by Joe Buhler and Ron Graham

Fountains, Showers, and Cascades

Juggling's quintessential combinations of algebra and acrobatics

What could be simpler than a game of catch? Anyone can follow a single ball thrown back and forth between two people. But add another ball or two and at once the game turns magical—the juggled balls take on a life of their own. It becomes difficult to tell whether the balls or the jugglers are in command, where the catch ends and the throw begins, or even how many balls there are. Suddenly, simple motions and common objects blur into one stunning display after another.

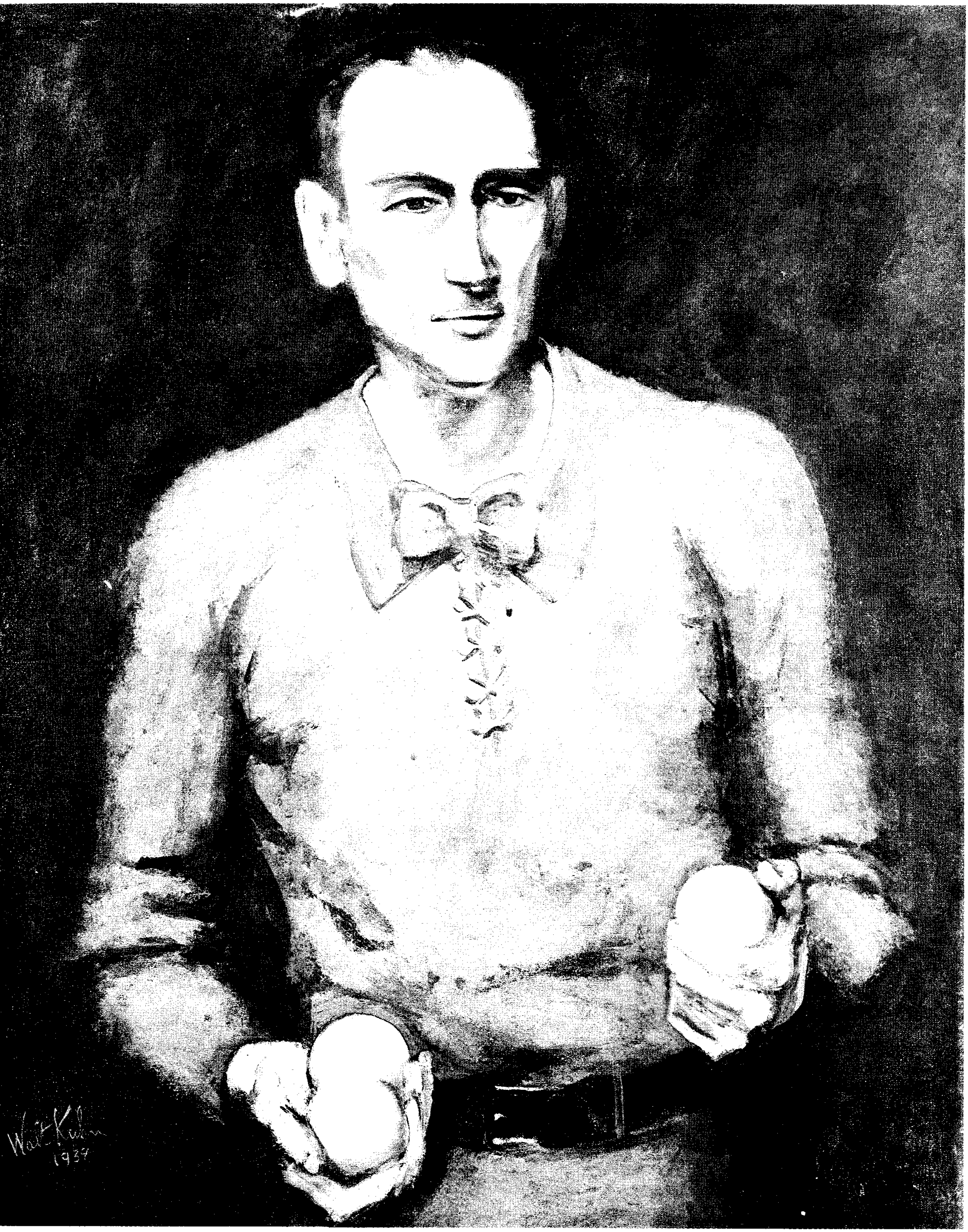
In the past decade, the art of juggling, which had declined since the days of traveling vaudeville troupes and circuses, has made a comeback. Street performers and skilled amateurs have been transmuting and permuting the game of catch in parks, backyards, and campus quadrangles around the globe. The membership of the International Jugglers Association—almost entirely amateur—has grown sixfold in the past five years. And many of the new aficionados are scientists and mathematicians who are drawn to juggling in part because it demands a talent for manipulating, inventing, and experimenting. Like music-making, it is a common ground between abstract form and physical dexterity; like mathematics, it is a form of pure play. And the same curiosity that impels scientists to discover the laws of natural forms is driving some of them to explore the fundamental laws and constraints that govern juggling.

Until recently, no one had troubled to formulate the rules of the game, to dissect juggling as a phenomenon dictated by the laws of physics. Most jugglers, even those who are highly talented, do not think analytically about what they do. For most of its four-thousand-year history, juggling was an intuitive art—in some places religious or mystical, in others purely frivolous. The earliest known evidence of it comes from an Egyptian tomb, dating from about 1900 B.C., that contains a painting of a woman juggling three balls. In ancient Egypt, as in India, China, Japan, Iran, and the Americas, juggling was part of religious ritual, performed only by a shaman or a priest—someone with divine connections. The Greeks and Romans seem to have taken juggling more lightly, lumping it in the same category as gymnastics and magic tricks. In medieval Europe, wandering minstrels were often jugglers, and some say the very term derives from these *joculatores* and *jongleurs*. But juggling was just a small part of the jongleurs' shows, which included songs, comedy routines, and sleight of hand.

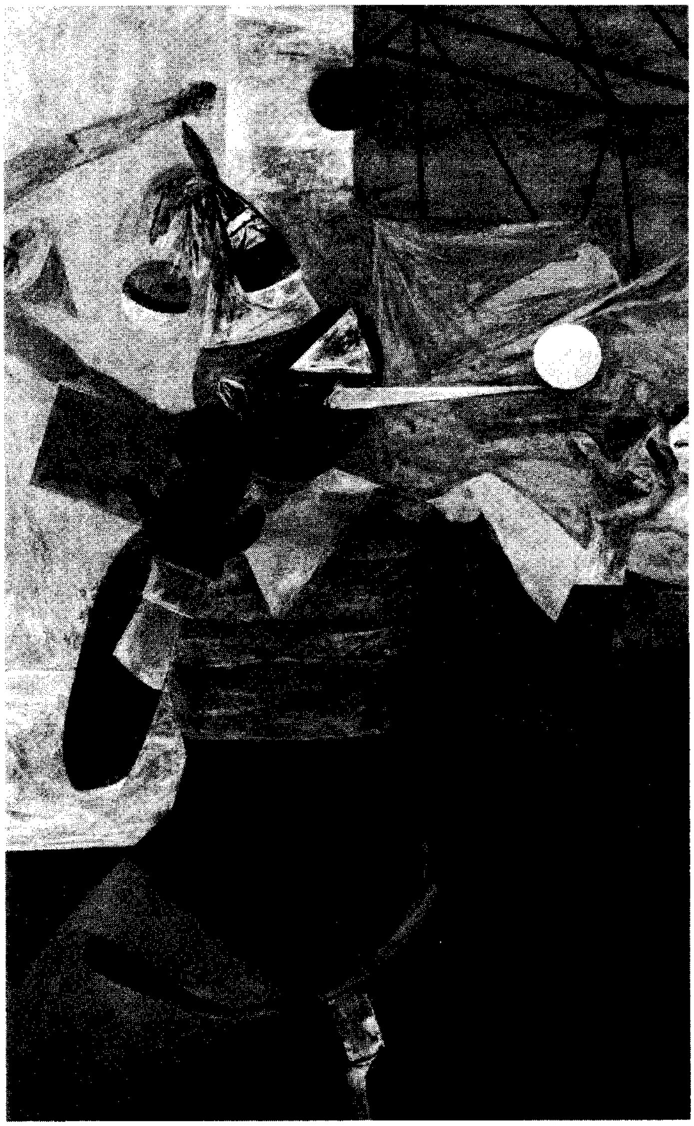
In the nineteenth century, juggling began to gain legitimacy as an art unto itself. European ethnographers in search of the "exotic" discovered Samoan islanders juggling oranges, and Burmese tossing and catching balls with their shoulders and feet. The amazing feats of juggling imported from the Orient—in particular those of the East Indian Ramo Samee, who juggled strings of beads and hollow brass balls with his mouth, and the Japanese artist Takashima, who could manipulate a cotton ball with a stick held in his teeth—convinced European audiences that juggling could be extraordinary show business.

By the turn of the century, every vaudeville show and circus had a juggling act, some featuring such remarkable performers as Enrico Rastelli, Bobby May, and Francis Brunn. Rastelli, probably the greatest juggler of all time, was known to practice ten hours a day. By the time of his death at age thirty-five, he had taught himself to juggle eight clubs, eight plates, or, briefly, ten balls; he could even bounce three balls continuously on the crown of his head. His fanatical virtuosity lured many an innocent to the sport, including the young W.C. Fields. Juggling's golden age, however, was nearly as short-lived as Rastelli himself. The traveling circuses and vaudevilles were eclipsed by Hollywood, radio, and television, and, until its recent revival, juggling all but vanished from the public eye.

Overheard in the corridors of a famous mathematics department: "The narrower pattern of the fountain," said one careful voice, "makes it intrinsically more difficult than the cascade, though the flight-to-dwell ratio is the same." "But having two independent feedback loops," said a second, "accommodates minor variations more easily. Rastelli preferred the fountain."



Walt Kuhn, **The Juggler,** *1934*



Yasuo Kuniyoshi, **Amazing Juggler,** *1952*

The two were debating, not hydrodynamics or some obscure branch of mathematics, but the analysis of juggling.

Juggling can be analyzed mathematically in at least two ways: through dynamics, the study of objects in motion; and through combinatorics, the study of ways objects and groups of objects can be combined. Dynamics can help explain the behavior of the bewildering variety of objects that jugglers use today, which include whips, batons, plastic swimming pools, Rubik's Cubes, fruit, playing cards, bowling balls, cigar boxes, and flaming torches. The Flying Karamazov Brothers, of San Francisco (who also juggle social commentary), have been known to throw a running chain saw back and forth in their act. Despite the proliferation of juggling objects, there are three main categories: balls (including limes and other fruit), rings (including plates), and clubs (including axes and flaming torches).

The dynamics of rotating solid bodies dictates the best type of throw for each object. A ring is best thrown so that it spins like a wheel in midair, and when thrown properly it is quite stable, wobbling only very slightly. A club is most stable if thrown so that it spins end over end. When it comes to juggling balls, there is no choice; all their axes are alike. Thus the balls' rotation is not very stable (neither, for that matter, are the rotations of the planets as they whirl around the sun). But because balls are spherically symmetrical, their wobbling does not ruin the pattern.

Jugglers trying for the largest possible number of objects usually use rings, which allow for a tighter traffic pattern and are stable when thrown to great heights. One can stop the action by thrusting one's hand through the rings, which is easier than catching five balls at once (needless to say, the hand-thrusting technique should not be attempted with plates). Several contemporary jugglers have been reliably reported to juggle ten or eleven rings or plates, and there are rumors that some are even working on twelve or thirteen. Clubs are probably the most visually pleasing objects to juggle, and are especially suited for passing back and forth between two or more performers. Because clubs take up a lot of space when they rotate and must be caught at one end, however, juggling even five of them is very tricky. Few performers have managed seven clubs with witnesses present, not even for a few seconds. For beginners, balls are best, because one does not have to worry about which end of the ball to catch, and because balls are familiar to the hand.

As far as we know, peoples of all cultures, from South Sea islanders to Aztec Indians, whether juggling sticks, stones, or *tui-tui* nuts, have used the same fundamental patterns. Combinatorics can help to quantify and codify these patterns, the most basic of which are the cascade, the shower, and the fountain. This branch of mathematics has been applied to routing and scheduling problems, which can be colossally complicated for a large company with many orders to ship, an airport with many planes in holding patterns, or a computer with many messages to juggle and store. For such intricate problems, combinatorics cannot usually provide the optimum strategy, but it *can* help find a highly efficient one, one in which little time, energy, or money is wasted. In a sense, the traditional juggling patterns, the cascade, the shower, and the fountain, represent problems in combinatorics that have already been intuitively solved: they are brilliant ways of scheduling the departures and arrivals of *tui-tui* nuts.

In the cascade, each ball travels from one hand to air to the other hand to air and back again, following a looping path that looks rather like a figure eight lying on its side (or the mathematical symbol for infinity). The juggler often starts with two balls in his right hand, using a sort of scooping motion and releasing a ball when his throwing hand is level with his navel. As the first ball reaches its apogee, he scoops and releases a second ball with his left hand, and as the second ball reaches its apogee he throws the third. Once the balls are in motion, the juggler never holds more than one ball in a hand at a time. Skilled jugglers can keep three, five, or even seven balls going in a cascade pattern, but never four or six: with an even num-

***Joe Buhler** is an associate professor of mathematics at Reed College, in Portland, Oregon, and a juggler. **Ron Graham** is director of the Mathematics and Statistics Research Center at Bell Laboratories, in Murray Hill, New Jersey, and a past president of the International Jugglers Association.*

ber of balls, each pair would collide at the intersection of the figure eight.

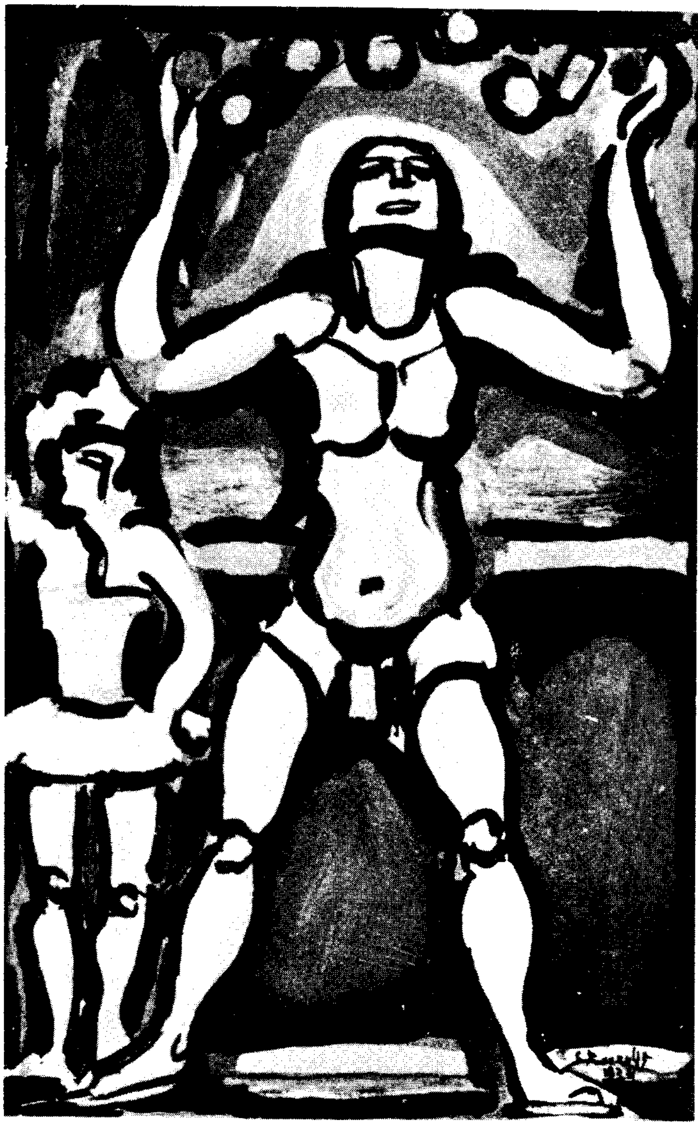
One can perform a cascade in two basic ways: the normal cascade, and the reverse cascade, which looks like a movie of the first played backward. Since the two are symmetrical—time reversals of each other—one would expect them to be equally easy, but a simple analysis shows why they are not. In the first, the balls pass close together *early* in their paths; in the second, they do so quite *late*. It is much easier to throw two balls so that they pass close together on their way up and come down far apart than to throw them so that they go up far apart and come down close together—in the latter throw the two balls are much more likely to collide.

The shower is a more difficult maneuver than the normal cascade, though among some peoples, such as the Tongans, of the South Sea islands, it is the only pattern known. The balls in a shower follow a more or less circular path as they are thrown upward by the right hand, caught by the left, and then quickly passed back to the right. Since the right hand does all the long-distance throwing, the shower is inherently asymmetrical and, therefore, inefficient; it is difficult for more than three objects. It does, however, permit a juggler to manipulate an odd *or* even number of balls.

The juggler wanting to keep a large, even number of balls in motion must rely on the third basic pattern, the fountain (also called the waterfall). To create a four-ball fountain, the juggler starts with two balls in each hand, and the two hands juggle them independently in a circular motion. For symmetry's sake this technique is usually performed with an even number of balls, but, as in the shower, any number can be used—the juggler is limited only by his own dexterity. If the hands throw alternately and the two circular patterns interlock, it is surprisingly hard for an onlooker to discern that the fountain is made of two separate components and not one large, interconnected pattern.

Practicing jugglers have an intuitive sense of the constraints of these patterns. They know what it is like to struggle against gravity with too few hands and too many balls. And they know all too well that if one ball or club is thrown too early, too late, too high, or too low, an entire performance can degenerate into a stageful of bouncing paraphernalia. The constraints can be expressed with mathematical precision, using algebra and combinatorics.

IN MATHEMATICAL TERMS, a juggler juggles five basic variables. He is free to vary the number of *balls* (b) he is juggling; the number of *hands* (h) with which he juggles; the *flight* time (f) of each ball between his hands; the length of time that a hand is *empty* (e) between catches; and the length of time that a ball *dwells* (d) in a hand between throws. For convenience, assume that two balls are never in the same hand at the same time, that the pattern in which they are thrown is periodic (in the sense that each configuration of balls recurs at fixed time intervals), and that each ball meets each hand, by following the same path, and always taking the same amount of time to complete a cycle. (These assumptions hold for the cascade, but not for the shower or fountain, which require more complicated mathematical descriptions.)



Georges Rouault, **Juggler**, *1934*

Associated American Artists, New York

In one *period* (p)—the time that it takes each ball to meet each hand—each ball will "dwell" in and "fly" out of each hand once. If there are h hands, the duration of the ball's journey will increase by $d+f$, or the combined dwell and flight times, for each hand it meets. Thus, one period equals $h(d+f)$. We can also calculate the length of one period from the hand's perspective. During a given period, each hand will hold and let go of (or be "empty" of) each ball. If there are b balls, the duration of the period for the hand will increase by $d+e$, the combined time that the hand holds and is empty of a ball, for every ball it meets. For the hand, then, one period equals $b(d+e)$. Of course, the time it takes for each hand to meet each ball is equal to the time it takes for each ball to meet each hand. Therefore, $b(d+e) = h(d+f)$, or $b/h = (d+f)/(d+e)$. A good name for this equation would be Shannon's Theorem, since it derives from the thinking of the mathematician Claude Shannon, formerly of Bell Laboratories and the Massachusetts Institute of Technology, and now retired; but several of the seminal theorems in coding and information theory already bear that name. With the use of this particular Shannon's Theorem, then, we can see that each variable in the juggler's art is related to the others, and is affected by changes in any of the others.

For example, say we deprive the juggler of one of his basic freedoms: he cannot alter how long each ball is in flight. What sort of freedoms are left? The juggler can still slow the pattern slightly by holding each ball longer (increasing dwell time), or speed it up by releasing each ball faster (increasing empty time). Either of these liberties can be carried too far: at one extreme is delayed juggling, in which the juggler holds onto one ball as long as possible before catching the next one (empty time = 0), and at the opposite extreme is "hot potatoes" juggling, in which the juggler barely touches one ball before tossing it away (dwell time = 0). Neither extreme is physically possible, but the relationship between them defines a juggler's freedom to vary the period length, given a fixed flight time.

To derive a measure of delayed juggling, we need only set $e=0$ (since the hand is never empty) and solve for d. Using Shannon's Theorem, we find that $d=(h/f)/(b-h)$. To find a measure of "hot potatoes" juggling, we do the reverse; by making $d=0$ (since the ball never dwells in the hand) and solving for e, we find that $e = hf/b$. The ratio of these two extremes expresses the juggler's freedom to vary his speed between the slowest and fastest juggling times. According to the ratio $[hf/(b-h)]/[hf/b]$, which equals $b/(b-h)$, then, the range of possible juggling speeds (the ratio of the largest to the smallest possible period lengths) increases with the number of hands and decreases with the number of balls. For instance, if a two-handed juggler juggles three balls, the ratio between the slowest and fastest possible speed is $3/(3-2)$, or three to one. But a two-handed juggler juggling six balls has less play; the ratio between his slowest and fastest speeds is $6/(6-2)$, or three to two. It is clear just how much the juggler's limited number of hands constrains his repertoire. If he had *four* hands to juggle six balls, the ratio between the slowest and fastest juggling speeds would be $6/(6-4)$, or three to one; it would be as easy as juggling three balls with two hands.

With constraints like these, it is not surprising that jugglers find the thought of extra arms quite tantalizing. Picture, for instance, a distant planet inhabited by a race of intelligent humanoids with three arms. So blessed, it is hard to believe that such creatures would *not* juggle. What would their basic juggling patterns be like? What, for example, would be the three-handed analogue of our two-handed cascade?

A three-armed humanoid with an arm extending from the middle of his chest could create a five-ball cascade in which each ball followed a double figure-eight pattern. In this pattern the middle hand would have to do twice as much work as the outer two. It would throw to the right

A Lesson in Juggling

Courtesy Hovey Burgess, New York

Mr. Paul Cinquevalli, a pitch-and-toss champion

THE BEST WAY TO UNDERSTAND the essence of juggling is to learn to do it yourself. As in learning mathematics, the student of juggling starts from basic facts and skills and is led step by step to new and ever more complex permutations. Mastering these requires new perceptual skills, beyond the mere motor ability to catch and throw. To many a neophyte juggler, the balls seem to fall so quickly that catching them seems impossible. But with practice, the juggler learns to see the balls differently. Eventually, they actually look as if they were falling much more slowly.

Some adults, perhaps as many as five percent, can learn the three-ball cascade in minutes. Most people, however, acquire the basic idea with a little coaching and then need from one to sixty days of practice to achieve a reasonably stable pattern. The novice will be comforted to know that once learned, juggling, like bicycle riding, is almost impossible to forget.

Beware of spending too much time in one practice session. It is better to try for ten minutes on several different occasions than to frustrate yourself with a two-hour binge.

As for equipment, three lacrosse balls, tennis balls, or beanbags will do nicely.

STEP 1: ONE BALL

Practice throwing a ball from your right hand to your left and back again. The ball should be thrown higher than eye level but no higher than your arms can reach. Most people do best with a height just a bit above the tops of their heads. Try to make the ball follow the path of a figure eight lying on its hand at the same time that the left hand threw to it, and then it would throw to the left hand as the right hand threw to it. Such a three-armed creature could also create a shower, though a rather cumbersome one: one outer arm would throw long passes to the other, while the middle arm would act as the "middleman," quickly relaying the balls from the catching arm to the throwing arm. For a fountain pattern three arms would be a real asset; for instance, a six-ball fountain would require each hand to juggle only two balls.

Of all the constraints that bind a juggler on any planet, the force of gravity is by far the most confining, because gravity brings any object down from a given height in a precisely fixed period of time. The strength of Earth's gravitational field rigidly defines how long an object will stay in flight once it is thrown. If a juggler throws a ball with one hand to a *height* (H) and catches it with the other, the ball's total flight time on its parabolic path can be described as $2\sqrt{2H/g}$, where g equals the downward acceleration of an object, due to the gravitational field. On Earth, where g equals about 9.8 meters per second squared, the juggler has a very short time to catch and throw one ball before another drops in his hand. He can slow down the process slightly by throwing the balls higher, but, unfortunately, higher throws also tend to be less accurate. Furthermore, the amount of flight time gained by a higher throw increases only as the square root of the extra height. For instance, a juggler who throws a ball two meters in the air has roughly 1.3 seconds from the time he throws it to the time he must catch it. But a juggler who throws a ball four times that height —eight meters—only doubles the flight time while losing a great deal of accuracy.

Since gravity imposes such sharp constraints, mathematically minded jugglers often daydream of the feats they could accomplish if the acceleration due to gravity were not so great. One way to explore these possibilities, of course, would be to send a juggler to the moon or some other celestial body with a weak gravitational force, to see just how many balls he could juggle. On the moon's surface, where the gravitational force is about one-sixth that on Earth, the flight time of a juggler's throw would be about two-and-a-half times as long as that of the same throw on this planet. Using Shannon's Theorem, we estimate that a juggler who can keep a seven-ball cascade going here on Earth could sustain fifteen balls or maybe even more on the moon. (Incidentally, the same calculation can be made for any planet simply by deriving a ratio of flight times for projectiles on Earth and on the planet in question.)

To sample the pleasures of low-gravity juggling here on Earth, jugglers have gone as far as taking lacrosse balls underwater and trying to toss them in the conven-

side: you can do this by slightly "scooping" the ball before throwing it, and releasing it near the navel. Catch the ball at the side of your body, and then repeat the sequence of scoops, throws, and catches.

Step 2: Two balls

Put one ball in each hand. Throw the ball in the left hand as in Step 1, and then, just as the ball passes its high point, throw the right-hand ball. (Left-handed people should reverse this sequence and all succeeding steps.) The sequence of throws is thus left, then right, with a noticeable pause between throws. Two very common problems are not waiting long enough to release the second throw, and not throwing the balls to approximately equal heights.

At first it may be difficult to catch the balls. Don't worry. Try to focus instead on the accuracy of the throws and on their height. The catching skill will appear naturally as soon as the throws are on target. Keep the two throws in a plane parallel to your body. It is important to throw the second ball so that it passes the first with a bit of room to spare. If things seem hectic, try increasing the height of the throws.

Step 3: Two balls reversed

Next, reverse the order of throws so that the sequence is first right, then left. Throw the second ball as high as the first. Do not pass it directly across to the right hand. Do not throw the second, or left-hand, ball too soon.

Step 4: Three balls

Now put two balls in your right hand and one in your left. Try to complete Step 2 while simply holding the extra ball in your hand. Pause, and then go on to Step 3. The sequence should flow left, right, pause, right, left.

The third ball can make it difficult to catch the second throw. To solve this problem, throw the third ball before the second throw arrives (in fact, just after the second throw reaches its high point). The sequence is thus right, left, right. At first it may be difficult to persuade your right hand to make the second throw. Just concentrate on making the three throws; the catches are irrelevant at this point. Throw high, accurately, and *slowly*. It is important to make sure that your left-hand throws rise to the same height as your right-hand throws. Don't rush the tempo and don't forget the figure-eight pattern.

Step 5: Three balls reversed

Put two balls in your left hand, one in your right, and throw left, right, left.

Step 6: Four throws

Starting with two balls in the right hand, throw right, left, right, left.

Step 7: More and more!

Continue in this way slowly to increase the number of throws you make. Concentrate on height and accuracy. If you find yourself moving forward to make the catches (and almost everyone does at the beginning), try harder not to throw the balls outward. Don't let your hands rise much above the level of your navel. Persist over a period of days and become a teacher yourself! —*J.B. & R.G.*

A Greek terra-cotta figure of a juggler, ***third century B.C.***

tional patterns (underwater waterfalls?). One would think that juggling even ten lacrosse balls would be an easy feat at the bottom of a pool, because they are only slightly denser than water and sink very slowly. But, unfortunately, the juggler's hand motions churn the water so much that the balls' descents are erratic and unpredictable. Juggling even three balls underwater is a challenge. The same, alas, is true of juggling while in free fall, as skydiving jugglers have discovered (sun-showers?). Although a ball falls only very slowly with respect to the juggler (since the juggler is falling, too), fierce air currents, not to mention the difficulty of recovering a "dropped" throw, make the game nearly impossible.

Faced with these setbacks, jugglers have settled for less extravagant ways of defying gravity. Some juggle balloons or silk handkerchiefs. Others practice the "Galilean technique," named for Galileo's experiments, in Pisa, with inclined planes and rolling balls. Rather than juggling balls in the air, Galilean jugglers roll balls up an inclined, flat surface like a tabletop and catch them as they roll back down. If the tilt of the table is slight, the pattern is traced out by the balls very slowly, and the juggler has plenty of time to refine his technique. As he gradually steepens the tilt, he begins to approximate "real time" juggling. At least one juggler, by gradually increasing the tilt of a table, and thus increasing the speed of the balls as they rolled back, ultimately mastered the five-ball cascade.

A JUGGLER, like a mathematician, is never finished: there is always another great unsolved problem. There is always one more ball. Shannon's Theorem makes it clear how challenging it is to add objects to a routine, how painfully limited we are by the underlying physics of juggling and by the fact that we have only two hands. Jugglers are always attracted by the excitement of what is called numbers juggling, but each additional object requires much higher, faster, *and* more accurate throws for the whole pattern.

Most people assume that a skilled juggler can manage perhaps ten or twenty objects. In fact, a five-ball cascade is very difficult, and the average juggler requires about a year of persistent practice to achieve any proficiency. Only a very few have perfected a seven-ball cascade; at the 1983 International Jugglers Association competition in seven-object juggling, the winning time (that is, the longest) was under twenty seconds.

Fortunately for the performer, the uninitiated audience can rarely count the number of objects being juggled; sometimes seven balls or rings look like a dozen or so. In an amusing passage in Xenophon's *The Banquet*, Socrates praises the dexterity of a girl who, to entertain the dinner guests, juggled twelve rings. If the story is true, that girl would hold the record for the largest number of objects juggled stably. But professionals are understandably skeptical of such reports—even from observers as reputable as Xenophon and Socrates.

Even with just three balls, the number of possible variations on the cascade, shower, and fountain appears to be nearly limitless. If rhythms and patterns are varied, the effects can be stunning. Indeed, audiences can usually appreciate an especially skilled three-ball routine much more easily than a routine involving seven. The most successful street performers are often those who can do the most eccentric, novel, humorous, or visually pleasing things with a mere three objects.

Just for the fun of it, one can always add constraints instead of objects. Consider the problem of blindfolded juggling. At first the stunt seems impossible, because one cannot see how far off course a ball is and correct for it—since one gets no feedback until the balls strike, or miss, the palms. However, by making short, accurate throws, an accomplished juggler can (often to his own great surprise) learn to juggle quite stably while blindfolded. The feedback comes from the "feel" of an errant throw, and of catching a slightly off-center ball.

Can one teach a machine to juggle? That depends on one's standards. With enough care, it might be possible to build a machine whose throws were so accurate that it would have no need for feedback at all, but this would not really be juggling. A true juggling machine would use a feedback mechanism, visual or tactile, to correct for minor perturbations. With the microprocessors now available, it should already be possible to design such a machine, though building the throwing, catching, and perceiving implements would be prohibitively costly. In his retirement, Claude Shannon, the distinguished mathematician and connoisseur of juggling, is working toward such a machine. He has built an intricate contraption that should someday be able to bounce steel balls on a steel drum and catch them. So far, the machine has yet to get the hang of it. ■

Indian jugglers, ***nineteenth century***