On Isometric Embeddings of Graphs

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ABSTRACT

For a finite connected undirected graph $G=(V,E)$ one can associate a metric d_G by defining $d_G(x,y)$ to be the number of edges in a shortest path between x and y, where x and y are any two vertices in V, the vertex set of G. If (M,d_M) is an arbitrary metric space we say that an embedding $\lambda: V \to M$ is *isometric* if for all $x, y \in V$, $d_M(\lambda(x),\lambda(y))=d_G(x,y)$.

In this note we survey recent results on this topic, especially in the case in which M is formed from the cartesian product of graphs.

1. Introduction

With a finite connected undirected graph* $G = (V,E)$ one can associate a metric $d_G: V \times V \to \mathbb{N}$ (the set of nonnegative integers) by defining $d_G(x,y)$ to be the number of edges in the shortest path between x and y for all $x,y \in V$, the vertex set of G. If (M,d_M) is an arbitrary metric space we say that an embedding $\lambda: V \to M$ is *isometric* † if for all $x,y \in V$,

$$(1) \qquad d_M(\lambda(x),\lambda(y)) = d_G(x,y).$$

We denote this by $G \overset{d}{\to} M$.

A fair number of papers have appeared in the past few years which deal with various properties of graphs which have isometric embeddings in certain metric and semi-metric§ spaces (e.g., see [As1],

*In general, we follow the terminology of [BM].
†In the literature this is also sometimes said to be "distance preserving."
§i.e., the triangle inequality may fail (in French, écart).

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[As2], [As3], [AD 1], [ADl2], [ADzl], [ADz2], [Av1], [Av2], [Dew], [Dez1], [Dez2], [DR], [Dj], [F], [GP1], [GP2], [HL], [GP2], [HL], [K1], [K2], [K3]).

In this note we will describe some very recent work in this subject, and in particular, when the "host" metric space is itself derived from a graph.

We should remark here that many of the results we discuss actually apply to general metric and semi-metric spaces. However, we will usually restrict ourselves to metrics induced by graphs.

2. Some Background

Some of the early interest in questions of this type were motivated by the investigation of routing algorithms in data networks ([P], [GP1], [BGK]). Several of the first results involved the (semi-metric) space S consisting of the three symbols 0, 1 and *, with the distance d_S given by:

$$d_S(x,y) = \begin{cases} 1 & \text{if } x = 0,\ y = 1 \textit{ or } x = 1,\ y = 1, \\ 0 & \textit{otherwise} \end{cases}$$

In general, if (M, d_M) is a (semi)-metric space there is a natural metric d_M on the cartesian product M^n given by:

$$\textit{For } \bar{x} = (x_1, \ldots, x_n),\ \bar{y} = (y_1, \ldots, y_n) \in M^n,$$

$$(2) \qquad d_{M^n}(x,y) := \sum_{k=1}^{n} d_M(x_k, y_k).$$

FACT ([GPI]). Every finite metric space M can be embedded isometrically into S^N for a suitable integer $N = N(M)$.

DEFINITION. For a graph G, denote by $N(G)$ the *least* integer N so that $G \overset{d}{\to} S^N$.

Let $D(G)$ denote the distance matrix of G. That is, if $V = V(G) = \{v_1, \ldots, v_n\}$ then $D(G) = (d_{ij})$ is the n by n matrix defined by setting $d_{ij} = d_G(v_i, v_j)$. Observe that since $D(G)$ is real and symmetric then the eigenvalues of $D(G)$ are real. Denote the number of positive, negative and zero eigenvalues of $D(G)$ by $n_+(G), n_-(G)$ and $n_0(G)$, respectively.

A basic result concerning $N(G)$ is the following.

THEOREM ([GPI]). For every (connected) graph G,

(3) $$N(G) \geq \max\{n_+(G), n_-(G)\}.$$

PROOF: Suppose $\lambda : G \to S^N$ is an isometry (= isometric embedding). Thus, since

$$d_{ij} = \sum_{k=1}^{N} d_s(\lambda(v_i)_k, \lambda(v_j)_k)$$

where

$$(x) = (\lambda(x)_1, \ldots, \lambda(x)_N)$$

then by the definition of d_S, we have the basic decomposition

(4) $$\sum_{i,j} d_{ij} x_i x_j = \sum_{k=1}^{N} \left(\sum_{a \in A_k} x_a \right) \left(\sum_{b \in B_k} x_b \right)$$

where

$$A_k = \{a : \lambda(v_a)_k = 0\},$$
$$B_k = \{b : \lambda(v_b)_k = 1\}$$

and the x_k are indeterminates.

We can rewrite (4) as

(4′) $$\sum_{i,j} d_{ij} x_i x_j = \frac{1}{4} \sum_{k=1}^{N} \left\{ \left(\sum_{a \in A_k} x_a + \sum_{b \in B_k} x_b \right)^2 - \left(\sum_{a \in A_k} x_a - \sum_{b \in B_k} x_b \right)^2 \right\}$$

The significance of (4′) is that it represents a decomposition of the quadratic form $\sum_{i,j} d_{ij} x_i x_j$ into a sum and difference of squares. Thus, by Sylvester's "law of inertia", $N = \geq n_+(G)$ and $N \geq n_-(G)$, i.e.,

$$N \geq \max\{n_+(G), n_-(G)\}$$

and the theorem is proved.□

In [GP1], [GP2], N(G) was determined for a number of classes of graphs. In particular, it was shown that for complete graphs K_n, trees T_n and cycles C_n, with n vertices:

$$N(K_n) = n-1,$$
$$N(T_n) = n-1,$$

$$N(C_n) = \begin{cases} n-1 \text{ for } n \text{ odd}, \\ \frac{n}{2} \text{ for } n \text{ even}. \end{cases}$$

On the basis of evidence of this type, it was conjectured in [GP1] that for every graph G with n vertices,

$$(5) \qquad N(G) \leq n-1$$

This was finally very recently established by P. Winkler [Win1] (improving earlier estimates of Yao [Y]).

We point out that the assertion $N(K_n) = n-1$ has the following equivalent combinatorial interpretation.

FACT ([GP1]). Suppose the edge set of K_n is decomposed into t disjoint complete bipartite subgraphs. Then $t \geq n-1$.

At present no purely combinatorial proof of this is known. However, Tverberg [Tv] has recently given the following nice algebraic argument.

Denote the hypothesized bipartite graphs' vertex sets by A_k and B_k, $1 \leq k \leq t$. Thus,

$$(6) \qquad \sum_{1 \leq i \leq j \leq n} x_i x_j = \sum_{k=1}^{t} \left(\sum_{a \in A_k} x_a \right) \left(\sum_{b \in B_k} x_b \right)$$

Consider the following system of $t+1$ homogeneous linear equations in the n variables x_i:

$$\sum_{a \in A_k} x_a = 0, \quad 1 \leq k \leq t,$$

and

$$\sum_{i=1}^{n} x_i = 0.$$

If $(y_1, \ldots, y_n)$ is any solution to this system then we must have

$$0 = \left\{ \sum_{i=1}^{n} y_i \right\}^2 = \sum_{i=1}^{n} y_i^2 + 2 \sum_{i<j} y_i y_j$$

$$= \sum_{i=1}^{n} y_i^2 + 2 \sum_{k=1}^{t} \left(\sum_{a \in A_k} x_a \right) \left(\sum_{b \in B_k} x_b \right)$$

$$= \sum_{i=1}^{n} y_i^2$$

i.e., $y_i = 0$ for all i. Hence, the number of equations in the system must be as large as the number of variables, i.e., $t+1 \leq n$. □

3. A Geometric Interpretation of the Tree Theorem

For a tree T_n with n vertices, the equality $N(T_n) = n-1$ relied on the fact that any tree T has $n_+(T) = 1$ (and consequently, $n_-(T) = n-1$ since $D(T_n)$ has trace 0). This follows from the surprising fact that det $D(T_n)$ depends only on n and is otherwise independent of the structure of T_n. Specifically, we have:

THEOREM ([GP1]). For a tree T_n with n vertices

$$\det D(T_n) = (-1)^{n-1}(n-1)2^{n-2}. \tag{7}$$

A straightforward proof is not difficult. One can simply (arbitrarily) choose a root in T_n and successively subtract the row and column of each "son" from those of his "father," always selecting the unprocessed vertices furthest from the root. At the termination of this process the resulting matrix $D' = (d'_{ij})$ satisfies

$$d'_{ij} = \begin{cases} 1 & \text{if } i = 1,\ j \neq 1 \text{ or } j = 1,\ i \neq 1, \\ -2 & \text{if } i = 1,\ j \neq 1, \\ 0 & \text{otherwise} \end{cases}$$

and (7) follows easily.

However, one strongly suspects from the form of (7) that something deeper must be involved. The fact that the factor $n-1$ represents the number of edges of T_n was observed in [GHH] where the following result was proved.

Let $cof(G)$ denote the sum of the cofactors of the matrix $D(G)$ for a graph of G.

THEOREM ([GHH]). If G has blocks* $G_1, \ldots, G_r$ then

$$cof(G) = \prod_{k=1}^{r} cof(G_k) \tag{i}$$

*i.e., maximal 2-connected subgraphs

$$\det D(G) = \sum_{k=1}^{r} \det D(G_k) \prod_{i \neq k} cof(G_i) \tag{ii}$$

Note that when all $cof(G_k) \neq 0$ then (ii) can be rewritten in the attractive form

$$\frac{\det D(G)}{cof(G)} = \sum_{k=1}^{r} \frac{\det D(G_k)}{cof(G_k)} \qquad \text{(ii)'}$$

Thus, for T_n, all blocks consist of a single edge K_2 which has $\det D(K_2) = -1$, $cof(K_2) = -2$ and consequently (7) follows at once.

The next result we give generalizes (7) and gives a geometrical explanation for the factor 2^{n-2}. Consider the set Q^n of vertices of the (usual) unit n-cube in $\mathbb{R}^n$, i.e.,

$$Q^n = \{0,1\}^n = \{\bar{a} = (a_1, \ldots, a_n) : a_k = 0 \text{ or } 1,\ 1 \leq k \leq n\}.$$

There is a natural metric d_H on Q^n, called the Hamming metric, given by

$$d_H((a_1, \ldots, a_n),(b_1, \ldots, b_n)) = \sum_{k=1}^{n} |a_k - b_k|.$$

i.e., the distance between $\bar{a}$ and $\bar{b}$ in Q^n is just equal to the number of coordinate positions in which they differ. Let us call a set $S \subseteq Q^n$ *full-dimensional* if the convex hull of S has positive n-dimensional volume.

THEOREM 1. If $\{\bar{a}_0, \ldots, \bar{a}_n\}$ is any full-dimensional set of $n+1$ points of Q^n then

$$\det(d_H(\bar{a}_i, \bar{a}_j)) = (-1)^n n \cdot 2^{n-1} \tag{8}$$

PROOF: For $\bar{a}_i = (a_{i1}, \ldots, a_{in})$ write

$$a_{ik} = \frac{1}{2} + \frac{1}{2}\alpha_{ik}$$

where $\alpha_{ik} = \pm 1$. Thus

$$\begin{aligned} d_H(\bar{a}_i, \bar{a}_j) &= \sum_{k=1}^{n} |a_{ik} - a_{jk}| \\ &= \frac{1}{2}\sum_{k=1}^{n} |\alpha_{ik} - \alpha_{jk}| \\ &= \frac{1}{2}\sum_{k=1}^{n} (1 - \alpha_{ik}\alpha_{jk}) \end{aligned} \tag{9}$$

$$= \frac{1}{2}(n - \overline{\alpha}_i \cdot \overline{\alpha}_j)$$

where $\overline{\alpha}_i \cdot \overline{\alpha}_j$ denotes the inner product of the vectors $\overline{\alpha}_i = (\alpha_{i1}, \ldots, \alpha_{in})$ and $\overline{\alpha}_j = (\alpha_{j1}, \ldots, \alpha_{jn})$. It follows from elementary linear algebra that for any square matrix $M = (m_{ij})$, if J is the matrix with all entries equal to 1 then

$$\text{(10)} \qquad \det(M + xJ) = \det M + x \det(m_{ij} - m_{1j} - m_{i1} + m_{11}).$$

Thus, from (9) we have

$$\text{(11)} \quad \det(d_H(a_i, a_j)) = \det\left(\frac{1}{2}(n - \overline{\alpha}_i \cdot \overline{\alpha}_j)\right)$$

$$= \frac{1}{(-2)^{n+1}} \det(\overline{\alpha}_i \cdot \overline{\alpha}_j - n)$$

$$= \frac{1}{(-2)^{n+1}} \{\det(\overline{\alpha}_i \cdot \overline{\alpha}_j) - n \det(\overline{\alpha}_i \cdot \overline{\alpha}_j - \overline{\alpha}_0 \cdot \overline{\alpha}_j - \overline{\alpha}_i \cdot \overline{\alpha}_0 + \overline{\alpha}_0 \cdot \overline{\alpha}_0)\}$$

$$= \frac{1}{(-2)^{n+1}} \{\det(\overline{\alpha}_i \cdot \overline{\alpha}_j) - n \det((\overline{\alpha}_i - \overline{\alpha}_0) \cdot (\overline{\alpha}_j - \overline{\alpha}_0))\}$$

The determinants which appear in (11) of the form $\det(\overline{x}_i \cdot \overline{x}_j)$, called *Gramians*, occur frequently in linear algebra. One of their particularly useful properties is the following.

FACT (see [Ga]). For a set of vectors $\overline{x}_1, \ldots, \overline{x}_n$ in $\mathbb{R}^n$, the Gramian $\det(\overline{x}_i \cdot \overline{x}_j)$ is just the square of the (m-dimensional) volume of the parallelepiped spanned by the $\overline{x}_k$.

In particular, in (11) since the $n+1$ $\overline{\alpha}_k$'s all lie in $\mathbb{R}^n$ then $\det(\overline{\alpha}_i \cdot \overline{\alpha}_j) = 0$. On the other hand, the vectors $\overline{\alpha}_k - \overline{\alpha}_0$ by hypothesis span an n-dimensional space. It is clear that the parallelepiped they span has width 2 in each dimension (since all $\alpha_{ik} = \pm 1$), and consequently, has volume 2^n.

Thus, continuing (11), we obtain

$$\det d_H(\overline{a}_i, \overline{a}_j) = \frac{1}{(-2)^{n+1}}(0 - n \cdot (2^n)^2)$$

$$= (-1)^n n \cdot 2^{n-1}$$

and the theorem is proved. □

Symmetric differences. It is often natural to interpret binary n-tuples as characteristic functions on the set $[n] = \{1, \ldots, n\}$ so that each

$\bar{a} = (a_1, \ldots, a_n)$ corresponds to a subset $S(\bar{a}) \subseteq [n]$ by

$$k \in S(\bar{a}) <=> a_k = 1.$$

With this association it is easy to see that

$$d_H(\bar{a},\bar{b}) = |S(\bar{a}) \,\Delta\, S(\bar{b})|$$

where $X\Delta Y$ denotes the symmetric difference $X/Y \cup Y/X$ of X and Y.

Let us say that a family of subsets of $[n]$ is full-dimensional if the corresponding n-tuples are. We can restate Theorem 1 in these terms.

THEOREM 1′. Suppose $\{S_o, \ldots, S_n\}$ is a full-dimensional family of subsets of [n]. Then

$$\det(|S_i \Delta S_j|) = (-1)^n n \cdot 2^{n-1}.$$

The advantage of this formulation is that it can be readily extended to the following more general situation. Suppose μ is a discrete measure on $2^{[n]}$, i.e.,

$$\mu(k) \geq 0, \; k \in [n]$$

$$\mu(X) = \sum_{x \in X} \mu(x), \; X \subseteq [n].$$

THEOREM 2. Suppose $\{S_0, \ldots, S_n\}$ is a full-dimensional family of subsets of [n]. Then

$$\text{(12)} \qquad \det(\mu(S_i \,\Delta S_j)) = (-1)^{n} 2^{n-1} \sum_k \mu(k) \prod_k \mu(k).$$

The proof of (12), which we will not give here, depends on an extension of $(ii)'$ appearing in [GHH]. Of course, when μ is just the counting measure, i.e., $\mu(k) = 1$ for all $k \in [n]$, then (12) reduces to the previous result.

4. Embedding in the m-Cube.

The cartesian product K_2^n of K_2 (the complete graph on two vertices) is usually called the n-cube in the graph theory literature. The induced metric $d_{K_2^n}$ on K_2^n is just the Hamming metric d_H. While we have seen earlier that every graph G embeds isometrically in $\{0,1,*\}^N = S^N$ for some N this is certainly not true for K_2^N. The problem of characterizing those G for which $G^d K_2^m$ for some m was settled by the following result of Djoković. First we need a definition.

For two vertices x and y on a graph G define $N(x,y) := \{z \in V(G): d_G(x,z) < d_G(y,z)\}$ (i.e., $N(x,y)$ is the set of points nearer to x than to y).

THEOREM (Djoković [Dj]). $G \overset{d}{\to} K_2^m$ for some m if and only if

(i) G is bipartite,

(ii) For each edge $\{x,y\}$ of G, if $a,b \in N(x,y)$ and $d_G(a,c) + d_G(c,b) = d_G(a,b)$ then $c \in N(x,y)$. (Of course, this also applies to $N(y,x)$).

What (ii) says is that $N(x,y)$ is closed under taking shortest paths. Note that because of (i), $N(x,y) \cup N(y,x)$ is actually a partition of the vertices of G whenever $\{x,y\}$ is an edge.

In [Dj], Djoković introduces the following equivalence relation θ on the edge set $E(G)$ of G by defining:

$$\{x,y\} = e \overset{\theta}{\sim} e' <=> e' \textit{ intersects both } N(x,y) \text{ and } N(y,x).$$

The corresponding set $E(G)/\theta$ of θ-equivalence classes has the following property. Denote by $dim(G)$ the least m such that $G \overset{d}{\to} K_2^m$ (when G satisfies (i) and (ii)).

THEOREM (Djoković [Dj])

$$dim(G) = card(E(G)/\theta).$$

In fact, as pointed out by P. Winkler [Win2], this approach actually shows that any isometry of G into K_2^m can only have $m = dim(G)$ (for a larger m none of the additional coordinates are used) and furthermore, the embedding is unique up to symmetries of the m-cube.

The next result ties dim(G) directly to the distance matrix $D(G)$.

THEOREM 3.

$$(13) \qquad dim(G) = n_-(G)$$

PROOF: First, recall from (3) that

$$N(G) \ge n_-(G).$$

Next, we claim

$$G \overset{d}{\to} K_2^m => n_+(G) = 1.$$

This can be seen by observing (as was done in [BG]) in (4′) that when no *'s are used, $A_k \cup B_k$ is then a partition of $V(G)$ and consequently, the quadratic form $\sum_{i,j} d_{ij} x_i x_j$ is expressible as a sum of *one* positive square and some negative squares. This implies $n_+(G) = 1$.

Suppose $\lambda: G \to K_2^m$ is isometry.

CLAIM: rank $(D(G)) = m+1$.

PROOF OF CLAIM: On one hand

$$\begin{aligned} rank(D(G) &= n_-(G) + n_+(G) \\ &= n_-(G) + 1 \\ &\le N(G) + 1 \\ &\le m + 1 \end{aligned}$$

On the other hand, since G is connected there must exist $v_0, v_1, \ldots, v_m \in V(G)$ such that the set $\{\lambda(v_0), \lambda(v_1), \ldots, \lambda(v_m)\}$ is full-dimensional in K_2^m. Thus the submatrix

$$(d_G(v_i, v_j)) = (d_H(\lambda(v_i), \lambda(v_j))$$

is nonsingular (by Theorem 1) and so,

$$rank(D(G)) \ge m+1$$

Consequently,

$$rank(D(G)) = m+1.$$

which proves the claim, and

$$n_-(G) = m = N(G) = dim(G)$$

which proves the theorem.□

We note that it follows from these considerations, for example, that $G \to K_2^m$ then $\det(D(G)) \neq 0$ iff G is a tree.

5. Embedding in Products of Graphs.

A natural extension of the questions raised in the preceding section is to attempt to characterize those G for which $G \overset{d}{\to} H^m$ or $G \overset{d}{\to} H_1 \times \cdots \times H_m$ hold, for various choices of H and $H_1, \ldots, H_m$. Unfortunately, our knowledge for these more general questions is rather incomplete at present. In this section we will mention several results (without proof) which are known.

To begin with, we should observe that no graph H can be a "universal host" (i.e., such that every graph G embeds isometrically into some power of H) since, for example, it is not hard to show that

$$G \overset{d}{\to} H^m \Rightarrow \chi(G) \le \chi(H)$$

where χ denotes chromatic number. (Compare with condition (i) of

Djoković's first theorem.) One reason that it was possible for $G \overset{d}{\to} S^N$ to hold for any G is that $S = \{0,1,*\}$ is not a metric space, since

$$d_S(0,1) = 1 > 0 = d_S(0,*) + d_S(*,1)$$

A more substantial restriction on G is given by the following considerations. For $x,y \in V(G)$ define the set

$$S(x,y) = \{z \in V(G): d_G(x,z) = d_G(y,z)\}$$

(the set of points of G at the *same* distance from x and y). With $N(x,y)$ and $N(y,x)$ defined as before, define the relation $\hat{\theta}$ on $E(G)$ as follows: If $e = \{x,y\}$ and e' are edges of G then

$$e \overset{\hat{\theta}}{\sim} e' \iff e' \textit{ intersects at least two of the sets } N(x,y), N(y,x), S(x,y)$$

It can be checked that $\hat{\theta}$ is symmetric and reflexive but not transitive (consider a 5-cycle).

Define θ to be the *transitive closure* of $\hat{\theta}$. Note that θ is actually the same equivalence relation given by Djoković in the case that G is bipartite.

A basic result concerning θ is the following.

FACT. Suppose $\lambda: G \overset{d}{\to} H_1 \times \cdots \times H_m$. If $\{x,y\} \overset{\theta}{\sim} \{x',y'\}$ and $\lambda(x)$ and $\lambda(y)$ differ only in their k^{th} components then $\lambda(x')$ and $\lambda(y')$ differ only in their k^{th} components.

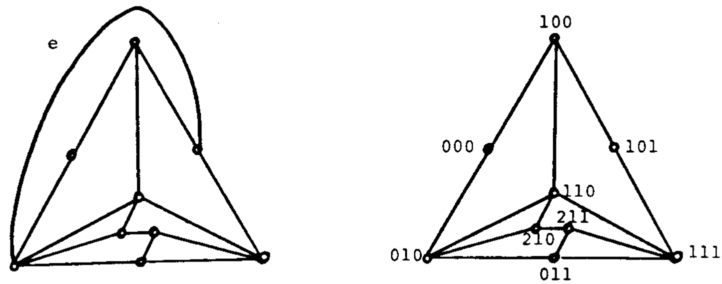
Of course, if $\{x,y\}$ is an edge then $d_G(x,y) = 1$ and so $\lambda(x)$ and $\lambda(y)$ can only differ in one component. This fact can be used to prove the following result.

THEOREM 4. If $G \overset{d}{\to} H^m$ then each connected component C of G induced by the θ-equivalence classes must satisfy $C \overset{d}{\to} H$.

This is a rather strong restriction. It implies, for example, that if $G \overset{d}{\to} K_3^m$ then all such C are either K_2's or K_3's. Let us call G *prime* if $G \overset{d}{\to} H_1 \times \cdots \times H_m \implies G \overset{d}{\to} H_k$ for some k. The preceding results can be used to prove the following.

THEOREM 5. If G has a single θ-equivalence class then G is prime.

For example, it is easy to check that this is the case for the 5-cycle C_5 although it seems to be a rather delicate condition in general. For example, for the graphs G and $G' = G-\{e\}$ shown in Figure 1, G has a single θ-equivalence and so, is prime while G' has *three* θ-equivalence classes and in fact, embeds isometrically in K_3^3. (We have shown the appropriate images next to each vertex, where we take $V(K_3) = \{0,1,2\}$).



(*a*) *G is prime* (*b*) $G' = G - \{e\} \overset{d}{\to} K_3^3$

Figure 1

6. Concluding Remarks

The problem of embedding graphs isometrically into other graphs is a special case the more general topic of embedding (finite) metric spaces isometrically into other metric (or semi-metric) spaces. This topic has an extensive literature, some of which can be found in [ADz1], [ADz2], [Av3], [K2], [K3]. Of course, many of these more general results impinge on our studies. For example, it follows from these considerations that if $G \overset{d}{\to} K_3^m$ (or indeed, if $G \overset{d}{\to} H^m$ for any graph H with at most four vertices) then $n_+(G) = 1$. The reason for this is as follows.

Lets say that an n by n distance matrix $D = (d_{ij})$ is of *negative type* if

$$(14) \qquad x_1 + \cdots + x_n = 0,\ x_k \in \mathbb{R} \Rightarrow \sum_{i,j} d_{ij} x_i x_j \leq 0.$$

Similarly, call D *hypermetric* if

$$(15) \qquad x_1 + \cdots + x_n = 1,\ x_k \in \mathbb{Z} \Rightarrow \sum_{i,j} d_{ij} x_i x_j \leq 0.$$

Although (14) and (15) are similar, (15) is actually much stronger. Not only does it imply (14) but also that the space actually satisfies the triangle inequality (and many stronger ones), something that (14) does not do. It is not hard to show that

$$D \text{ is of negative type} \Rightarrow n_+(D) = 1.$$

No example is currently known of a graph G for which $n_+(G) = 1$ and $D(G)$ is *not* of negative type although such graphs undoubtedly exist.

It turns out that the properties of hypermetricity and negative type are preserved under taking products and isometric subsets. Thus,

$$K_3 \text{ is of negative type (easy to check)}$$

$$\Rightarrow K_3^m \text{ is of negative type}$$

$$\Rightarrow G \xrightarrow{d} K_3^m \text{ is of negative type}$$

$$\Rightarrow n_+(G) = 1$$

An interesting related question is the following. Suppose X is a semi-metric space with distance matrix C. Let $D^{(k)}$ denote the distance matrix corresponding to the product space X^k. As just remarked, if X is of negative type then so is X^k and consequently $n_+(D^{(k)}) = 1$ for any k. Does the converse hold? In other words, does $n_+(D^{(k)}) = 1$ for all k imply that X is of negative type?

It was conjectured at one time by Deza [Dez2] that hypermetricity was a sufficient condition for isometric embeddability into l_1(*i.e.*, $\mathbb{R}^n$ *with* $d(\bar{x},\bar{y}) = \sum_i |x_i - y_i|$). This was shown *not* to be the case by Avis [Av3] (see also Assouad [As1]), who proved that the graph $K_7 - P_3$ is hypermetric but not isometrically embeddable into l_1. However, it is true [Dez1], [K1] that hypermetricity is a necessary condition for l_1-embeddability. In the same spirit it is easy to show that the graph $K_3 + \{e\}$ (an edge) is of negative type but not hypermetric (see [AsD2]).

Finally, we arrive at the graph $K_{3,2}$, which is exceptional in several respects. Since $n_+(K_{3,2}) = 2$, $K_{3,2}$ is not of negative type and therefore not isometrically embeddable into any K_2^m or even $\mathbb{R}^n$. In fact, it is not even a *subgraph* of K_2^m. It also turns out that

$$N(K_{3,2}) = 4 > max(n_+(K_{3,2}), n_-(K_{3,2})) = 3.$$

showing that equality does not have to hold in (3).

At present no necessary condition is known for a graph to be l_1-embeddable, hypermetric or of negative type. It would seem fruitful to study the characteristic polynomials of the associated distance matrices of various spaces rather than just the signs of the eigenvalues. This has been initiated for trees in [EGG] and [GL]. It seems quite likely that our understanding of this whole general area would increase substantially if the corresponding results were known for more general graphs, e.g., those $G \overset{d}{\to} K_2^m$.

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Added in proof: The converse of Theorem 5 has now been proved by Peter Winkler. Also, H.J. Landau has shown that $n_+(D^{(2)})=1$ already complies that the underlying space X is of negative type.

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