2^{\frac{1}{k} \cdot 2^k} \tag{5.6}$$

and

$$\cup(k, 2) \geqslant 2^k / \log 2k .$$

The proofs are based on a standard use of the probabilistic method (cf. [ES74]). The lower bound for $\cup(k, 2)$ is stated in [T81]. Here, we give only the proof of the first inequality.

Let us call a set $\{x_1, x_2, \ldots, x_k\} \subseteq [1, n]$ *good* if

$$|\Sigma\{x_1, x_2, \ldots, x_k\}| = 2^k - 1 .$$

To each such set associate the set of all possible sequences $(x_{i_1}, x_{i_1} + x_{i_2}, \ldots, x_{i_1} + x_{i_2} + \ldots + x_{i_k})$. It is clear that:

(i) Each sequence is associated with at most one set;

(ii) A good set is associated with $k!$ different sequences.

Thus, since there are at most $\binom{n}{k}$ possible sequences (they are all increasing) then there are at most

$\frac{1}{k!}\binom{n}{k}$ good sets.

Now 2-color the integers in $[1, n]$ randomly so that each integer is independently colored red or blue with probability 1/2. Then, if

$$\frac{1}{k!}\binom{n}{k}2^{-2^k+2} < 1 \tag{5.7}$$

then there must exist some 2-coloring without a monochromatic set of the form ΣX satisfying $|X| = k$ and $|\Sigma X| = 2^k - 1$. A simple computation shows that (5.7) holds provided

$$n \leqslant \frac{k^2}{e^2} \cdot 2^{\frac{1}{k}\cdot 2^k}$$

and the assertion is proved. ■

If not all the subset sums are required to be distinct, then the corresponding quantity $S^*(k, 2)$ is only known (by unpublished results of Erdös and Spencer) to satisfy

$$S^*(k, 2) > \exp(ck^2/\log k) .$$

As we noted at the beginning of this section, one can give an upper bound for $S(3, t)$ of the form $c^{t \log t}$. It would be very interesting to know if a similar "small" upper bound exists for $S(4, t)$.

6. van der Waerden's Theorem

The celebrated theorem of van der Waerden on arithmetic progressions forms a cornerstone in the edifice of Ramsey theory. In this section we will discuss various numerical aspects of this result. For completeness, we also give a statement and short proof of the theorem.

Theorem 6.1 (van der Waerden [W27]).

For every pair of integers k and r, there exists a least integer $W = W(k, r)$ such that for every r-coloring of $[W] = \{1, 2, \dots, W\}$, some monochromatic arithmetic progression of k terms must be formed.

As often happens, it turns out that it is easier to prove a somewhat stronger statement (which we take from [GR74]). First, we need some notation.

Let $[0, \ell]^m$ denote the set of m-tuples of nonnegative integers not exceeding ℓ. Let us call two m-tuples $(x_1, \dots, x_m)$, $(x_1', \dots, x_m')$ *ℓ-equivalent* if for some $i \geqslant 0$:

$$j < i \Rightarrow x_j = x_j' ,$$

$$x_i = x_i' = \ell ,$$

$$j > i \Rightarrow x_j < \ell , \; x_j' < \ell .$$

If $i = 0$, then only the last condition applies. For any ℓ, m, consider the statement:

$S(\ell, m)$: For any r, there exists $N = N(\ell, m, r)$ so that for any r-coloring χ: $[N] \longrightarrow [r]$ there exist positive integers $a, d_1, \dots, d_m$ such that $\chi(a + \sum_{i=1}^{m} x_i d_i)$ is constant on each ℓ-equivalence class.

Theorem 6.2 [GR74]

$$S(\ell, m) \text{ holds for all } \ell, m \geqslant 1 .$$

Proof: (i) $\underline{S(\ell, m) \text{ for some } m \geqslant 1 \Rightarrow S(\ell, m+1)}$. For a fixed r, let $M = N(\ell, m, r)$, $M' = N(\ell, 1, r^M)$ and suppose χ: $[MM'] \longrightarrow [r]$ is given. Define the induced coloring χ': $[M'] \longrightarrow [r^M]$ so that

$$\chi'(k) = \chi'(k') \text{ iff } \chi(kM - j) = \chi(k'M - j) \text{ for } 0 \leqslant j < M .$$

By the induction hypothesis, there exist a' and d' such that $\chi'(a' + xd')$ is constant for $x \in [0, \ell - 1]$. Since $S(\ell, m)$ also applies to the interval $[(a' - 1)M + 1, a'M] := I$ then by the choice of M, there exist $a, d_1, \ldots, d_m$ with all sums $a + \sum_{i=1}^{m} x_i d_i$, $x_i \in [0, \ell]$, in I and with $\chi(a + \sum_{i=1}^{m} x_i d_i)$ constant on ℓ-equivalence classes. Set $d_i' = d_i$ for $i \in [m]$ and $d_{m+1}' = d'M$. Then $S(\ell, m+1)$ holds with these choices.

(ii) $\underline{S(\ell, m) \text{ for all } m \geqslant 1 \Rightarrow S(\ell + 1, 1)}$. For a fixed r, let χ: $[2N(\ell, r, r)] \longrightarrow [r]$ be arbitrarily given. Thus, there exist $a, d_1, \ldots, d_r$ such that for $x_i \in [0, \ell]$, $a + \sum_{i=1}^{r} x_i d_i$ is bounded above by $N(\ell, r, r)$ and $\chi(a + \sum_{i=1}^{r} x_i d_i)$ is constant on ℓ-equivalence classes. By the pigeon-hole principle there exist $u, v \in [0, r]$ with $u < v$ such that

$$\chi(a + \sum_{i=1}^{u} \ell d_i) = \chi(a + \sum_{i=1}^{v} \ell d_i) .$$

Therefore,

$$\chi((a + \sum_{i=1}^{u} \ell d_i) + t(\sum_{i=u+1}^{v} d_i))$$

is constant for $t \in [0, \ell]$. This proves $S(\ell + 1, 1)$. Since $S(1, 1)$ clearly is true then the theorem holds by induction. ■

Of course, Theorem 6.1 is the special case $m = 1$ in Theorem 6.2.

The upper bound on $W(k, r)$ resulting from this proof is quite large. Essentially, it is given inductively by a function in two variables, and grows like the Ackermann function (cf. [Sp83]). In fact, no proof is known which yields an upper bound on $W(k, r)$ which is even primitive recursive! The same also applies to the special case $W(k) := W(k, 2)$.

On the other hand, the strongest lower bounds for $W(k)$ are much more modest, namely, just exponential in k (cf. Theorems 6.3, 6.4). What the truth really is here represents a central open question in this whole area.

Theorem 6.3 [Ber68]

If p is prime, then

(6.1) $$W(p + 1) \geqslant p \cdot 2^p .$$

Proof: For simplicity, we only prove a slightly weaker result:

(6.2) $$W(p + 1) \geqslant p(2^p - 1) .$$

Let $GF(2^p)$ denote the finite field with 2^p elements, and fix a primitive element $\alpha \in GF(2^p)$. Let $v_1, \ldots, v_p$ be a basis for $GF(2^p)$ over $GF(2)$. For any integer j, set

$$\alpha^j = a_{1j} v_1 + a_{2j} v_2 + \ldots + a_{pj} v_p , \quad a_{ij} \in GF(2) .$$

Let

$$C_0 = \{j:\ a_{1j} = 0\,,\quad 1 \leqslant j \leqslant p(2^p - 1)\}\,,$$

$$C_1 = \{j:\ a_{1j} = 1\,,\quad 1 \leqslant j \leqslant p(2^p - 1)\}\,.$$

We claim that neither C_0 nor C_1 contains a $(p+1)$-term arithmetic progression. Suppose, on the contrary, that $\{a, a+d, \ldots, a+pd\} \subseteq C_i$ for some i. Set $\beta = \alpha^a$, $\gamma = \alpha^d$. Since

$$1 \leqslant a < a + pd \leqslant p(2^p - 1)$$

then $d < 2^p - 1$ and so, since α is primitive, we have $\gamma \neq 1$. Therefore, $\beta, \beta\gamma, \ldots, \beta\gamma^{p-1}$ are all distinct and since $\beta\gamma^\ell = \alpha^{a+\ell d}$ then the elements $\beta, \beta\gamma, \ldots, \beta\gamma^p$ all have the same first coordinate, considered as vectors.

Case 1. $i = 0$. Then $\beta, \beta\gamma, \ldots, \beta\gamma^{p-1}$ are p vectors in a $(p-1)$-dimensional space (since the first coordinate is 0), and hence, they are dependent. Thus, there exist $a_0, a_1, \ldots, a_{p-1} \in GF(2)$, not all 0, such that

$$\sum_{i=0}^{p-1} a_i(\beta\gamma^i) = 0$$

and so,

$$\sum_{i=0}^{p-1} a_i\gamma^i = 0$$

which is impossible for $\gamma \in GF(2^p)$, $\gamma \neq 0, 1$ (cf. [MS78, Ch. 4, Th. 10]).

Case 2. $i = 1$. Thus, $\beta(\gamma - 1), \beta(\gamma^2 - 1), \ldots, \beta(\gamma^p - 1)$ belong to a $(p-1)$-dimensional space, which implies

$$\sum_{i=0}^{p} a_i\beta(\gamma^i - 1) = 0$$

for $a_i \in GF(2)$ not all 0. Dividing by $\beta(\gamma - 1)$, we again see that γ satisfies a polynomial of degree at most $p - 1$, a contradiction. ∎

For general k, a slightly weaker lower bound is available (cf. [GRS80]).

Theorem 6.4

(6.3) $$W(k) > \frac{2^k}{2e\,k}\,(1 + o(1))\,.$$

The proof is based on the following lemma, which is a consequence of Theorem 2.2.

Lemma 6.5. Let $A_1, A_2, \ldots, A_n$ be events with $Pr(A_i) \leqslant p$ for all i, and with a dependence graph having maximum degree d. Then

(6.4) $$e\,p(d+1) < 1 \ \Rightarrow\ Pr(\bigcap_i \bar{A}_i) > 0\,.$$

Proof: We apply Theorem 2.2 with $x_1 = \ldots = x_n = d/(d+1)$. The hypothesis of Theorem 2.2 then becomes

(6.5) $$p < \frac{d^d}{(d+1)^{d+1}}\,.$$

Since

$$p(d+1)\left(1 + \frac{1}{d}\right)^d < e\,p(d+1) < 1$$

then (6.5) is satisfied and thus, by Theorem 2.2, $Pr(\bigcap_i \overline{A}_i) > 0$. ∎

Proof of Theorem 6.4: We red-blue color $[1, n]$ randomly so that each integer in $[1, n]$ is independently assigned the color red or blue with probability 1/2. To each k-term arithmetic progression P associate the event A_p: "P is monochromatic". Two vertices P and Q in the dependence graph Γ form an edge if $P \cap Q \neq \varnothing$. The maximum degree d in Γ clearly satisfies $d \leqslant nk$. Thus, by Lemma 6.5, if $n \leqslant \frac{2^k}{2e\,k}(1 - o(1))$, then $Pr(\bigcap_P \overline{A}_P) > 0$. This implies that there are 2-colorings of $[1, n]$ having no monochromatic k-term arithmetic progressions and the claim is proved. ∎

The estimation of $W(k)$ is closely related to the following problem:

Estimate

$$v_k(n) := \max\{|S|:\ S \subseteq [1, n],\ S \text{ contains no } k\text{-term arithmetic progression}\}\ .$$

The first major result for this problem was due to Roth [Ro53] who showed

$$v_3(n) = O\left(\frac{n}{\log\log n}\right)\ . \tag{6.6}$$

This was followed by the result of Szemerédi [Sz67] that $v_4(n) = o(n)$, and finally by the celebrated theorem of Szemerédi [Sz75], settling a \$1000 conjecture of Erdös and Turán:

Theorem 6.5 [Sz75]

For all k,

$$v_k(n) = o(n)\ . \tag{6.7}$$

(We will *not* present a proof of (6.7) here.) In this section we will restrict our discussion to the case $k = 3$. In this case, the best current bounds are given in the following result.

Theorem 6.6

$$n \exp(-c_1 \sqrt{\log n}) < v_3(n) < c_2 n/(\log n)^{c_3} \tag{6.8}$$

for suitable positive constants c_1, c_2, c_3.

The lower bound is a classical result of Behrend [Beh46]; the upper bound is due to Szemerédi and Heath-Brown [H]. First, we give a proof of the lower bound.

For $d \geqslant 1$ and $x \leqslant n$, set

$$x = \sum_{j=0}^{k} x_j (2d+1)^j\ ,\quad 0 \leqslant x_j \leqslant 2d\ .$$

Let

$$N(x_0, x_1, \ldots, x_k) := \left(\sum_{i=0}^{k} x_i^2\right)^{1/2}$$

and define

$$X_{n,d,s} := \{x:\ 1 \leqslant x \leqslant n\ ,\quad 0 \leqslant x_i \leqslant d \quad \text{for all } i\ , \quad \text{and}$$
$$N(x_0, x_1, \ldots, x_k) = \sqrt{s}\}\ .$$

Claim: $X_{n,d,s}$ contains no 3-term arithmetic progression.

Proof: Suppose that

$$x = \sum_{i=0}^{k} x_i (2d+1)^i ,$$

$$y = \sum_{i=0}^{k} y_i (2d+1)^i ,$$

$$z = \sum_{i=0}^{k} z_i (2d+1)^i$$

satisfy $x + y = 2z$ where $x \neq y$, $z \in X_{n,d,s}$. Since x_i, y_i, z_i are all less than $d + 1$ then in fact we must have $x_i + y_i = 2z_i$ for all i. Furthermore,

$$N(x_0, x_1, \ldots, x_k) = N(y_0, y_1, \ldots, y_k) = N\left(\frac{x_0 + y_0}{2}, \ldots, \frac{x_k + y_k}{2}\right)$$

which implies

$$\sum_{i=0}^{k} (x_i - y_i)^2 = 0 ,$$

i.e., $x_i = y_i$ for all i, a contradiction.

For fixed d we have

$$k \sim \frac{\log n}{\log(2d+1)}$$

and there are at most $d^2 k$ possible values for s. The union of the $X_{n,d,s}$ over all s contains all sums $\sum_i x_i (2d+1)^i$, $0 \leqslant x_i \leqslant d$, which are all at most n. There are essentially $n 2^{-k}$ such integers. Thus, for some s

$$\nu_3(n) \geqslant |X_{n,d,s}| \geqslant \frac{n}{d^2 k 2^k} .$$

Setting $d = \exp(\sqrt{\log n})$ we deduce

$$|X_{n,d,s}| \geqslant n \exp(-c\sqrt{\log n})$$

for some $c > 0$, as required.

Instead of the upper bound in (6.8), we only show here that $\nu_3(n) = o(n)$. There are other relatively simple proofs of this fact (cf. [RS78], [G81]). The proof give here, based on ideas of Ruzsa and Szemerédi, is taken from [EFR86].

Theorem 6.7

(6.9) $$\nu_3(n) = o(n) .$$

Proof: Let $G = (V, E)$ denote a graph and let $A, B \subseteq V$ be a pair of disjoint non-empty subsets of V. The *density* of the pair (A, B) is defined to be the ratio

$$d(A, B) := e(A, B)/|A||B|$$

where $e(A, B)$ denotes the number of edges $\{a, b\}$ with $a \in A$, $b \in B$. The pair (A, B) is called ϵ*-uniform* if for all $A' \subseteq A$, $B' \subseteq B$ with $|A'| > \epsilon|A|$, $|B'| > \epsilon|B|$ we have

$$|d(A', B') - d(A, B)| < \epsilon .$$

A partition $V = C_0 \cup C_1 \cup \ldots \cup C_k$ is called ϵ*-uniform* if:

(1) $|C_0| < \epsilon |V|$;

(2) $|C_1| = |C_2| = \ldots = |C_k|$;

(3) All except at most $\epsilon \binom{k}{2}$ of the pairs (C_i, C_j), $1 \leqslant i < j \leqslant k$, are ϵ-uniform.

The following fundamental result (which we state here without proof) is due to Szemerédi.

Lemma 6.8. (Regularity Lemma [Sz76]). For every $\epsilon > 0$ and positive integer ℓ, there exist positive integers $n_0(\epsilon, \ell)$ and $k_0(\epsilon, \ell)$ such that every graph with at least $n_0(\epsilon, \ell)$ vertices has an ϵ-uniform partition into k classes, where k is some integer satisfying $\ell < k < k_0(\epsilon, \ell)$.

Now, let $A \subsetneq [1, n]$ and let X, Y, Z be three disjoint copies of the interval $[1, 3n]$. Consider the set S of all triples $\{x, y, z\}$, $x \in X, y \in Y, z \in Z$ such that

$$y - x = z - y = \frac{z - x}{2} \in A \,. \tag{6.10}$$

Further, let G be the graph consisting of all pairs contained in the triples of S. Thus, G has $9n$ vertices, $|E(G)| \geqslant 3|A|n$ edges, and $E(G)$ can be decomposed $\frac{1}{3}|E(G)|$ edge-disjoint triangles $\{x, y, z\}$, which we call "ordinary" triangles.

Claim 6.9. [RS78] If G contains a triangle which is not ordinary, then A contains a 3-term arithmetic progression.

To see this, let x', y', z' be such a triangle where $y' - x' \neq z' - y'$. Thus, for $a := y' - x'$ and $b := z' - y'$, we have $a \in A$, $b \in A$, and $\frac{a+b}{2} = \frac{z'-x'}{2} \in A$, which forms the required 3-term arithmetic progression.

Suppose now that $|A| = \alpha n$ for a fixed positive constant α independent of n. We will show that for n sufficiently large, A must contain a 3-term arithmetic progression. Set $m = 9n$ and set

$$|E(G)| = \beta \binom{m}{2} > 3\alpha n^2$$

where β is a fixed positive constant independent of n. Further, set $\epsilon = \beta/15$ and $\ell = \lceil \epsilon^{-1} \rceil$. We now apply the Regularity Lemma to G with these choices (and $n > n_0(\epsilon, \ell)$). The number of edges not contained in pairs with density at least $\beta/6$ is at most

$$k\binom{m/k}{2} + \frac{\beta}{6}\binom{k}{2}\left(\frac{m}{k}\right)^2 + \epsilon\binom{k}{2}\left(\frac{m}{k}\right)^2 + \epsilon m^2 < \frac{\beta}{3}\binom{m}{2} \,.$$

After the deletion of these edges we obtain a graph G' which still contains a triangle T (there were $\frac{\beta}{3}\binom{m}{2}$ edge-disjoint triangles in G). Moreover, all three edges of this triangle are contained in pairs which are ϵ-uniform and which have density at least $\beta/6$. Let C_p, C_q and C_r be the three partition classes containing the three endpoints of T.

Claim 6.10. If all three pairs (C_p, C_q), (C_p, C_r) and (C_q, C_r) are ϵ-uniform with density at least $\beta/6$ then there is a vertex $x \in C_r$ which is contained in at least $\left(\frac{\beta}{10}\right)^3 |C_p||C_q|$ triangles.

Proof: If both (C_p, C_r) and (C_q, C_r) are ϵ-uniform then we can find at least $(1 - 2\epsilon)|C_r|$ vertices $x \in C_r$ which are joined to at least $\left(\frac{\beta}{6} - \epsilon\right)|C_i|$ vertices of C_i, for $i = p$ and $i = q$. Fix one such vertex x and let N_x^i be the set of neighbors of x in C_i. Since $\frac{\beta}{6} - \epsilon = \frac{\beta}{10} > \epsilon$, we find there are at least $\left(\frac{\beta}{10}\right)^3 |C_p||C_q|$ edges joining vertices of N_x^p and N_x^q. Each such edge clearly corresponds to a triangle containing x. ■

To complete the proof of Theorem 6.7, note that by Claim 6.10, for m sufficiently large (and k fixed) there are at least

$$(6.11) \qquad \left(\frac{\beta}{10}\right)^3 |C_p||C_q| > |C_p| = |C_q|$$

triangles containing the vertex x. Since no two ordinary triangles in G share two vertices then there are at most $|C_p| = |C_q|$ ordinary triangles containing the vertex x. Thus, by (6.11), G contains a triangle which is not ordinary, and so by Claim 6.9, A contains a 3-term arithmetic progression. This completes the proof of Theorem 6.7. ■

We conclude this section with several remarks. To begin with, as an indication of the extent of our ignorance on the true order of growth of the van der Waerden function $W(k)$, the first author has made for some time the following offer.

Conjecture ($1000) For all k,

$$W(k) \leqslant \left.2^{2^{2^{\cdot^{\cdot^{2}}}}}\right\} k\,.$$

The known values are:

$$W(2) = 3\,,\quad W(3) = 9\,,\quad W(4) = 35\,,\quad W(5) = 178\,.$$

An interesting variation which was considered in [G83] is the following. Define $W^*(k)$ to be the least integer such that there exists a set $X(k) \subseteq \{1, 2, \ldots\}$ with $|X(k)| = W^*(k)$ so that any 2-coloring of $X(k)$ always forms a monochromatic k-term arithmetic progression.

Problem 6.11

$$\text{Does } W^*(k)/W(k) \longrightarrow 0 \text{ as } k \longrightarrow \infty\,?$$

It is known [G83] that $W^*(2) = W(2)$, $W^*(3) = W(3)$ but $W^*(4) \leqslant 27 < 35 = W(4)$.

7. Concluding remarks

This paper has dealt almost exclusively with asymptotic bounds for various results of Ramsey type. The *exact* values for the associated functions are invariably much more difficult to obtain. As an indication of this difficulty, we list in Table 1, all known (non-trivial) values of $r(k, \ell)$, $k \leqslant \ell$, together with the best bounds currently available for several other values of $r(k, \ell)$. It would appear, for example, that the determination of $r(5, 5)$ will require some significant new ideas. The reader is referred to [CG83] or [RK] for a fuller discussion.

$k \backslash \ell$	3	4	5	6	7	8	9	10
3	6	9	14	18	23	28–29	36	39–44
4	–	18	25–28	39–44				
5	–	–	42–55	57–94				
6	–	–	–	102–169				

Small values of $r(k, \ell)$

Table 1

In the other direction, the best upper bound we currently have for the van der Waerden function $W(k)$ grows like the Ackermann function. While this is not known to be the true order of growth of $W(k)$ (and, in the opinion of many combinatorialists, is a gross over-estimate), it turns out that there are in fact a number of natural problems of Ramsey type which *do* have functions which grow this rapidly, and indeed, *much* more rapidly. The earliest example of this phenomenon was exhibited in the celebrated result of Paris and Harrington [PH77]. To state their result, we need one definition. Let us call a finite set S of positive integers *large* if $\min(S) > |S|$. Consider the following statement:

(7.1) For all k and t, there exists a least number $PH(k, t)$ so that if $n \geqslant PH(k, t)$ then in any t-coloring of the k-element subsets of $[1, n]$ there must exist a monochromatic *large* set $B \subseteq [k+1, n]$

The truth of (7.1) follows easily from the infinite form of Ramsey's theorem. What was unexpected was that while (7.1) is a perfectly well-defined statement in Peano Arithmetic (PA) (that first order theory of numbers which includes the basically finitistic methods of number theory), it is in fact *unprovable* in PA. One way of proving this, as pointed out by Ketonen and Solovay [KS81], is to show that the function $PH(k, t)$ grows so rapidly that it cannot even be defined in PA. A nice-description of this work is given in [Sm80], [Sm82] and [Sp83].

In some sense, even more striking is a very recent result of Friedman, which can be considered as a finite form of a well known result of Kruskal [Kr60] (which asserts that the set of finite trees is well-quasi-ordered under homeomorphic embedding). This finite form is given by the following statement:

(7.2) For each $k > 1$, there is a least number $F(k)$ such that if $n \geqslant F(k)$ and $T_1, T_2, \ldots, T_n$ is a sequence of trees with T_i having at most $k + i$ vertices then there exist $i < j$ such that that T_i is homeomorphically embeddable in T_j.

Although (7.2) only deals with finite sets of finite objects, any proof of (7.2) must, in a certain precise sense, invoke the concept of uncountability. Again, this is due from one point of view to the extremely rapid growth rate of the function $F(k)$. A full account of these fascinating developments can be found in [Sm82], [Sp83] and [NT87].

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