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Recent Developments in Ramsey Theory

Introduction

Mathematics has often been called the science of order. From this viewpoint the guiding principle of Ramsey theory is perhaps best summed up by the statement of T. S. Motzkin: "Complete disorder is impossible". Ramsey theory is basically the study of structure preserved under partitions. Before stating some background material, we first introduce the following notation. We will adopt the usual convention of identifying the positive integer n with the set of its predecessors $\{0, 1, \ldots, n-1\}$, where 0 corresponds to $\emptyset$. The symbol ω denotes $\{0, 1, 2, \ldots\}$, the set of natural numbers. For $X \subseteq \omega$, $k \in \omega$, $[X]^k$ denotes the set of k-element subsets of X, and $[X]^\omega$ denotes the set of infinite subsets of X (if there are any). The generic result in Ramsey theory is due (not surprisingly) to F. P. Ramsey [49]:

Ramsey's Theorem (1930)

For any $k, r \in \omega$, if $[\omega]^k = C_1 \cup \ldots \cup C_r$ then there exists $X \in [\omega]^\omega$ such that $[X]^k \subseteq C_i$ for some i.

An earlier result of Ramsey type was given by I. Schur [52] in 1916:

If $\omega = C_1 \cup \ldots \cup C_r$ then there exist $x, y, z \in C_i$ for some i such that $x + y = z$.

The result of Schur was generalized successively as follows.

THEOREM (Rado [47], Folkman [17], Sanders [51]). *For all $m \in \omega$, if $\omega = C_1 \cup \ldots \cup C_r$ then there exists $X \in [\omega]^m$ such that for some i and all nonempty $F \subseteq X$, $\sum_{a \in F} a \in C_i$.*

THEOREM (Hindman [34]). *If* $\omega = C_1 \cup \ldots \cup C_r$ *then there exists* $X \in [\omega]^\omega$ *such that for some* i *and all finite nonempty* $F \subseteq X$, $\sum_{a \in F} a \in C_i$.

A much weaker form of the Rado–Folkman–Sanders theorem was actually given by Hilbert in 1892:

THEOREM [33]. *For all* $m \in \omega$, *if* $\omega = C_1 \cup \ldots \cup C_r$ *then there exists* $X \in [\omega]^\omega$ *and* $t \in \omega$ *such that for some* i *and all nonempty* $F \subseteq X$, $t + \sum_{a \in F} a \in C_i$.

Finally, we mention the result which will motivate much of what we discuss in this paper. This is:

VAN DER WAERDEN'S THEOREM (1927) [63]. *If* $\omega = C_1 \cup \ldots \cup C_r$ *then for some* i, C_i *contains arbitrarily long arithmetic progressions.*

The theorem of van der Waerden has proved to be an extremely fertile seed from which a major part of modern combinatorics has developed, especially through the work of Rado [47], [48], Erdös [16], [15], Roth [50], Szemerédi [60], [61], Deuber [9] and many others (see [14], [31], [10]). A particularly important generalization was given in 1963 by Hales and Jewett. For a fixed finite set A, call a subset $L \subseteq A^N$ a *combinatorial line* if for some nonempty $I \subseteq N$, L can be written as

$$L = L_I = \bigcup_{a \in A} \{(x_0, x_1, \ldots, x_{N-1})\colon x_i = a \text{ if } i \in I \text{ and } x_i = b_i \in A \text{ if } i \notin I\}.$$

Thus, $|L| = |A|$.

HALES–JEWETT THEOREM [32]. *For all finite* A *and* r, *there exists* $N(A, r)$ *such that if* $N \geqslant N(A, r)$ *and* $A^N = C_1 \cup \ldots \cup C_r$ *then some* C_i *must contain a combinatorial line.*

To see that this implies van der Waerden's Theorem, simply take $A = t = \{0, 1, \ldots, t-1\}$ and identify the point $\bar{x} = (x_0, \ldots, x_{N-1}) \in A^N$ with the integer $|\bar{x}| = \sum_{i \in N} x_i t$. The t points in any combinatorial line clearly correspond to t integers in an arithmetic progression. Since t was arbitrary, a standard compactness argument yields van der Waerden's Theorem.

The Hales–Jewett Theorem also implies the higher-dimensional analogues of van der Waerden's Theorem, first proved by Gallai (see [47]) and Witt [66].

THEOREM. *If* $\omega^n = C_1 \cup \ldots \cup C_r$ *then some* C_i *must contain for all* $k \in \omega$ *a homothetic copy of* $\{0, 1, \ldots, k-1\}^n$, *i.e., all* k^n *points*

$$\{(ai_1 + b_1, ai_2 + b_2, \ldots, ai_n + b_n)\colon 0 \leqslant i_1, \ldots, i_n < k\}$$

for suitable a, $b_i \in \omega$.

A much stronger "density" form of van der Waerden's theorem was conjectured by Erdös and Turán [16] nearly 50 years ago: If $A \subseteq \omega$ satisfies

$$\limsup_{n\to\infty} \frac{|A \cap n|}{n} > 0 \tag{$*$}$$

then A contains arbitrarily long arithmetic progressions.

It was shown by Roth [50] in 1953 that $(*)$ implies A has a 3-term arithmetic progression and by Szemerédi [60] in 1969 that $(*)$ implies A has a 4-term arithmetic progression. Finally, Szemerédi [61] in 1974 in a brilliant combinatorial tour de force established the full conjecture. Szemerédi's Theorem and the higher-dimensional density analogues of van der Waerden's Theorem have fairly recently been proved by quite different techniques from ergodic theory and topological dynamics. This exciting work of Furstenberg, Katznelson, Weiss and others (see [22], [24], [20], [21]) has furnished a very stimulating link between these two branches of mathematics which is just beginning to reveal its full potential.

It is very natural to ask whether there is a corresponding *density* version for the Hales–Jewett Theorem. We can phrase this as follows:

CONJECTURE.[1] *For all finite* A *and* $\varepsilon > 0$ *there exists* $N(A, \varepsilon)$ *such that if* $N \geqslant N(A, \varepsilon)$ *and* $R \subseteq A^N$ *satisfies* $|R| \geqslant \varepsilon |A^N|$ *then* R *must contain a combinatorial line.*

The conjecture, if true, clearly implies Szemerédi's Theorem. It is known to be true if $|A| = 2$ by the following argument. Assume without loss of generality that $A = \{0, 1\}$. Identify with each point $\bar{x} = (x_0, x_1, \ldots, x_{N-1}) \in A^N$ the subset $S(\bar{x}) \subseteq N$ by $i \in S(\bar{x})$ iff $x_i = 1$ (i.e., $\bar{x}$ is the characterstic function for $S(\bar{x})$). Thus, a combinatorial line in A^N corresponds to a pair of distinct subset $X, Y \subseteq N$ with $X \subset Y$. However, a well-known result of Sperner [59] asserts that any family $\boldsymbol{F}$ of subsets of N in which $X, Y \in \boldsymbol{F}$, $X \neq Y$ implies $X \not\subset Y$ can have cardinality at most

$$\binom{N}{[N/2]} \sim \left(\frac{2}{\pi N}\right)^{1/2} \cdot 2^N.$$

Thus, for ε fixed, if N is sufficiently large then $(2/\pi N)^{1/2} < \varepsilon$ and the assertion follows.

If A is taken to be the finite field $GF(3)$, then Brown and Buhler [5] have recently shown that any subset R of the affine space A^N having at

[1] The author is currently offering US $1000 for a proof or disproof of this conjecture.

least $\varepsilon \cdot 3^N$ points must contain an *affine* line, provided $N \geqslant N(\varepsilon)$. (Combinatorial lines correspond to very special kinds of affine lines.) More generally, Furstenberg and Katznelson have now proved (unpublished) the following weakened form of the conjecture. Let us write $A = \{a_0, a_1, \ldots, a_{t-1}\}$. Call a set $\tilde{L}$ of t points of A^N a *twisted* combinatorial line if for some nonempty $I \subseteq N$ and $d_i \in t$, $i \in I$, $\tilde{L}$ can be written as

$$\tilde{L} = \bigcup_{j \in t} \{(x_0, x_1, \ldots, x_{N-1}) \colon x_i = a_{j+d_i} \text{ if } i \in I \text{ and } x_i = b_i \in A \text{ if } i \notin I\}$$

where index addition is modulo t.

Thus, in a twisted line, the entries in each of the coordinates which vary have been cyclically permuted.

THEOREM [23]. *For all finite A and $\varepsilon > 0$ there exists $\tilde{N}(A, \varepsilon)$ such that if $N \geqslant \tilde{N}(A, \varepsilon)$ and $R \subseteq A^N$ satisfies $|R| \geqslant \varepsilon |A^N|$ then R must contain a twisted combinatorial line.*

This result implies as a corollary the fact that any subset $R \subseteq GF(q)^N$ with $|R| \geqslant \varepsilon q^N$ always contains an affine line, provided N is sufficiently large (as a function of q and ε).

Partitions into infinitely many classes

If we allow partitions of ω of the form $\omega = \bigcup_{i \in \omega} C_i$ then it is clear that the conclusion of van der Waerden's Theorem does not have to hold. For example, we could take $C_i = \{i\}$. However, in this case we have arbitrarily long arithmetic progressions which hit each C_i in *at most one* element. The following result of Erdös and Graham shows that one of these two possibilities must always occur.

THEOREM [14], [11]. *If $\omega = \bigcup_{i \in \omega} C_i$ then either some C_i contains arbitrarily long arithmetic progressions or there are arbitrarily long arithmetic progressions hitting each C_i in at most one element.*

The idea behind the proof is basically this. If some C_i has positive upper density then by Szemerédi's Theorem, C_i has the desired progressions. If not, then for N large the number of arithmetic progressions which have at least two elements in a single C_i is $o(N^2)$. Since there are at least $c_k N^2$ arithmetic progressions of length k for a fixed $c_k > 0$, the desired conclusion follows.

This result is an example of a so-called "canonical" partition theorem, first introduced by Erdös and Rado for Ramsey's Theorem [15]. Other theorems of this type have recently been given by Baumgartner [2], Taylor [62], Voigt [64], [46] and others. One of the most striking theorems of this type is the canonical partition theorem for the n-dimensional analogues of van der Waerden's theorem. As an illustration of the increased range of behavior the canonical partitions can have, consider the case $n = k = 2$. Suppose $\omega^2 = \bigcup_{i\in\omega} C_i$. Let us say that $(x, y) \sim (x', y')$ if (x, y) and (x', y') belong to the same C_i. Consider the following six partitions:

(i) $(x, y) \sim (x', y')$ iff $(x, y) = (x', y')$,
(ii) $(x, y) \sim (x', y')$ for all $(x, y), (x', y') \in \omega^2$,
(iii) $(x, y) \sim (x', y')$ iff $x = x'$,
(iv) $(x, y) \sim (x', y')$ iff $y = y'$.
(v) $(x, y) \sim (x', y')$ iff $x+y = x'+y'$,
(vi) $(x, y) \sim (x', y')$ iff $x-y = x'-y'$.

In Figure 1, we show the six different possibilities for the four vertices of a square in ω^2 (where, $\alpha, \beta, \ldots$ denote distinct classes).

$\alpha.\ \ \beta.$ $\gamma.\ \ \delta.$ (i)	$\alpha.\ \ \alpha.$ $\alpha.\ \ \alpha.$ (ii)	$\alpha.\ \ \beta.$ $\alpha.\ \ \beta.$ (iii)
$\alpha.\ \ \alpha.$ $\beta.\ \ \beta.$ (iv)	$\alpha.\ \ \beta.$ $\gamma.\ \ \alpha.$ (v)	$\alpha.\ \ \beta.$ $\beta.\ \ \gamma.$ (vi)

Fig. 1. The six canonical partitions of $\{0, 1\}^2$.

It follows from the following theorem that these are a *complete* set of canonical partitions, i.e., in *any* partition $\omega^2 = \bigcup_{i\in\omega} C_i$ at least one of these patterns must occur.

THEOREM (Deuber, Graham, Prömel, Voigt [11]). *All canonical partitions of ω^n are given as follows: For a subspace $V \subseteq \boldsymbol{R}^n$ over $\boldsymbol{R}$, partition $\boldsymbol{R}^n$ into disjoint translates of V by*

$$\boldsymbol{R}^n = \bigcup_{a\in A} (V+a).$$

This induces a partition of $\omega^n = \bigcup_{\beta\in B} C_\beta$ (where B is countable). These partitions form a complete set of canonical partitions of ω^n.

We remark that the only proof known for this result requires the use of the deep Furstenberg–Katznelson density version of the Gallai–Witt Theorem.

Hales–Jewett revisited

In order to describe the next series of results we will first recast the Hales–Jewett Theorem into a different format. As usual, we fix a finite set A and assume $A \cap \omega = \emptyset$. For $X \subseteq \omega$, $k \in \omega$, we let $(X)^k$ denote the set of *partitions* of X into k nonempty blocks. Furthermore, we let $(X)^k_A$ denote the set of partitions of $X \cup A$ into $k+|A|$ nonempty blocks so that each block contains at most one element of A. Such partitions will be called *A-partitions* of $X \cup A$. Finally, if $Y \in (X)^k_A$ and $m \leqslant k$ then $(Y)^m_A$ denotes the set of A-partitions Z of $X \cup A$ having $m+|A|$blocks such that every block of Y is contained in a block of Z. Thus, Y is a *refinement* of Z.

The theorem of Hales and Jewett can be restated as follows:

THEOREM. *For all finite A and r if $N \geqslant N(A, r)$ and $(N)^0_A = C_1 \cup \ldots \cup C_r$ then there exists $X \in (N)^1_A$ such that $(X)^0_A \subseteq C_i$ for some i.*

This was generalized by Graham and Rothschild in 1971:

THEOREM [29]. *For all finite A and $k, m, r \in \omega$, $m \leqslant k$, there exists $N(A, k, m, r)$ such that if $N \geqslant N(A, k, m, r)$ and $(N)^m_A = C_1 \cup \ldots \cup C_r$ then there exists $X \in (N)^k_A$ such that $(X)^m_A \subseteq C_i$ for some i.*

A very beautiful generalization of this has now just been proved by Carlson and Simpson. It deals with *infinite* partitions of ω. To state the result we first introduce the following topology on $(\omega)^\omega$, the partitions of ω having *infinitely* many blocks. Any partition $X \in (\omega)^\omega$ induces an equivalence relation on $\omega \times \omega$ by having $x, y \in \omega$ equivalent iff they belong to the same block. The set of all binary relations $2^{\omega \times \omega}$ can be endowed with the usual product topology, where each factor has the discrete topology. In this way $(\omega)^\omega$ becomes a topological space under the topology inherited from $2^{\omega \times \omega}$. The following result can in a certain sense be considered a *dual* to the usual Ramsey Theorem.

CARLSON–SIMPSON THEOREM [7]. *For any $k \in \omega$, if $(\omega)^\omega = C_1 \cup \ldots \cup C_r$ where each C_i is Borel then there exists $X \in (\omega)^\omega$ such that $(X)^k \subseteq C_i$ for some i.*

Carlson and Simpson in fact prove the stronger analogous result for A-partitions of $(\omega)^\omega_A$ and which can properly be considered as an infinite generalization of the Graham–Rothschild Theorem. It should be pointed out that *some* condition on the C_i is necessary since otherwise a counterexample for $(\omega)^2 = C_1 \cup C_2$ can be easily constructed using transfinite induction.

In addition to the preceding results, dual forms are proved in [7] for the Galvin–Prikry extension [25] of Ramsey's Theorem for the case of infinite subsets of ω, as well as for Ellentuck's generalization [13] of it, but space limitations prevent us from discussing them further.

In another direction, Carlson (see [44]) has very recently obtained a beautiful theorem which unifies a large number of known Ramsey-type theorems, both finite and infinite. Again, space restrictions do not allow us to give a full description of this striking achievement here. However, we will now describe a key ingredient used in the proof, which is of significant interest in its own right.

To begin with, for a fixed finite set A and a variable $v \in A$, denote by $W(v)$ the set of all "variable words" of A, i.e., the set of all finite strings $a_0, a_1, \ldots, a_m$ where $a_i \in A \cup \{v\}$ and $a_j = v$ for at least one index j. For $a \in A$ and $w(v) \in W(v)$ we can form the string $w(a)$ by simply replacing each occurrence of v in $w(v)$ by a (i.e., we just "evaluate" $w(v)$ at a). Let $S = S(A, v)$ denote the set of all infinite sequences $\bar{s} = (s_0(v), s_1(v), \ldots)$ where $s_i(v) \in W(v)$. By a *v-reduction* of $\bar{s}$ we mean any sequence $\bar{t} = (t_0(v), t_1(v), \ldots)$ formed from $\bar{s}$ in the following way. For each $i \in \omega$, $s_i(v)$ is replaced by $s_i(b_i)$ where $b_i \in A \cup \{v\}$. Disjoint blocks of consecutive $s_i(b_i)$'s are then concatenated, forming a sequence of strings $\bar{t} = (t_0(v), t_1(v), \ldots)$, where the symbol v must still occur at least once in each $t_i(v)$, (thus, $\bar{t} \in S$). Denote by $R(\bar{s})$ the set of all v-reductions of $\bar{s}$ and by $R_0(\bar{s})$ the set of all $t_0(v)$ for $\bar{t} = (t_0(v), t_1(v), \ldots) \in R(\bar{s})$.

MAIN LEMMA (Carlson). *For any $\bar{s} \in S$, if $R_0(\bar{s}) = C_1 \cup \ldots \cup C_r$ then there exists $\bar{t} \in R(\bar{s})$ such that $R_0(\bar{t}) \subseteq C_i$ for some i.*

This deceptively simple looking statement conceals much of its inherent strength. As a simple application, we derive Hindman's Theorem (following [6]). Let $\omega = C_1 \cup \ldots \cup C_r$ be given. Choose $A = \{0\}$ and partition $W(v) = C_1^* \cup \ldots \cup C_r^*$ by defining: $w(v) \in C_i^*$ iff $w(v)$ has m v's occurring in it and $m \in C_i$. Applying Carlson's result for $\bar{s} = (v, v, \ldots)$ we are guaranteed the existence of $\bar{t} = (t_0(v), t_1(v), \ldots) \in R_0(\bar{s}) = W(v)$ with $R_0(\bar{t}) \subseteq C_i^*$ for some i. For any finite subset $J \subseteq N$, the word $w_J(v) = t_0(b_0)t_1(b_1)\ldots t_{N-1}(b_{N-1}) \in C_i^*$ where

$$b_i = \begin{cases} v & \text{if } j \in J, \\ 0 & \text{if } j \notin J. \end{cases}$$

Thus, if n_i denotes the number of v's occurring in $t_i(v)$ then this implies $\sum_{j \in J} n_j \in C_i$ for all finite J, which is just Hindman's Theorem.

We should note here that Voigt [65] has very recently independently also obtained infinite generalizations of the Hales–Jewett and Graham–Rothschild Theorems which are similar to, though somewhat weaker than, Carlson's Main Lemma. His proofs however are more combinatorial in nature whereas Carlson relies on the intricate use of idempotent ultrafilter arguments (which are often quite effective for problems of this type, e.g., see [35]).

Van der Waerden again

The finite form of van der Waerden's Theorem (for two classes) asserts the following: For all $k \in \omega$, there exists a least $W(k)$ so that if $\{1, 2, \ldots, W(k)\} = C_1 \cup C_2$ then some C_i must contain a k-term arithmetic progression.

The determination of the values and, in fact, even the growth rate of $W(k)$ has proved to be extremely frustrating for combinatorialists. The known exact values are listed in Table 1.

Table 1

k	1	2	3	4	5	6
$w(k)$	1	3	9	35	178	?

The best lower bound known is due to Berlekamp [4]:

$$w(k+1) > k \cdot 2^k \quad \text{if } k \text{ is a prime power.}$$

There is currently no known upper bound for $W(k)$ which is primitive recursive. This is because all available proofs leading to upper bounds involve at some point a (perhaps intrinsic) *double* induction, with k as one of the variables. This leads naturally to rapidly growing functions like the Ackermann function which may help to explain the enormous gap in our knowledge here. The possibility that $W(k)$ might in fact actually have this Ackermann-like growth has been strengthened by the work of Paris and Harrington [43], Ketonen and Solovay [37], and more recently Friedman [18], who show that some natural combinatorial questions do indeed have *lower* bounds which grow this rapidly (and even much more rapidly, e.g., see [54], [55]). In spite of this potential evidence to the contrary, I am willing to make the following:

CONJECTURE.

$$W(k) \leqslant 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$$

for $k \geqslant 1$, where the number of 2's is k.

It should be pointed out that while any partition of the set $\{1, 2, \ldots, 9\} = C_1 \cup C_2$ always results in some C_i containing a 3-term arithmetic progression (and this is true for any set homèthetic to $\{1, 2, \ldots, 9\}$), other sets also have this property, e.g., $\{1, 3, 4, 5, 6, 7, 8, 9, 11\}$. However, it can be shown [53] that no 8-element set has the property. In general, define

$$W^*(k) = \min\{|X|: X \subseteq \omega, X = C_1 \cup C_2 \Rightarrow \text{some } C_i \text{ contains a } k\text{-term arithmetic progression}\}.$$

Thus,

$$W^*(3) = W(3) = 9$$

and, in general,

$$W^*(k) \leqslant W(k).$$

It turns out perhaps unexpectedly that $W^*(k)$ can be strictly smaller than $W(k)$. In particular, recent computations have yielded $W^*(4) \leqslant 27$, compared to $W(4) = 35$. The characteristic function of a set which achieves the bound of 27 is given by:

100100110111111111111111111011001001.

It would be of great interest to know if $W^*(k)$ is in general significantly smaller than $W(k)$, e.g., does

$$W^*(k)/W(k) \to 0 \quad \text{as } k \to \infty?$$

As an abbreviation, let us write $X \to AP(k)$ to denote the fact that for any partition of $X = C_1 \cup C_2$, some C_i contains a k-term arithmetic progression. Going in the other direction from $W^*(k)$, one might naturally ask whether there exist *arbitrarily large* sets $X(k)$ with the properties:

(i) $X(k) \to AP(k)$;

(ii) $Y \not\to AP(k)$ for any proper subset $Y \subset X(k)$.

In fact, the existence of arbitrarily large "critical" sets for both k-term arithmetic progressions as well as more general combinatorial lines in A^N has just recently been established by Graham and Nešetřil [28]. From this work, it appears that even the structure of sets $X(3)$ which satisfy (i) and (ii) for $k = 3$ can be exceedingly complex.

Concluding remarks

As mentioned earlier, we did not have the opportunity here to give more than a brief sketch of a few of the large number of exciting recent developments in Ramsey theory. The interested reader will find more of these developments reported in the following references: [35], [57], [41], [27], [8], [1], [36], [42], [46], [38].

Finally, I remark that essentially no progress has occurred on the following (by now) old conjecture of Erdös on arithmetic progressions, which would imply Szemerédi's theorem and for which Erdös currently offers US $ 3000:

CONJECTURE. *If* $A \subseteq \omega$ *and* $\sum_{a\in A} 1/a = \infty$ *then* A *contains arbitrarily long arithmetic progressions.*

A related perhaps easier conjecture is this:

CONJECTURE. *If* $A \subseteq \omega^2$ *and* $\sum_{(i,j)\in A} 1/(i^2+j^2) = \infty$ *then* A *contains the* 4 *vertices of a square.*

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