COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
37. FINITE AND INFINITE SETS, EGER (HUNGARY), 1981.

ON IRREGULARITIES OF DISTRIBUTION

F.R.K. CHUNG — R.L. GRAHAM

INTRODUCTION

A fundamental problem in the study of the distribution of sequences is the quantitative estimation of the extent that an arbitrary sequence must deviate from some appropriately defined standard of regularity. Among the many results available on this topic, those of K.F. Roth and W.M. Schmidt are particularly noteworthy. An excellent survey of much of what is currently known may be found in the book [4] of Kuipers and Niederreiter.

In 1949, de Bruijn and Erdős published a paper [2] which considered the following measure of irregularity. Let $\bar{x} = (x_1, x_2, \ldots)$ be an arbitrary real sequence with $x_k \in [0, 1]$. Define

$$\omega(\bar{x}) = \liminf_{n \to \infty} n \inf_{1 \leqslant i < j \leqslant n} |x_i - x_j|.$$

Then

$$\omega(\bar{x}) \leqslant \frac{1}{\log 4} = 0.72135 \ldots . \tag{1}$$

Furthermore, the bound in (1) is best possible as shown, for example, by taking $x_n = \left\{\frac{\log(2n-1)}{\log 2}\right\}$ where $\{x\}$ denotes the fractional part of x. (The reader should also consult [6], [7], [8] and [9] for further references on this result.)

In this paper we consider a much more sensitive measure of clustering. Observe that a sequence $\bar{x}$ could gave two consecutive terms very nearly equal infinitely often and still have $\omega(\bar{x})$ large, provided that the pairs occur sufficiently far out. In fact, this is exactly what happens for $x_n = \left\{\frac{\log(2n-1)}{\log 2}\right\}$. We remedy this by introducing the measure $C(\bar{x})$, suggested by a question of D. J. Newman (see [3]):

$$C(\bar{x}) \equiv \inf_n \liminf_{m \to \infty} n\,|x_{m+n} - x_m|.$$

The rationale behind this measure of irregularity of distribution is clear. If $\bar{x}$ were somehow perfectly spread out, we might hope that $|x_{m+n} - x_m| \geqslant \frac{1}{n}$ for all m and n (and indeed, there are sequences $\bar{x}$ for which this happens for all m and all but finitely many n).

Our first result furnishes a precise bound for $C(\bar{x})$.

Theorem 1. *For any sequence* $\bar{x}$ *in* $[0, 1]$,

$$C(\bar{x}) \leqslant \Big(1 + \sum_{k \geqslant 1} \frac{1}{F_{2k}}\Big)^{-1} \equiv \alpha = 0.39441967\ldots, \tag{2}$$

where F_n *denotes the* n*-th Fibonacci number, defined by* $F_0 = 0$, $F_1 = 1$ *and* $F_{n+2} = F_{n+1} + F_n$, $n \geqslant 0$.

The bound (2) is best possible, as shown by the next result. For each integer $n \geqslant 0$, let $\epsilon(n)$ denote the unique sequence $(\epsilon_1(n), \epsilon_2(n), \ldots)$ satisfying (see Lemma 1):

(i) $n = \sum_{i \geqslant 1} \epsilon_i(n) F_{2i}$;

(ii) For all i, $\epsilon_i(n) = 0, 1$ or 2;

(iii) If $\epsilon_i(n) = \epsilon_j(n) = 2$, $i < j$, then for some k with $i < k < j$, $\epsilon_k(n) = 0$.

Define the sequence $\bar{x}^* = (x_0^*, x_1^*, \ldots)$ by

$$x_n^* = \alpha \sum_{i \geq 1} \frac{\epsilon_i(n)}{F_{2i}}.$$

Note that $x_n^* \in [0, 1]$ and that $\bar{x}^*$ is nowhere dense.

Theorem 2.

$$C(\bar{x}^*) = \alpha. \tag{3}$$

In fact,

$$\inf_{n \geq 1} \inf_{m \geq 0} n \, | x_{m+n}^* - x_m^* | = \alpha.$$

The preceding results are intimately related to the following extremal theorem on the set S_m of permutations on $\{1, 2, \ldots, m\}$. Define

$$u_m = \min_{\pi \in S_m} \max_I \sum_{k=1}^{r-1} | \pi(i_{k+1}) - \pi(i_k) |^{-1},$$

where I ranges over all increasing subsequences

$$\{i_1 < i_2 < \ldots < i_r\} \subseteq \{1, 2, \ldots, m\}.$$

Theorem 3.

$$u_m = \begin{cases} 1 + \displaystyle\sum_{k=1}^{t} \frac{1}{F_{2k}} & \text{if} \quad F_{2t+3} \leq m < F_{2t+4}, \\ 1 + \displaystyle\sum_{k=1}^{t} \frac{1}{F_{2k}} + \frac{1}{F_{2t+3}} & \text{if} \quad F_{2t+4} \leq m < F_{2t+5}. \end{cases} \tag{4}$$

We remark here that permutations for which (4) is achieved can be generalized by the ordering of the first m terms of $\bar{x}^*$. These are the same permutations formed by arranging the first m terms of the well known sequence $\{k\tau\}$, $k = 1, 2, \ldots$, where $\tau = \frac{1}{2}(1 + \sqrt{5})$, in increasing order.

The proofs of these three theorems are somewhat involved. In what follows we first establish various preliminary results which will then be applied to prove the asserted theorems. We will conclude with remarks concerning possible extensions to other metric spaces, and in particular, to the unit square in $\mathbf{R}^2$.

PRELIMINARIES

Let F_n denote the n-th Fibonacci number defined in the preceding section. Thus, setting $\sigma = \frac{1-\sqrt{5}}{2}$ (so that $\sigma + \tau = 1$ and $\sigma\tau = -1$), F_n has the explicit representation

$$(5) \qquad F_n = \frac{1}{\sqrt{5}} (\tau^n - \sigma^n).$$

It follows that

$$(6) \qquad \sum_{k=1}^{n} F_{2k} = F_{2n+1} - 1.$$

Let $\|x\|$ denote the distance from x to the nearest integer. Then

$$(7) \qquad \begin{aligned} \|F_i\tau\| &= |F_i\tau - F_{i+1}| = \frac{1}{\sqrt{5}} |(\tau^i - \sigma^i)\tau - \tau^{i+1} + \sigma^{i+1}| = \\ &= \frac{1}{\sqrt{5}} |\sigma^{i+1} + \sigma^{i-1}| = \frac{1}{\sqrt{5}} |\sigma^i| \left| \sigma + \frac{1}{\sigma} \right| = |\sigma^i| = (-1)^i \sigma^i. \end{aligned}$$

Thus,

$$(8) \qquad F_i \|F_i\tau\| = \frac{1}{\sqrt{5}} (\tau^i - \sigma^i)(-1)^i\sigma^i = \frac{1}{\sqrt{5}} (1 - (-1)^i\sigma^{2i}).$$

In particular,

$$(9) \qquad \begin{aligned} &\frac{3-\sqrt{5}}{2} = F_2\|F_2\tau\| < F_4\|F_4\tau\| < \ldots < \frac{1}{\sqrt{5}} < \ldots \\ &\ldots < F_5\|F_5\tau\| < F_3\|F_3\tau\|. \end{aligned}$$

It is known from results in diophantine approximation (e.g., see [4] or [5]) that

$$(10) \qquad \min_{1 \leqslant k \leqslant n} \|k\tau\| = \|F_t \tau\|,$$

where

$$F_t \leqslant n < F_{t+1},$$

and, in fact,

$$(11) \qquad \min_{k > F_{2i}} k\|k\tau\| = F_{2i+2}\|F_{2i+2}\tau\| = \frac{1}{\sqrt{5}}(1 - \sigma^{4i+4}).$$

Lemma 1. *Every nonnegative integer* n *can be uniquely represented as a sum*

$$n = \sum_{i \geqslant 1} \epsilon_i F_{2i}, \quad \epsilon_i = 0, 1 \text{ or } 2,$$

so that if $\epsilon_i = \epsilon_j = 2$ *with* $i < j$ *then for some* k, $i < k < j$, *we have* $\epsilon_k = 0$.

Proof. The lemma holds by inspection for $n = 0, 1$ and 2. Assume that for some $t \geqslant 2$ the lemma holds for all $n < F_{2t}$. We will consider those n in the range $F_{2t} \leqslant n < F_{2t+2}$. Let n' be defined by

$$n' = \begin{cases} n - F_{2t} & \text{if} \quad F_{2t} \leqslant n < 2F_{2t}, \\ n - 2F_{2t} & \text{if} \quad 2F_{2t} \leqslant n < F_{2t+2}. \end{cases}$$

Since $0 \leqslant n' < F_{2t}$ then by the induction hypothesis n' has the valid representation

$$n' = \sum_{i \geqslant 1} \epsilon_i' F_{2i},$$

where, of course, $\epsilon_i' = 0$ for $i \geqslant t$. We claim that the unique valid representation for n is given by

$$n = \begin{cases} F_{2t} + \sum_{i \geqslant 1} \epsilon_i' F_{2i} & \text{if} \quad F_{2t} \leqslant n < 2F_{2t}, \\ 2F_{2t} + \sum_{i \geqslant 1} \epsilon_i' F_{2i} & \text{if} \quad 2F_{2t} \leqslant n < F_{2t+2}. \end{cases}$$

By construction, the indicated expressions do sum to n. Furthermore, they are valid since, in the first case, taking $\epsilon_t = 1$ causes no trouble and, in the second case, taking $\epsilon_t = 2$ also causes no trouble because in this case

$$n' = n - 2F_{2t} < F_{2t+2} - 2F_{2t} = F_{2t-1},$$

and consequently, the largest index $j \leqslant t-1$ with $\epsilon_j' \neq 1$ must have $\epsilon_j = 0$ (since by (6)

$$F_{2t-2} + F_{2t-4} + \ldots + F_{2j+2} + 2F_{2j} = F_{2t-1} + F_{2j-2} > n').$$

The uniqueness of the representation for n follows by similar considerations. ∎

Lemma 2. *For all* $1 \leqslant s \leqslant t$,

$$(12) \qquad \frac{1}{F_{2s+1}} < \sum_{i=s+1}^{t} \frac{1}{F_{2i}} + \frac{1}{F_{2t}} - \frac{1}{F_{2t+2}} \leqslant \frac{1}{F_{2s}} - \frac{1}{F_{2s+2}}.$$

Proof. We first need several auxiliary results.

(i) $\dfrac{1}{F_{2s-1}} < \dfrac{1}{F_{2s}} + \dfrac{1}{F_{2s+1}},$

(ii) $\dfrac{1}{F_{2s}} > \dfrac{1}{F_{2s+1}} + \dfrac{1}{F_{2s+2}},$

(iii) $\dfrac{3}{F_{2s}} < \dfrac{1}{F_{2s-2}} + \dfrac{1}{F_{2s+2}}, \quad s > 1.$

Inequalities (i) and (ii) follow at once from

$$(13) \qquad F_{n-1}F_{n+1} - F_n^2 = (-1)^n$$

which is an immediate consequence of (5). To prove (iii), it suffices to show

$$\left(\frac{1}{F_{2s-2}} - \frac{1}{F_{2s}}\right) - \left(\frac{1}{F_{2s}} - \frac{1}{F_{2s+2}}\right) \geqslant \frac{1}{F_{2s}},$$

i.e.,

$$\frac{F_{2s-1}}{F_{2s-2}F_{2s}} - \frac{F_{2s+1}}{F_{2s}F_{2s+2}} \geqslant \frac{1}{F_{2s}},$$

or

$$\frac{F_{2s-1}}{F_{2s-2}} - \frac{F_{2s+1}}{F_{2s+2}} \geqslant 1,$$

or

$$F_{2s-1}F_{2s+2} \geqslant F_{2s-2}(F_{2s+1} + F_{2s+2}) = F_{2s-2}F_{2s+3}.$$

However, it follows from (5) that

$$F_{2s-1}F_{2s+2} = F_{2s-2}F_{2s+3} + 3$$

so that (iii) holds. By (ii), (12) holds for $t = s$. Suppose now that (12) holds for a fixed value of $t \geqslant s$. Thus,

$$\sum_{i=s+1}^{t} \frac{1}{F_{2i}} + \frac{1}{F_{2t}} - \frac{1}{F_{2t+2}} \leqslant \frac{1}{F_{2s}} - \frac{1}{F_{2s+2}}.$$

By (iii),

$$-\frac{1}{F_{2t}} + \frac{3}{F_{2t+2}} - \frac{1}{F_{2t+4}} < 0.$$

Adding these inequalities, we obtain

$$\sum_{i=s+1}^{t+1} \frac{1}{F_{2i}} + \frac{1}{F_{2t+2}} - \frac{1}{F_{2t+4}} \leqslant \frac{1}{F_{2s}} - \frac{1}{F_{2s+2}}$$

which is just the right-hand side of (12) for $t + 1$. Thus, the right-hand side of (12) holds for all s and t, $s < t$.

To establish the left-hand side of (12), first note that it holds for $s = t$ by (ii).

Suppose now that the left-hand side of (12) holds for a fixed s, $1 < s \leqslant t$. Thus

$$\frac{1}{F_{2s+1}} < \sum_{i=s+1}^{t} \frac{1}{F_{2i}} + \frac{1}{F_{2t}} - \frac{1}{F_{2t+2}}.$$

By (i),

$$\frac{1}{F_{2s-1}} - \frac{1}{F_{2s+1}} < \frac{1}{F_{2s}}$$

so that adding the two inequalities we obtain the left-hand side of (12) for $s-1$, i.e.,

$$\frac{1}{F_{2s-1}} < \sum_{i=s}^{t} \frac{1}{F_{2i}} + \frac{1}{F_{2t}} - \frac{1}{F_{2t+2}}.$$

This completes the induction step and Lemma 2 is proved. ∎

The final result we consider in this section is an inequality relating the "alternating" arithmetic and harmonic means of an increasing sequence.

Lemma 3. *If* $0 < x_0 \leqslant x_1 \leqslant \ldots \leqslant x_{2n-1} \leqslant x_{2n}$ *then*

$$(14) \qquad \Big(\sum_{k=0}^{2n} (-1)^k x_k\Big)\Big(\sum_{k=0}^{2n} \frac{(-1)^k}{x_k}\Big) \geqslant 1.$$

Proof. For $n = 1$, (14) is immediate. Assume (14) holds for all values of $n < N$. Considering the function

$$f(x_0) = \Big(\sum_{k=0}^{2N} (-1)^k x_k\Big)\Big(\sum_{k=0}^{2N} \frac{(-1)^k}{x_k}\Big)$$

as a function of the variable x_0, we find

$$\frac{df}{dx_0} = \sum_{k=0}^{2N} \frac{(-1)^k}{x_k} - \sum_{k=0}^{2N} (-1)^k x_k \frac{1}{x_0^2} =$$

$$= \sum_{k=1}^{2N} \frac{(-1)^k}{x_k} - \sum_{k=1}^{2N} (-1)^k x_k \frac{1}{x_0^2} \leqslant 0$$

with equality only if $x_{2i-1} = x_{2i}$ for all i. In this case the lemma holds so we can assume strict inequality holds above. Thus, the minimum value of f occurs when x_0 is as large as possible, i.e., $x_0 = x_1$. However, in this case the desired inequality now holds by induction. ∎

AN UPPER BOUND ON u_m

For a permutation $\pi \in S_m$, define

$$(15) \qquad u(\pi) = \max_{I} \sum_{k} |\pi(i_{k+1}) - \pi(i_k)|^{-1}$$

where I ranges over all increasing subsequences

$$\{i_1 < i_2 < \ldots < i_r\} \subseteq \{1, 2, \ldots, m\}.$$

Thus,

$$u_m = \min_{\pi \in S_m} u(\pi).$$

In this section we exhibit a permutation $\rho \in S_m$ which achieves the bound in (4). Specifically, for a fixed $m \geqslant 2$, define the sequence $x(k)$, $1 \leqslant k \leqslant m$, by $\{\{k\tau\}: 1 \leqslant k \leqslant m\} = \{x(1) < x(2) < \ldots < x(m)\}$ where, as ususal, $\tau = \dfrac{1+\sqrt{5}}{2}$. Further, define $\rho = \rho_m \in S_m$ by

$$x(k) = \{\rho(k)\tau\}, \qquad 1 \leqslant k \leqslant m.$$

Claim. *For* $t \geqslant 0$,

$$(16) \qquad u(\rho_m) \leqslant \begin{cases} 1 + \displaystyle\sum_{k=1}^{t} \frac{1}{F_{2k}} & \text{if} \quad F_{2t+3} \leqslant m < F_{2t+4}, \\ 1 + \displaystyle\sum_{k=1}^{t} \frac{1}{F_{2k}} + \frac{1}{F_{2t+3}} & \text{if} \quad F_{2t+4} \leqslant m < F_{2t+5}. \end{cases}$$

Proof of Claim. First, suppose $t = 0$. If $F_3 \leqslant m < F_4$ then $m = 2$, $\rho(1) = 2$, $\rho(2) = 1$ and $u(\rho) = 1$. If $F_4 = 3 \leqslant m < F_5 = 5$ then either $m = 3$, $\rho(1) = 3$, $\rho(2) = 1$, $\rho(3) = 2$ and $u(\rho) = \frac{3}{2}$ or $m = 4$, $\rho(1) = 3$, $\rho(2) = 1$, $\rho(3) = 4$, $\rho(4) = 2$ and again, an easy computation shows $u(\rho) = \frac{3}{2}$. This proves that (16) holds for $t = 0$, i.e., $m \leqslant 4$.

Assume $m \geqslant 5$ is fixed. It follows from the definition of the $x(k)$ that for $i > j$,

$$(17) \qquad \begin{aligned} &x(i) - x(j) = \\ &= \{\rho(i)\tau\} - \{\rho(j)\tau\} \geqslant \|\{\rho(i)\tau\} - \{\rho(j)\tau\}\| = \|\,|\rho(i) - \rho(j)|\tau\|. \end{aligned}$$

Hence, if $I = \{i_1 < i_2 < \ldots < i_r\} \subseteq \{1, 2, \ldots, m\}$ then

$$(18) \qquad x(i_r) - x(i_1) = \sum_k (x(i_{k+1}) - x(i_k)) \geqslant$$

$$\geqslant \sum_k \| |\rho(i_{k+1}) - \rho(i_k)|\tau\| = \sum_k \| d_k \tau \|,$$

where $d_k = |\rho(i_{k+1}) - \rho(i_k)|$.

It is important to note at this point that if (16) holds for $m = F_{n+1} - 1$ then in fact (16) holds for all m satisfying $F_n \leqslant m < F_{n+1}$. Hence, we now assume that for some $n \geqslant 5$, $m = F_{n+1} - 1$. Thus, by (18)

$$\sum_k \| d_k \tau \| \leqslant \max_{1 \leqslant j < F_{n+1}} \{j\tau\} - \min_{1 \leqslant j < F_{n+1}} \{j\tau\} \leqslant \tag{19}$$

$$\leqslant 1 + (-1)^n (\sigma^{n-1} - \sigma^n) = 1 - (-1)^n \sigma^{n-2}.$$

There are two cases.

Case 1. n *is odd.*

From what we have shown so far it follows that

$$\{F_n \tau\} < \{(F_n - 1)\tau\} < \{(F_n - 1 - F_2)\tau\} < \ldots$$

$$\ldots < \{(F_n - 1 - F_2 - \ldots - F_{2k})\tau\} < \ldots < \{F_{n-1}\tau\},$$

i.e.,

$$(-\sigma)^n < (-\sigma)^n + 1 + \sigma < (-\sigma)^n + 1 + \sigma + \sigma^2 < \ldots$$

$$\ldots < (-\sigma)^n + 1 + \sigma + \sigma^2 + \ldots + \sigma^{2k} < \ldots < 1 - \sigma^{n-1},$$

where we have used (6) and (7) in deriving this.

Thus, choosing

$$(\rho(i_1), \rho(i_2), \rho(i_3), \ldots) = (F_n, F_n - 1, F_n - 1 - F_2, \ldots$$

$$\ldots, F_n - 1 - F_2 - \ldots - F_{n-5}, F_{n-1})$$

we have $d_1 = 1$, $d_2 = F_2$, $d_3 = F_4, \ldots,$ and

$$\sum_k |\rho(i_{k+1}) - \rho(i_k)|^{-1} = \sum_k \frac{1}{d_k} = 1 + \sum_{2j \leqslant n-3} \frac{1}{F_{2j}}. \tag{20}$$

Define

$$m_j = \begin{cases} 2 & \text{if} \quad j = 1, \\ 1 & \text{if} \quad j = F_{2i}, \qquad 2 < 2i \leqslant n - 3, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\sum_k \frac{1}{d_k} = \sum_{1 \leqslant j < F_{n+1}} \frac{m_j}{j}. \tag{21}$$

Suppose now that for some other choice of I, say

$$I = \{i'_1 < i'_2 < \ldots\} \subseteq \{1, 2, \ldots, F_{n+1} - 1\},$$

we have

$$\sum_k |\rho(i'_{k+1}) - \rho(i'_k)|^{-1} = \sum_{1 \leqslant j < F_{n+1}} \frac{c_j}{j} > \sum_{1 \leqslant j < F_{n+1}} \frac{m_j}{j}. \tag{22}$$

Let k denote the least index for which $c_k \neq m_k$. There are several possibilities.

(a) Suppose $c_k > m_k$.

Let i satisfy $F_{2i} \leqslant k < F_{2i+2}$. Then

$$1 + \sigma^{n-2} \geqslant \sum_j c_j \| j\tau \| \geqslant \sum_{j \leqslant F_{2i}} m_j \| j\tau \| + (c_k - m_k) \| k\tau \| \geqslant$$

$$\geqslant \sum_{j \leqslant i} \sigma^{2j} + \sigma^2 + \| F_{2i+1} \tau \| = \frac{\sigma^2 - \sigma^{2i+2}}{1 - \sigma^2} + \sigma^2 - \sigma^{2i+1} =$$

$$= -\sigma + \sigma^{2i+1} + \sigma^2 - \sigma^{2i+1} = 1$$

which is impossible since n is odd and $\sigma < 0$.

(b) Suppose $c_k < m_k$.

Thus, $k = F_{2i}$ for some i with $2i \leqslant n - 3$.

First we deal with the case $k = 1$. Suppose $c_1 = 1$. Then

$$\sum_{j \geqslant 1} \frac{c_j}{j} = 1 + \sum_{j \geqslant 2} \frac{c_j}{j} \leqslant 1 + \sum_{j \geqslant 2} \frac{c_j \| j\tau \|}{3 \| 3\tau \|} \quad \text{by (9) and (10)} \tag{23}$$

$$\leqslant 1 + \frac{1}{3\|3\tau\|}(1 + \sigma^{n-2} - 1 - \sigma) =$$

$$= 1 + \frac{1}{3\|3\tau\|}(-\sigma + \sigma^{n-2}).$$

If $n = 5$ then (23) implies

$$\sum_{j\geqslant 1} \frac{c_j}{j} \leqslant 1 + \frac{1}{3\|3\tau\|}(-\sigma + \sigma^3) = 1 + \frac{\sigma^3}{3\|3\tau\|} = 1.87268\ldots,$$

while

$$\sum_{j\geqslant 1} \frac{m_j}{j} \leqslant 1 + \frac{1}{F_2} = 2$$

which is a contradiction.

If $n = 7$ then (23) implies

$$\sum_{j\geqslant 1} \frac{c_j}{j} \leqslant 1 + \frac{1}{3\|3\tau\|}(-\sigma + \sigma^5) = 2.20601\ldots,$$

while

$$\sum_{j\geqslant 1} \frac{m_j}{j} = 1 + \frac{1}{F_2} + \frac{1}{F_4} = \frac{7}{3} = 2.3333\ldots$$

which is a contradiction.

If $n \geqslant 9$ then (23) implies

$$\sum_{j\geqslant 1} \frac{c_j}{j} \leqslant 1 + \frac{1}{3\|3\tau\|}(-\sigma + \sigma^{n-2}) < 1 - \frac{\sigma}{3\|3\tau\|} = 2.41202\ldots,$$

while

$$\sum_{j\geqslant 1} \frac{m_j}{j} \geqslant 1 + \frac{1}{F_2} + \frac{1}{F_4} + \frac{1}{F_6} = \frac{59}{24} = 2.45833\ldots$$

which is a contradiction.

Similarly, if $c_1 = 0$ then

$$\sum_{j\geqslant 1} \frac{c_j}{j} \leqslant \sum_{j\geqslant 2} \frac{c_j\|j\tau\|}{3\|3\tau\|} \leqslant \frac{1}{3\|3\tau\|}(1 + \sigma^{n-2}).$$

If $n = 5$ then

$$\sum_{j\geqslant 1} \frac{c_j}{j} \leqslant \frac{1}{3\|3\tau\|}(1 + \sigma^3) \leqslant 1.74536\ldots$$

and

$$\sum_{j\geqslant 1} \frac{m_j}{j} = 1 + \frac{1}{F_2} = 2$$

which is a contradiction.

If $n \geqslant 7$ then

$$\sum_{j\geqslant 1} \frac{c_j}{j} \leqslant \frac{1}{3\|3\tau\|}(1 + \sigma^{n-2}) \leqslant \frac{1}{3\|3\tau\|} = 2.28470\ldots$$

and

$$\sum_{j\geqslant 1} \frac{m_j}{j} \geqslant 1 + \frac{1}{F_2} + \frac{1}{F_4} = \frac{7}{3} = 2.3333\ldots$$

which is a contradiction. This completes the analysis for the case $k = 1$.

Suppose $k \geqslant 2$. Thus, $m_k = 1$ and $c_k = 0$, and

$$\sum_{j\geqslant k} \frac{c_j}{j} \leqslant \sum_{j>F_{2i}} \frac{c_j\|j\tau\|}{F_{2i+2}\|F_{2i+2}\tau\|} \quad \text{by (9) and (10)}$$

$$= \frac{\sqrt{5}}{1-\sigma^{4i+4}} \sum_{j>F_{2i}} c_j\|j\tau\| \quad \text{by (8)}$$

$$\leqslant \frac{\sqrt{5}}{1-\sigma^{4i+4}} \Big(1 + \sigma^{n-2} - \sum_{j<k} c_j\|j\tau\|\Big) =$$

$$= \frac{\sqrt{5}}{1-\sigma^{4i+4}} \Big(1 + \sigma^{n-2} - \sum_{j<k} m_j\|j\tau\|\Big) =$$

$$= \frac{\sqrt{5}}{1-\sigma^{4i+4}} \times$$

$$\times\ (1 + \sigma^{n-2} - (\|\tau\| + \|F_2\tau\| + \|F_4\tau\| + \ldots + \|F_{2i-2}\tau\|)) =$$

$$= \frac{\sqrt{5}}{1-\sigma^{4i+4}}(1+\sigma^{n-2}-1-\sigma^{2i-1}) = \frac{\sqrt{5}(\sigma^{n-2}-\sigma^{2i-1})}{1-\sigma^{4i+4}}.$$

On the other hand

$$\sum_{j \geqslant k} \frac{m_j}{j} = \sum_{2i \leqslant 2k \leqslant n-3} \frac{1}{F_{2k}}.$$

If $n = 2i + 3$ then this sum is just

$$\frac{1}{F_{2i}} = \frac{\sqrt{5}\,\sigma^{2i}}{1-\sigma^{4i}}$$

which is clearly greater than

$$\frac{\sqrt{5}(\sigma^{n-2}-\sigma^{2i-1})}{1-\sigma^{4i+4}} = \frac{\sqrt{5}(\sigma^{2i+1}-\sigma^{2i-1})}{1-\sigma^{4i+4}} = \frac{\sqrt{5}\,\sigma^{2i}}{1-\sigma^{4i+4}},$$

a contradiction. On the other hand, if $n \geqslant 2i + 5$ then

$$\sum_{j \geqslant k} \frac{m_j}{j} = \frac{1}{F_{2i}} + \frac{1}{F_{2i+2}} + \sum_{2i+2<2\nu \leqslant n-3} \frac{1}{F_{2\nu}} =$$

$$= \frac{1}{F_{2i}} + \frac{1}{F_{2i+2}} + \sum_{2i+2<2\nu \leqslant n-3} \frac{\sqrt{5}\,\sigma^{2\nu}}{1-\sigma^{4\nu}} \geqslant$$

$$\geqslant \frac{1}{F_{2i}} + \frac{1}{F_{2i+2}} + \frac{\sqrt{5}}{1-\sigma^{2n-6}} \sum_{2i+2<2\nu \leqslant n-3} \sigma^{2\nu} =$$

$$= \frac{1}{F_{2i}} + \frac{1}{F_{2i+2}} + \frac{\sqrt{5}}{1-\sigma^{2n-6}} \frac{\sigma^{2i+4}-\sigma^{n-1}}{1-\sigma^{2}} =$$

$$= \frac{1}{F_{2i}} + \frac{1}{F_{2i+2}} + \frac{\sqrt{5}(\sigma^{n-2}-\sigma^{2i+3})}{1-\sigma^{2n-6}}.$$

Therefore,

$$\frac{1}{\sqrt{5}}\Big(\sum_j \frac{m_j}{j} - \sum_j \frac{c_j}{j}\Big) \geqslant \frac{\sigma^{2i}}{1-\sigma^{4i}} + \frac{\sigma^{2i+2}}{1-\sigma^{4i+4}} + \tag{24}$$

$$+ \frac{\sigma^{n-2}-\sigma^{2i+3}}{1-\sigma^{2n-6}} - \frac{\sigma^{n-2}-\sigma^{2i-1}}{1-\sigma^{4i+4}}.$$

We obtain a contradiction if we can show the right-hand side of (24) is at least zero. This is so provided

$$\frac{\sigma^{2i}(1-\sigma^{4i+4})+\sigma^{2i+2}(1-\sigma^{4i})}{(1-\sigma^{4i})(1-\sigma^{4i+4})}+$$

$$+\frac{\sigma^{n-2}-\sigma^{2i+3}}{1}-\frac{\sigma^{n-2}-\sigma^{2i-1}}{1-\sigma^{4i+4}} \geqslant 0.$$

In turn, this holds if

$$\frac{\sigma^{2i}+\sigma^{2i+2}-\sigma^{6i+4}-\sigma^{6i+2}}{1-\sigma^{4i}}+$$

$$+\sigma^{n-2}-\sigma^{n+4i+2}-\sigma^{2i+3}+\sigma^{6i+7}-\sigma^{n-2}+\sigma^{2i-1} \geqslant 0$$

i.e., if

$$\sigma^{2i}+\sigma^{2i+2}-\sigma^{6i+4}-\sigma^{6i+2}-\sigma^{n+4i+2}-\sigma^{2i+3}+\sigma^{6i+7}+$$

$$+\sigma^{2i-1}+\sigma^{n+8i+2}+\sigma^{6i+3}-\sigma^{10i+7}-\sigma^{6i-1} \geqslant 0.$$

However, this holds if

$$\sigma^{2i-1}+\sigma^{2i}+\sigma^{2i+2}-\sigma^{2i+3}-$$

$$-(\sigma^{6i-1}+\sigma^{6i+2}-\sigma^{6i+3}+\sigma^{6i+4}-\sigma^{6i+7})-$$

$$-\sigma^{10i+7}+\sigma^{n+8i+2}-\sigma^{n+4i+2} \geqslant 0.$$

But

$$1+\sigma+\sigma^3-\sigma^4 = 1+\sigma-\sigma^2 = 0,$$

and

$$-(\sigma^{10i+7}+\sigma^{n+4i+2}(1-\sigma^{4i})) \geqslant 0.$$

Thus, the desired inequality holds provided

$$1+\sigma^3-\sigma^4+\sigma^5-\sigma^8 \geqslant 0.$$

A straightforward computation now shows this to indeed be the case. This implies the desired contradiction and Case 1 is finished.

Case 2. *Suppose* n *is even.*

Thus $n \geqslant 6$. It follows as before from what we have shown that

$$\{F_{n-1}\tau\} < \{(F_n - 1)\tau\} < \{(F_n - 1 - F_2)\tau\} < \ldots$$
$$\ldots < \{(F_{n-1} - 1 - F_2 - \ldots - F_{2k})\tau\} < \ldots$$
$$\ldots < \{(F_{n-1} - 1 - F_2 - \ldots - F_{n-4})\tau\} < \{F_n\tau\}.$$

Thus, choosing

$$(\rho(i_1), \rho(i_2), \ldots) = (F_{n-1}, F_{n-1} - 1, F_{n-1} - 1 - F_2, \ldots$$
$$\ldots, F_{n-1} - 1 - F_2 - F_{n-4}, F_n)$$

we have

$$d_1 = 1, \quad d_2 = F_2, \quad d_3 = F_4, \ldots, d_{r-1} = F_{n-4}$$

and

$$d_r = F_n - (F_{n-1} - 1 - F_2 - \ldots - F_{n-4}) = F_{n-1},$$

and consequently,

$$\sum_k |\rho(i_{k+1}) - \rho(i_k)|^{-1} = \sum_k \frac{1}{d_k} = 1 + \sum_{2j \leqslant n-4} \frac{1}{F_{2j}} + \frac{1}{F_{n-1}}.$$

Define

$$m_j = \begin{cases} 2 & \text{if} \quad j = 1, \\ 1 & \text{if} \quad j = F_{2i}, \qquad 2 < 2i \leqslant n-4, \\ 1 & \text{if} \quad j = F_{n-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then from above we have

$$\sum_k \frac{1}{d_k} = \sum_{1 \leqslant j < F_{n+1}} \frac{m_j}{j}.$$

Suppose now that for some other choice of I, say,

$$I = \{i''_1 < i''_2 < \ldots\} \subseteq \{1, 2, \ldots, F_{n+1} - 1\}$$

we have

$$\sum_k |\rho(i''_{k+1}) - \rho(i''_k)|^{-1} = \sum_{1 \leqslant j < F_{n+1}} \frac{c_j}{j} > \sum_{1 \leqslant j < F_{n+1}} \frac{m_j}{j}.$$

Let k denote the least integer such that $c_k \neq m_k$. There are several possibilities.

(a) Suppose $c_k > m_k$.

Assume i satisfies $F_{2i} \leqslant k < F_{2i+2}$ and $2i \leqslant n - 4$. Then

$$1 - \sigma^{n-2} \geqslant \sum_j c_j \| j\tau \| \geqslant \sum_{j \geqslant F_{2i}} m_j \| j\tau \| + (c_k - m_k) \| k\tau \| \geqslant 1$$

by exactly the argument given in $I(a)$. This is a contradiction since n is even.

On the other hand suppose $F_{n-2} \leqslant k < F_{n-1}$. Then

$$1 - \sigma^{n-2} \geqslant$$

$$\geqslant \sum_j c_j \| j\tau \| \geqslant \sum_{j \leqslant F_{n-4}} m_j \| j\tau \| + \| k\tau \| \geqslant 1 + \sigma^{n-3} + \sigma^{n-2}$$

which is a contradiction since $\sigma < -\frac{1}{2}$.

Finally, suppose $F_{n-1} \leqslant k < F_{n+1}$. Then

$$1 - \sigma^{n-2} \geqslant \sum_j c_j \| j\tau \| \geqslant \sum_{j \leqslant F_{n-4}} m_j \| j\tau \| + \| F_{n-1}\tau \| + \| F_n \tau \| =$$

$$= 1 + \sigma^{n-3} - \sigma^{n-1} + \sigma^n$$

which is again a contradiction.

(b) Suppose $c_k < m_k$.

As in Case 1, we first deal with the case $k = 1$. For $c_k = 0$ we have

$$\sum_j \frac{c_j}{j} \leqslant \sum_{j > 1} \frac{c_j \| j\tau \|}{3 \| 3\tau \|} \leqslant \frac{1}{3 \| 3\tau \|} (1 - \sigma^{n-2}).$$

For $n = 6$ this yields

$$\sum_j \frac{c_j}{j} \leqslant 1.95137\ldots$$

and

$$\sum_j \frac{m_j}{j} = 1 + \frac{1}{F_2} + \frac{1}{F_5} = 2.2$$

which is a contradiction.

For $n \geqslant 8$ this gives

$$\sum_j \frac{c_j}{j} \leqslant \frac{1}{3\|3\tau\|} = 2.28470\ldots$$

and

$$\sum_j \frac{m_j}{j} \geqslant 1 + \frac{1}{F_2} + \frac{1}{F_4} = \frac{7}{3} = 2.3333\ldots$$

which again is a contradiction.

For $c_k = 1$,

$$\sum_j \frac{c_j}{j} \leqslant 1 + \sum_{j>1} \frac{c_j\|j\tau\|}{3\|3\tau\|} \leqslant \frac{1}{3\|3\tau\|}(-\sigma - \sigma^{n-2}).$$

If $n = 6$ this yields

$$\sum_j \frac{c_j}{j} \leqslant \frac{1}{3\|3\tau\|}(-\sigma - \sigma^4) = 2.07869\ldots$$

and

$$\sum_j \frac{m_j}{j} = 1 + \frac{1}{F_2} + \frac{1}{F_5} = 2.2$$

which is a contradiction.

If $n = 8$ then

$$\sum_j \frac{c_j}{j} \leqslant 1 + \frac{1}{3\|3\tau\|}(-\sigma - \sigma^6) = 2.28470\ldots$$

and

$$\sum_j \frac{m_j}{j} \geqslant 1 + \frac{1}{F_2} + \frac{1}{F_4} + \frac{1}{F_7} = 2.41026\ldots$$

which is a contradiction.

If $n \geqslant 10$ then

$$\sum_j \frac{c_j}{j} \leqslant 1 + \frac{1}{3\|3\tau\|}(-\sigma) = 2.41202\ldots$$

and

$$\sum_j \frac{m_j}{j} \geqslant 1 + \frac{1}{F_2} + \frac{1}{F_4} + \frac{1}{F_6} = 2.45833\ldots$$

which is a contradiction.

Thus, we may assume $k > 1$. This implies $m_k = 1$ and $c_k = 0$.

(i) Suppose $k = F_{2i}$ where $2i \leqslant n - 6$. Then

$$\sum_j \frac{m_j}{j} - \sum_j \frac{c_j}{j} = \sum_{j \geqslant F_{2i}} \frac{m_j}{j} - \sum_{j > F_{2i}} \frac{c_j}{j}. \tag{25}$$

We now estimate the two sums on the right-hand side of (25). First,

$$\sum_{j > F_{2i}} \frac{c_j}{j} \leqslant \sum_{j > F_{2i}} \frac{c_j \|j\tau\|}{F_{2i+2}\|F_{2i+2}\tau\|} \leqslant$$

$$\leqslant \frac{\sqrt{5}}{1 - \sigma^{4i+4}}\Big(1 - \sigma^{n-2} - \sum_{j < F_{2i}} c_j \|j\tau\|\Big) =$$

$$= \frac{\sqrt{5}}{1 - \sigma^{4i+4}}(-\sigma^{n-2} - \sigma^{2i-1})$$

since $c_j = m_j$ for $j < F_{2i}$. Also

$$\sum_{j \geqslant F_{2i}} \frac{m_j}{j} = \frac{\sqrt{5}\,\sigma^{2i}}{1 - \sigma^{4i}} + \sum_{2i < 2j \leqslant n-4} \frac{\sqrt{5}\,\sigma^{2j}}{1 - \sigma^{4j}} - \frac{\sqrt{5}\,\sigma^{n-1}}{1 + \sigma^{2(n-1)}}.$$

Therefore,

$$\frac{1}{\sqrt{5}}\left(\sum_j \frac{m_j}{j} - \sum_j \frac{c_j}{j}\right) \geqslant$$

$$\geqslant \frac{\sigma^{2i}}{1-\sigma^{4i}} + \frac{\sigma^{2i+2}}{1-\sigma^{4i+4}} + \frac{\sigma^{2i-1}+\sigma^{n-2}}{1-\sigma^{4i+4}} +$$

$$+ \frac{1}{1-\sigma^{2n-8}} \sum_{2i+2<2j\leqslant n-4} \sigma^{2j} - \frac{\sigma^{n-1}}{1+\sigma^{2n-2}} =$$

$$= \frac{\sigma^{2i}(1-\sigma^{4i+4}) + (\sigma^{2i+2}+\sigma^{2i-1}+\sigma^{n-2})(1-\sigma^{4i})}{(1-\sigma^{4i})(1-\sigma^{4i+4})} +$$

$$+ \frac{\sigma^{n-3}-\sigma^{2i+3}}{1-\sigma^{2n-8}} - \frac{\sigma^{n-1}}{1+\sigma^{2n-2}} =$$

$$= \frac{\sigma^{2i+3} - (\sigma^{6i-1}+\sigma^{6i+2}+\sigma^{6i+4}) + \sigma^{n-2} - \sigma^{n+4i-2}}{(1-\sigma^{4i})(1-\sigma^{4i+4})} +$$

$$+ \frac{\sigma^{n-3}-\sigma^{2i+3}}{1-\sigma^{2n-8}} - \frac{\sigma^{n-1}}{1+\sigma^{2n-2}} \geqslant$$

$$\geqslant \frac{\sigma^{2i+3} - (\sigma^{6i-1}+\sigma^{6i+2}+\sigma^{6i+4}) + \sigma^{n-2} - \sigma^{n+4i-2}}{(1-\sigma^{4i})(1-\sigma^{4i+4})} +$$

$$+ \sigma^{n-3} - \sigma^{2i+3} - \frac{\sigma^{n-1}}{1+\sigma^{2n-2}} \quad \text{since} \quad n-3 \geqslant 2i+3$$

$$\geqslant \frac{\sigma^{2i+3} - (\sigma^{6i-1}+\sigma^{6i+2}+\sigma^{6i+4}) + \sigma^{n-2} - \sigma^{n+4i-2}}{(1-\sigma^{4i})(1-\sigma^{4i+4})} -$$

$$- \sigma^{2i+3} + \frac{\sigma^{n-3}-\sigma^{n-1}+\sigma^{3n-5}}{1+\sigma^{2n-2}} \geqslant$$

$$\geqslant \frac{\sigma^{2i+3} - (\sigma^{6i-1}+\sigma^{6i+2}+\sigma^{6i+4}) + \sigma^{n-2} - \sigma^{n+4i-2}}{(1-\sigma^{4i})(1-\sigma^{4i+4})} -$$

$$- \sigma^{2i+3} - \sigma^{n-2} + \sigma^{3n-5} \geqslant$$

$$\geqslant \frac{\sigma^{2i+3} - (\sigma^{6i-1}+\sigma^{6i+2}+\sigma^{6i+4}) - \sigma^{n+4i-2}}{(1-\sigma^{4i})(1-\sigma^{4i+4})} -$$

$$- \sigma^{2i+3} + \sigma^{3n-5} =$$

$$= \frac{\sigma^{2i+3} - (\sigma^{2i+3} - \sigma^{6i+3} - \sigma^{6i+7} + \sigma^{10i+7})}{(1 - \sigma^{4i})(1 - \sigma^{4i+4})} -$$

$$- \frac{(\sigma^{6i-1} + \sigma^{6i+2} + \sigma^{6i+4}) + \sigma^{n+4i-2}}{(1 - \sigma^{4i})(1 - \sigma^{4i+4})} + \sigma^{3n-5} =$$

$$= \frac{-\sigma^{6i-1}(1 + \sigma^3 - \sigma^4 + \sigma^5 - \sigma^8 + \sigma^{4i+8} + \sigma^{n-2i-1})}{(1 - \sigma^{4i})(1 - \sigma^{4i+4})} +$$

$$+ \sigma^{3n-5} \geqslant$$

$$\geqslant \frac{-\sigma^{6i-1}(1 + \sigma^3 - \sigma^4 + \sigma^5 - \sigma^8 + \sigma^5)}{(1 - \sigma^{4i})(1 - \sigma^{4i+4})} + \sigma^{3(2i+6)-5} >$$

$$> \frac{-\sigma^{6i-1}\sigma^2}{(1 - \sigma^{4i})(1 - \sigma^{4i+4})} + \sigma^{6i+13} > -\sigma^{6i+1} + \sigma^{6i+13} > 0$$

which finally is a contradiction.

(ii) Suppose $k = F_{2i}$ where $2i = n - 4$. Then

$$\sum_{j \geqslant F_{2i}} \frac{c_j}{j} \leqslant \sum_{j > F_{2i}} \frac{c_j \| j\tau \|}{F_n \| F_n \tau \|} \leqslant$$

$$\leqslant \Big(1 - \sigma^{n-2} - \sum_{j < F_{n-4}} c_j \| j\tau \|\Big) \frac{\sqrt{5}}{1 - \sigma^{2n}} \leqslant$$

$$\leqslant (1 - \sigma^{n-2} - 1 - \sigma^{n-5}) \frac{\sqrt{5}}{1 - \sigma^{2n}} = \frac{-(\sigma^{n-5} + \sigma^{n-2})\sqrt{5}}{1 - \sigma^{2n}}$$

and

$$\sum_{j \geqslant F_{2i}} \frac{m_j}{j} \geqslant \frac{1}{F_{n-4}} + \frac{1}{F_{n-1}} = \sqrt{5}\Big(\frac{\sigma^{n-4}}{1 - \sigma^{2n-8}} - \frac{\sigma^{n-1}}{1 + \sigma^{2n-2}}\Big).$$

Thus,

$$\frac{1}{\sqrt{5}}\Big(\sum_j \frac{m_j}{j} - \sum_j \frac{c_j}{j}\Big) \geqslant$$

$$\geqslant \frac{\sigma^{n-4}}{1 - \sigma^{2n-8}} - \frac{\sigma^{n-1}}{1 + \sigma^{2n-2}} + \frac{\sigma^{n-5}}{1 - \sigma^{2n}} + \frac{\sigma^{n-2}}{1 - \sigma^{2n}} \geqslant$$

$$\geqslant \frac{\sigma^{n-4}}{1-\sigma^{2n}} - \frac{\sigma^{n-1}}{1-\sigma^{2n}} + \frac{\sigma^{n-5}}{1-\sigma^{2n}} + \frac{\sigma^{n-2}}{1-\sigma^{2n}}$$

since

$$\sigma^{n-4}\left(\frac{1}{1-\sigma^{2n-8}} - \frac{1}{1-\sigma^{2n}}\right) +$$

$$+ \sigma^{n-1}\left(\frac{1}{1-\sigma^{2n}} - \frac{1}{1+\sigma^{2n-2}}\right) =$$

$$= \sigma^{n-4}\left(\frac{\sigma^{2n-8}-\sigma^{2n}}{(1-\sigma^{2n-8})(1-\sigma^{2n})}\right) + \frac{\sigma^{n-1}(\sigma^{2n-2}+\sigma^{2n})}{(1-\sigma^{2n})(1+\sigma^{2n-2})} \geqslant$$

$$\geqslant \sigma^{n-4}\sigma^{2n-6} + \sigma^{n-1}\sigma^{2n-4} = \sigma^{3n-10} + \sigma^{3n-5} > 0.$$

This computation makes use of the inequality

$$\frac{\sigma^{2n-2}+\sigma^{2n}}{(1-\sigma^{2n})(1+\sigma^{2n-2})} < \sigma^{2n-4},$$

i.e.,

$$\sigma^2 + \sigma^4 < (1-\sigma^{2n})(1+\sigma^{2n-2}) = 1 + \sigma^{2n-2} - \sigma^{2n} - \sigma^{4n-2}$$

which follows from

$$\sigma^2 + \sigma^4 < 1 \quad \text{since} \quad \sigma^{2n-2} - \sigma^{2n} - \sigma^{4n-2} \geqslant 0.$$

Therefore,

$$\frac{1}{\sqrt{5}}\left(\sum_j \frac{m_j}{j} - \sum_j \frac{c_j}{j}\right) \geqslant \frac{-\sigma^{n-1}+\sigma^{n-2}+\sigma^{n-5}+\sigma^{n-4}}{1-\sigma^{2n}} = 0$$

which is a contradiction.

(iii) Suppose $k = F_{n-1}$. In this case observe that

$$1-\sigma^{n-2} \geqslant \sum_j c_j \|j\tau\| \geqslant \sum_{j \leqslant F_{n-4}} m_j \|j\tau\| + \sum_{j > F_{n-1}} c_j \|F_n \tau\| =$$

$$= 1 + \sigma^{n-3} + \left(\sum_{j > F_{n-1}} c_j\right)\sigma^n.$$

Thus,

$$\sum_{j > F_{n-1}} c_j \leqslant \frac{-\sigma^{n-2} - \sigma^{n-3}}{\sigma^n} = \frac{-\sigma^{n-1}}{\sigma^n} = \tau < 2.$$

Since the c_j are nonnegative integers then there is exactly one j, say j_0, greater than F_{n-1} such that $c_{j_0} = 1$, and all other c_j's are 0. Therefore,

$$\sum_j \frac{m_j}{j} - \sum_j \frac{c_j}{j} = \sum_{j \geqslant F_{n-1}} \frac{m_j}{j} - \sum_{j > F_{n-1}} \frac{c_j}{j} = \frac{1}{F_{n-1}} - \frac{1}{j_0} > 0$$

which is a contradiction. This completes the analysis of Case 2 and (16) is proved. ∎

A LOWER BOUND ON u_m

We next turn our attention to lower bounds for u_m. In this section we prove that the upper bounds for $u(\rho_m)$ given in (16) are lower bounds for u_m in general. That is,

$$(26) \qquad u_m \geqslant \begin{cases} 1 + \sum_{k=1}^{t} \frac{1}{F_{2k}} & \text{if} \quad F_{2t+3} \leqslant m < F_{2t+4}, \\ 1 + \sum_{k=1}^{t} \frac{1}{F_{2k}} + \frac{1}{F_{2t+3}} & \text{if} \quad F_{2t+4} \leqslant m < F_{2t+5}. \end{cases}$$

Proof. Define $f_1 = f_2 = 0$, $f_3 = 1$ and

$$(27) \qquad \begin{aligned} f_{2n} &= f_{2n-1} + \frac{1}{F_{2n-1}}, \\ f_{2n+1} &= f_{2n-1} + \frac{1}{F_{2n-2}}. \end{aligned} \qquad n \geqslant 2$$

Thus,

$$f_{2n} = 1 + \sum_{k=1}^{n-2} \frac{1}{F_{2k}} + \frac{1}{F_{2n-1}},$$

$$f_{2n+1} = 1 + \sum_{k=1}^{n-1} \frac{1}{F_{2k}}.$$

Consider the statements:

$A(n)$: $u_{F_{2n}} \geqslant f_{2n}$.

$A'(n)$: *If* $\pi \in S_{F_{2n}}$ *and* $\pi(1),\ \pi(F_{2n}) > F_{2n-1}$ *then*

$$u(\pi) > f_{2n} + \frac{1}{F_{2n}}.$$

$B(n)$: $u_{F_{2n+1}} \geq f_{2n+1}$.

$B'(n)$: *If* $\pi \in S_{F_{2n+1}}$ *and* $\pi(1),\ \pi(F_{2n+1}) > F_{2n}$ *then*

$$u(\pi) > f_{2n+1} + \frac{1}{F_{2n+1}}.$$

We will prove these statements by induction on n.

First note that $A'(n)$ and $B'(n)$ apply under complementation in the following form:

$\bar{A}'(n)$: *If* $\pi \in S_{F_{2n}}$ *and* $\pi(1),\ \pi(F_{2n}) \leq F_{2n-2}$ *then*

$$u(\pi) > f_{2n} + \frac{1}{F_{2n}}.$$

$\bar{B}'(n)$: *If* $\pi \in S_{F_{2n+1}}$ *and* $\pi(1),\ \pi(F_{2n+1}) \leq F_{2n-1}$ *then*

$$u(\pi) < f_{2n+1} + \frac{1}{F_{2n+1}}.$$

(This can be seen by considering the maps $z \to F_m + 1 - z$ for $m = 2n$ and $m = 2n + 1$.

To begin the induction, a modest computation (left to the reader) shows that all the statements hold (occasionally vacuously) for $n = 1$ and 2.

Assume now that the statemenets hold for all values less than some value of $n \geq 3$. We prove that they also hold for the value n.

(a) Assume $\pi \in S_{F_{2n}}$ and let $x = \pi(1)$.

(a_1) Suppose $x \leq F_{2n-2}$.

Consider the block of integers $B = \{x + 1, \ldots, x + F_{2n-1}\}$. Let

y denote the left-hand most element of B in π, i.e., $y = \pi(i)$ and $z \in B$, $z \neq y$, $z = \pi(j)$ implies $j > i$. Thus

$$(28)\qquad \begin{aligned} u(\pi) &\geqslant \frac{1}{y-x} + u_{F_{2n-1}} \geqslant f_{2n-1} + \frac{1}{F_{2n-1}} \qquad \text{(by induction)} \\ &= f_{2n}. \end{aligned}$$

By complementation, (28) also holds if $\pi(1) > F_{2n-1}$ or if $\pi(F_{2n}) = = x' \leqslant F_{2n-2}$ or if $x' > F_{2n-1}$.

(a$_2$) Suppose $F_{2n-2} < x$, $x' \leqslant F_{2n-1}$.

Consider π', the permutation induced by π on $\{1, 2, \ldots, F_{2n-1}\}$. Then by $B'(n-1)$,

$$u(\pi) \geqslant u(\pi') > f_{2n-1} + \frac{1}{F_{2n-1}} = f_{2n}.$$

Hence, in either case, $u(\pi) \geqslant f_{2n}$, and $A(n)$ holds.

(b) Assume $\pi \in S_{F_{2n+1}}$

(b$_1$) Suppose $x = \pi(1) \leqslant F_{2n-1}$.

Consider the block $B = \{x+1, \ldots, x + F_{2n}\}$. Let y denote the left-hand most element of B in π.

(i) Suppose $y < x + F_{2n}$. Then

$$\begin{aligned} u(\pi) &\geqslant \frac{1}{y-x} + u_{F_{2n}} \geqslant \frac{1}{F_{2n} - 1} + f_{2n} = \\ &= f_{2n-1} + \frac{1}{F_{2n-1}} + \frac{1}{F_{2n} - 1} \end{aligned}$$

by the definition of f. Thus, $B(n)$ holds in this case provided

$$(29)\qquad \frac{1}{F_{2n-1}} + \frac{1}{F_{2n} - 1} \geqslant \frac{1}{F_{2n-2}}$$

i.e., if

$$\frac{1}{F_{2n} - 1} \geqslant \frac{1}{F_{2n-2}} - \frac{1}{F_{2n-1}} = \frac{F_{2n-3}}{F_{2n-2}F_{2n-1}},$$

or

$$F_{2n-2}F_{2n-1} \geq F_{2n-3}F_{2n} - F_{2n-3}.$$

However, since

$$F_{2n-1}F_{2n-2} = F_{2n}F_{2n-3} - 1$$

then the preceding inequality is equivalent to $F_{2n-3} \geq 1$, which holds since $n \geq 2$.

(ii) Suppose $y = x + F_{2n}$.

Consider the subblock $B' = \{x + 1, \ldots, x + F_{2n-1}\}$ of B. Let z be the left-hand most element of B' in π. There are two possibilities.

1. $z \leq x + F_{2n-2}$. Then

$$u(\pi) \geq \frac{1}{z - x} + u_{F_{2n-1}} \geq \frac{1}{F_{2n-2}} + f_{2n-1} = f_{2n+1}$$

and this subcase is done.

2. $z > x + F_{2n-2}$. Then

$$z > x + F_{2n-2} = y - F_{2n} + F_{2n-2} = y - F_{2n-1}$$

i.e.,

$$y < z + F_{2n-1}$$

and consequently,

$$u(\pi) \geq \frac{1}{y - x} + \frac{1}{y - z} + u_{F_{2n-1}} \geq \frac{1}{F_{2n}} + \frac{1}{F_{2n-1} - 1} + f_{2n-1}.$$

Hence, this case is taken care of provided

$$\frac{1}{F_{2n}} + \frac{1}{F_{2n-1} - 1} \geq \frac{1}{F_{2n-2}},$$

since then

$$u(\pi) \geq \frac{1}{F_{2n-2}} + f_{2n-1} = f_{2n+1}.$$

However, this follows from

$$\frac{1}{F_{2n-1}-1} \geqslant \frac{1}{F_{2n-2}} - \frac{1}{F_{2n}} = \frac{F_{2n-1}}{F_{2n-2}F_{2n}},$$

which in turn follows from the equality $F_{2n-2}F_{2n} = F_{2n-1}^2 - 1$ since $n \geqslant 1$. This completes case (b_1). As before, this argument also applies if $x > F_{2n}$ or $\pi(F_{2n}) = x' \leqslant F_{2n-1}$ or $x' > F_{2n}$.

(b_2) Suppose $F_{2n-1} < x, x' \leqslant F_{2n}$.

Assume without loss of generality that $x < x'$. Consider the block $B = \{x - F_{2n-1}, \ldots, x - 1\}$. Let y and y' denote the left-hand most and right-hand most elements, respectively, of B in π. Then

$$(30) \qquad u(\pi) \geqslant \frac{1}{x-y} + \frac{1}{x'-y} + u_{F_{2n-1}}.$$

Now, the least possible value the right-hand side of (30) can take is achieved when x' is as large as possible, i.e., $x' = F_{2n}$, and when x is small as possible (since $y, y' \in B$), i.e., $x = F_{2n-1} + 1$. Also, if $\{y, y'\} \neq$ $\neq \{x - F_{2n-1}, x - F_{2n-1} + 1\} = \{1, 2\}$ then decreasing either one of y or y' decreases the right-hand side of (3). Hence, we can assume either $y = 1$, $y' = 2$ or $y = 2$, $y' = 1$. In fact, the smaller value always occurs for the choice $y = 1$, $y' = 2$. Consequently, (30) implies

$$u(\pi) \geqslant \frac{1}{F_{2n-1}} + \frac{1}{F_{2n}-2} + f_{2n-1} \geqslant \frac{1}{F_{2n-2}} + f_{2n-1} \quad \text{by (29)}$$

$$= f_{2n+1}.$$

Thus, in both cases (b_1) and (b_2), $u(\pi) \geqslant f_{2n+1}$ and $B(n)$ holds for case (b).

(a') Assume $\pi \in S_{F_{2n}}$, and $x = \pi(1)$, $x' = \pi(F_{2n})$ with $x < x'$. Furthermore, assume $x, x' > F_{2n-1}$.

Consider the block $B = \{x - F_{2n-1}, \ldots, x - 1\}$. Let y and y' denote the left-hand most and right-hand most elements, respectively, of B in π. Then

$$u(\pi) \geqslant \frac{1}{x-y} + \frac{1}{x'-y'} + u_{F_{2n-1}} \geqslant$$

$$\geqslant \frac{1}{F_{2n-1}} + \frac{1}{F_{2n}-2} + f_{2n-1} \quad \text{by the argument in } (b_2)$$

$$= f_{2n} + \frac{1}{F_{2n}-2} > f_{2n} + \frac{1}{F_{2n}}.$$

This proves $A'(n)$.

(b') Assume $\pi \in S_{F_{2n+1}}$, $x = \pi(1)$, $x' = \pi(F_{2n+1})$, $x < x'$ and $x, x' > F_{2n}$.

As usual, let y and y' denote the left-hand most and right-hand most elements, respectively, of $B = \{x - F_{2n}, \ldots, x-1\}$ in π.

(b_1') Suppose $y, y' \leqslant x - F_{2n} + F_{2n-2} = x - F_{2n-1}$.

Then $\bar{A}'(n)$ applies in this case and we obtain

$$u(\pi) \geqslant \frac{1}{x-y} + \frac{1}{x'-y'} + f_{2n} + \frac{1}{F_{2n}} \geqslant$$

$$\geqslant \frac{1}{F_{2n}} + \frac{1}{F_{2n+1}-2} + \frac{1}{F_{2n}} + f_{2n} =$$

$$= \frac{2}{F_{2n}} + \frac{1}{F_{2n+1}-2} + \frac{1}{F_{2n-1}} + f_{2n-1} =$$

$$= f_{2n+1} + \left(\frac{2}{F_{2n}} + \frac{1}{F_{2n+1}-2} + \frac{1}{F_{2n-1}} - \frac{1}{F_{2n-2}}\right).$$

Thus, $B'(n)$ holds in this case provided we can show

$$\frac{2}{F_{2n}} + \frac{1}{F_{2n+1}-2} + \frac{1}{F_{2n-1}} - \frac{1}{F_{2n-2}} > \frac{1}{F_{2n+1}},$$

i.e.,

$$\frac{2}{F_{2n}} > \frac{1}{F_{2n-2}} - \frac{1}{F_{2n-1}} - \left(\frac{1}{F_{2n+1}-2} - \frac{1}{F_{2n+1}}\right).$$

However, this follows immediately from

$$\frac{2}{F_{2n}} > \frac{1}{F_{2n-2}} - \frac{1}{F_{2n-1}} =$$

$$= \frac{F_{2n-3}}{F_{2n-2}F_{2n-1}} = \frac{F_{2n-3}}{F_{2n}F_{2n-3} - 1}$$

since $n \geqslant 2$.

(b_2') Suppose $y > x - F_{2n-1}$.

Thus, $y \geqslant x - F_{2n-1} + 1$ and

(31) $$u(\pi) \geqslant \frac{1}{x-y} + \frac{1}{x'-y'} + u_{F_{2n}}.$$

Arguing as before, we can show that the right-hand side of (31) is minimized by choosing $x' = F_{2n+1}$, $x = F_{2n} + 1$, $y = F_{2n-2} + 2$ and $y' = 1$. Thus, (31) implies

$$u(\pi) \geqslant \frac{1}{F_{2n-1} - 1} + \frac{1}{F_{2n+1} - 1} + f_{2n} =$$

$$= f_{2n-1} + \frac{1}{F_{2n-1}} + \frac{1}{F_{2n-1} - 1} + \frac{1}{F_{2n+1} - 1} =$$

$$= f_{2n+1} +$$

$$+ \left(\frac{1}{F_{2n-1}} + \frac{1}{F_{2n-1} - 1} + \frac{1}{F_{2n+1} - 1} - \frac{1}{F_{2n-2}}\right) >$$

$$> f_{2n+1} + \frac{1}{F_{2n+1}} + \left(\frac{1}{F_{2n-1}} + \frac{1}{F_{2n-1} - 1} - \frac{1}{F_{2n-2}}\right) \geqslant$$

$$\geqslant f_{2n+1} + \frac{1}{F_{2n+1}}$$

since

$$\frac{1}{F_{2n-1}} + \frac{1}{F_{2n-1} - 1} > \frac{2}{F_{2n-1}} \geqslant \frac{1}{F_{2n-2}}$$

for $n \geqslant 2$.

(b_2'') Suppose $y' > x - F_{2n-1}$.

Thus, $y' \geqslant x - F_{2n-1} + 1$ and

(32) $$u(\pi) \geqslant \frac{1}{x-y} + \frac{1}{x'-y'} + u_{F_{2n}}.$$

Arguing as before, the right-hand side of (32) is minimized by choosing $x = F_{2n} + 1$, $x' = F_{2n+1}$, $y = 1$ and $y' = x - F_{2n-1} + 1 = F_{2n-2} + 2$. Thus, (32) implies

$$u(\pi) \geqslant \frac{1}{F_{2n}} + \frac{1}{F_{2n+1} - F_{2n-2} - 2} + f_{2n} =$$

$$= f_{2n-1} + \frac{1}{F_{2n-1}} + \frac{1}{F_{2n}} + \frac{1}{F_{2n+1} - F_{2n-2} - 2} =$$

$$= f_{2n+1} +$$

$$+ \left(\frac{1}{F_{2n-1}} + \frac{1}{F_{2n}} + \frac{1}{F_{2n+1} - F_{2n-2} - 2} - \frac{1}{F_{2n-2}}\right).$$

Therefore, $B'(n)$ holds in this case provided we can show

$$(33) \qquad \frac{1}{F_{2n-1}} + \frac{1}{F_{2n}} + \frac{1}{F_{2n+1} - F_{2n-2} - 2} - \frac{1}{F_{2n-2}} > \frac{1}{F_{2n+2}}.$$

However, since

$$\frac{1}{F_{2n-2}} - \frac{1}{F_{2n}} - \frac{1}{F_{2n-1}} = \frac{1}{F_{2n}F_{2n-1}F_{2n-2}}$$

then (33) is equivalent to

$$\frac{1}{F_{2n+1} - F_{2n-2} - 2} - \frac{1}{F_{2n+1}} > \frac{1}{F_{2n}F_{2n-1}F_{2n-2}}$$

i.e.,

$$(F_{2n-2} + 2)F_{2n}F_{2n-1}F_{2n-2} > F_{2n+1}(F_{2n+1} - F_{2n-2} - 2).$$

A brief computation shows that this inequality is indeed valid for $n \geqslant 3$ and consequently, $B'(n)$ holds in this case as well.

All cases have now been taken care of and the induction step has been completed. Therefore, $A(n)$, $B(n)$, $A'(n)$ and $B'(n)$ hold for all $n \geqslant 1$ and, by the monotonicity of u_m in m, (26) is proved. ∎

By combining (16) and (26) we finally obtain a proof of Theorem 3, namely,

$$u_m = \begin{cases} 1 + \sum_{k=1}^{t} \frac{1}{F_{2k}} & \text{if} \quad F_{2t+3} \leqslant m < F_{2t+4}, \\ 1 + \sum_{k=1}^{t} \frac{1}{F_{2k}} + \frac{1}{F_{2t+3}} & \text{if} \quad F_{2t+4} \leqslant m < F_{2t+5}. \end{cases}$$

As an immediate corollary of Theorem 3 we have:

Theorem 1. *For any sequence* $\bar{x}$ *in* $[0, 1]$,

$$C(\bar{x}) = \inf_n \liminf_{m \to \infty} n \,|\, x_{m+n} - x_m \,| \leqslant \Big(1 + \sum_{k \geqslant 1} \frac{1}{F_{2k}}\Big)^{-1} \equiv \alpha. \tag{34}$$

Proof. Suppose (34) does not hold. Thus, there exists a sequence $\bar{x}$ in $[0, 1]$ such that for all n there exists $\epsilon > 0$ such that for all sufficiently large m,

$$n \,|\, x_{m+n} - x_m \,| \geqslant (1 + \epsilon)\alpha.$$

By Theorem 3, we know that for all $\delta > 0$, if N is sufficiently large then for any $\pi \in S_N$ there is a subsequence

$$I = \{i_1 < i_2 < \ldots\} \subseteq \{1, 2, \ldots, N\}$$

such that

$$u(\pi) = \sum_k |\, \pi(i_{k+1}) - \pi(i_k) \,|^{-1} > \frac{1 - \delta}{\alpha}.$$

Let π be specified by the N consecutive terms $(x_{M+1}, x_{M+2}, \ldots$ $\ldots, x_{M+N})$ (M large) of $\bar{x}$ by

$$x_{M+\pi(1)} \leqslant x_{M+\pi(2)} \leqslant \ldots \leqslant x_{M+\pi(N)}$$

(ties are decided arbitrarily). Thus,

$$1 \geqslant \sum_k (x_{M+\pi(i_{k+1})} - x_{M+\pi(i_k)}) \geqslant$$

$$\geqslant \alpha(1 + \epsilon) \sum_k |\, \pi(i_{k+1}) - \pi(i_k) \,|^{-1} > (1 + \epsilon)(1 - \delta)$$

which is a contradiction for δ sufficiently small. This proves Theorem 1. ∎

AN EXTREMAL SEQUENCE

In this section we construct a sequence which achieves the upper bound in Theorem 1.

We begin by defining an infinite sequence $y(0), y(1), y(2), \ldots$ as follows:

For $n \geqslant 0$, write $n = \sum_{i \geqslant 1} \epsilon_i F_{2i}$ in the unique representation guaranteed by Lemma 1. Define

$$y(n) = \sum_{i \geqslant 1} \frac{\epsilon_i}{F_{2i}}.$$

Theorem. *For all* $a \neq b$,

(35) $$|(a-b)(y(a)-y(b))| \geqslant 1.$$

Proof. Write $a = \sum_{i \geqslant 1} \epsilon_i^{(a)} F_{2i}$, $b = \sum_{i \geqslant 1} \epsilon_i^{(b)} F_{2i}$. We first observe that (35) holds if $b = 0$. For in this case it is sufficient to show that $y(a) \geqslant \frac{1}{a}$. This inequality certainly holds for all a if it holds for those a of the form F_{2m}. However, since $y(F_{2m}) = \frac{1}{F_{2m}}$ then the claim is verified.

Hence, we may assume without loss of generality that $a > b > 0$.

Next, note that if (35) is violated for some a, b with $\epsilon_i^{(a)} \epsilon_i^{(b)} > 0$ for some i then it is violated for $a' = a - F_{2i}$ and $b' = b - F_{2i}$. Thus, we may assume

(36) $$\epsilon_i^{(a)} \epsilon_i^{(b)} = 0 \quad \text{for all } i.$$

Let t be the largest index satisfying $\epsilon_t^{(a)} > 0$. Then $\epsilon_i^{(b)} = 0$ (by (36)) and $\epsilon_j^{(a)} = \epsilon_j^{(b)} = 0$ for $j > t$.

Case 1. For some s, $\epsilon_s^{(a)} > \epsilon_s^{(b)} = 0$, and $\epsilon_i^{(a)} = \epsilon_i^{(b)} = 0$ for all $i < s$.

In this case it follows that $y(a) > y(b)$ and (35) is equivalent to

(37) $\quad (a-b)(y(a)-y(b)) \geqslant 1.$

Observe that if

$$(a'-b')(y(a')-y(b')) \geqslant 1$$

for some a', b' with $0 < a' \leqslant a$, $y(a') \leqslant y(a)$, and $b' \geqslant b$, $y(b') \geqslant y(b)$ then (37) also holds. Thus, we can normalize a and b as follows:

(a) Take $\epsilon_t^{(a)} = \epsilon_1^{(a)} = 1$ and all other $\epsilon_i^{(a)} = 0$.

(b) Take $\bar{\epsilon}^{(b)} = (\epsilon_t^{(b)}, \epsilon_{t-1}^{(b)}, \ldots, \epsilon_1^{(b)})$ to have the form

$$\bar{\epsilon}^{(b)} = (\overset{t}{0}, 1, 1, \ldots, \overset{t_r}{2}, \ldots, 1, 1, \overset{s_r}{0}, 1, 1, \ldots, \overset{t_{r-1}}{2}, 1, 1, \ldots, \overset{s_{r-1}}{0}, \ldots$$

$$\ldots, \overset{s_2}{0}, 1, 1, 1, \overset{t_1}{2}, 1, 1, 1, \ldots, 1, \overset{s_1}{0})$$

where $s = s_1 < t_1 < s_2 < t_2 < \ldots < s_r < t_r < t$. (That is, $\bar{\epsilon}^{(b)}$ consists of blocks of 1's separated alternatively by 0's and 2's.) We obtain this by noting that if $\epsilon_i^{(b)} = \epsilon_{i+1}^{(b)} = 0$ then we can replace $\bar{\epsilon}^{(b)}$ by $\bar{\epsilon}^{(b')}$, formed by changing $\epsilon_i^{(b)}$ to 1. Also, if a block of 1's in $\bar{\epsilon}^{(b)}$ is bounded at each end by 0's then we can change one of the 1's to a 2, thereby increasing both b and $y(b)$.

Thus, we can assume

$$a = F_{2t} + F_{2s_1},$$

$$b = (F_{2t-2} + \ldots + F_{2s_1+2}) + F_{2t_r} - F_{2s_r} + \ldots$$
$$\ldots + F_{2t_2} - F_{2s_2} + F_{2t_1} =$$

$$= F_{2t-1} - F_{2s_1+1} + (F_{2t_r} - F_{2s_r}) + \ldots$$
$$\ldots + (F_{2t_2} - F_{2s_2}) + F_{2t_1}.$$

Therefore,

$$y(a) = \frac{1}{F_{2t}} + \frac{1}{F_{2s_1}}$$

and

$$y(b) = \sum_{k=s_1+1}^{t-1} \frac{1}{F_{2k}} + \left(\frac{1}{F_{2t_r}} - \frac{1}{F_{2s_r}}\right) + \ldots$$

$$\ldots + \left(\frac{1}{F_{2t_2}} - \frac{1}{F_{2s_2}}\right) + \frac{1}{F_{2t_1}} \leqslant$$

$$\leqslant \left(\frac{1}{F_{2s_1}} - \frac{1}{F_{2s_2+2}} - \frac{1}{F_{2t-2}} + \frac{1}{F_{2t}}\right) + \left(\frac{1}{F_{2t_r}} - \frac{1}{F_{2s_r}}\right) + \ldots$$

$$\ldots + \left(\frac{1}{F_{2t_2}} - \frac{1}{F_{2s_2}}\right) + \frac{1}{F_{2t_1}}$$

by Lemma 2. Hence,

$$(a-b)(y(a)-y(b)) \geqslant$$

$$\geqslant (F_{2t} + F_{2s_1} - F_{2t-1} + F_{2s_1+1} - F_{2t_r} + F_{2s_r} - \ldots - F_{2t_1}) \times$$

$$\times \left(\frac{1}{F_{2t}} + \frac{1}{F_{2s_1}} - \frac{1}{F_{2s_1}} + \frac{1}{F_{2s_1+2}} + \frac{1}{F_{2t-2}} -\right.$$

$$\left. - \frac{1}{F_{2t}} - \frac{1}{F_{2t_r}} - \ldots - \frac{1}{F_{2t_1}}\right) =$$

$$= (F_{2t-2} - F_{2t_r} + F_{2s_r} - \ldots - F_{2t_1} + F_{2s_1+2}) \times$$

$$\times \left(\frac{1}{F_{2t-2}} - \frac{1}{F_{2t_r}} + \frac{1}{F_{2s_r}} - \ldots - \frac{1}{F_{2t_1}} + \frac{1}{F_{2s_1+2}}\right) \geqslant 1$$

by Lemma 3, since

$$t > t_r > s_r > \ldots > t_1 > s_1 > 0.$$

This completes the proof of (35) for Case 1.

Case 2. For some s, $\epsilon_s^{(b)} > \epsilon_s^{(a)} = 0$, and $\epsilon_i^{(a)} = \epsilon_i^{(b)} = 0$ for all $i < s$. Thus, $y(b) > y(a)$ and (35) is equivalent to

(38) $\qquad (a-b)(y(b)-y(a)) \geqslant 1.$

In this case, we can make the following normalizations:

(I) In $\bar{\epsilon}^{(a)}$, replace $\ldots, 1, 0, \ldots$ by $\ldots, 0, 1, \ldots,$

replace $\ldots, 2, 0, \ldots$ by $\ldots, 0, 1, \ldots,$

and replace $\ldots, 2, 1, \ldots$ by $\ldots, 1, 2, \ldots$.

(II) In $\bar{\epsilon}^{(b)}$, replace, $\ldots, 0, 1, \ldots$ by $\ldots, 1, 0, \ldots,$

replace, $\ldots, 0, 2, \ldots$ by $\ldots, 1, 0, \ldots,$

and replace, $\ldots, 1, 2, \ldots$ by $\ldots, 2, 1, \ldots$.

After these normalizations, $\bar{\epsilon}^{(a)}$ and $\bar{\epsilon}^{(b)}$ have the form:

$$\bar{\epsilon}^{(a)} = (\epsilon_t^{(a)}, 0, 0, \ldots, 0, 0, \overset{u}{1}, 1, 1, \ldots, 1, \epsilon_{s+1}^{(a)}, 0),$$

$$\bar{\epsilon}^{(b)} = (0, \epsilon_{t-1}^{(b)}, 1, \ldots, \overset{v}{1}, 0, 0, 0, \ldots, 0, 0, \epsilon_s^{(b)})$$

where $\epsilon_t^{(a)}, \epsilon_{s+1}^{(a)}, \epsilon_{t-1}^{(b)}, \epsilon_s^{(b)} > 0$, and $\epsilon_{u+1}^{(a)} = 0$, $\epsilon_u^{(a)} = 1$, $\epsilon_v^{(b)} = 1$, $\epsilon_{v-1}^{(b)} = 0$. Then

$$a - b \geqslant F_{2t} - 2F_{2t-2} - F_{2t-4} - \ldots - F_{2v} \geqslant$$

$$\geqslant F_{2t} - F_{2t-1} + F_{2v-1} - F_{2t-2} = F_{2v-1}.$$

Also,

$$y(b) - y(a) \geqslant \frac{1}{F_{2s}} - \frac{2}{F_{2s+2}} - \frac{1}{F_{2s+4}} - \ldots - \frac{1}{F_{2u}} \geqslant$$

$$\geqslant \frac{1}{F_{2u+1}} \geqslant \frac{1}{F_{2v-1}}$$

by Lemma 2 and (i) since $v > u$. Therefore

$$(a - b)(y(b) - y(a)) \geqslant 1$$

and (35) holds in this case.

(ii)

$$\bar{\epsilon}^{(a)} = (\epsilon_t^{(a)}, 1, 1, \ldots, 1, \epsilon_{s+1}^{(a)}, 0),$$

$$\bar{\epsilon}^{(b)} = (0, 0, 0, \ldots, 0, 0, \epsilon_s^{(b)})$$

where $\epsilon_t^{(a)}, \epsilon_{s+1}^{(a)}, \epsilon_s^{(b)} > 0$ and $\bar{\epsilon}^{(a)}$ does not have two 2's in it.

Subcase 1. $t = s + 1$.

Then

$$\bar{\epsilon}^{(a)} = (\epsilon_{s+1}^{(a)}, 0),$$

$$\bar{\epsilon}^{(b)} = (0, \epsilon_s^{(b)})$$

and

$$(a-b)(y(b)-y(a)) = \tag{39}$$

$$= (\epsilon_{s+1}^{(a)} F_{2s+2} - \epsilon_s^{(b)} F_{2s}) \left(\frac{\epsilon_s^{(b)}}{F_{2s}} - \frac{\epsilon_{s+1}^{(a)}}{F_{2s+2}}\right).$$

For $\epsilon_{s+1}^{(a)} = \epsilon_s^{(b)} = \epsilon$, the right-hand side of (39) is

$$\epsilon^2 (F_{2s+2} - F_{2s}) \left(\frac{1}{F_{2s}} - \frac{1}{F_{2s+2}}\right) = \frac{\epsilon^2 F_{2s+1}^2}{F_{2s} F_{2s+2}} > 1$$

for $\epsilon \geqslant 1$.

For $\epsilon_{s+1}^{(a)} = 2$, $\epsilon_s^{(b)} = 1$, the right-hand side of (39) is

$$(2F_{2s+2} - F_{2s}) \left(\frac{1}{F_{2s}} - \frac{2}{F_{2s+2}}\right) = \frac{F_{2s-1} F_{2s+3}}{F_{2s} F_{2s+2}} > 1.$$

For $\epsilon_{s+1}^{(a)} = 1$, $\epsilon_s^{(b)} = 2$, the right-hand side of (39) is

$$(F_{2s+2} - 2F_{2s}) \left(\frac{2}{F_{2s}} - \frac{1}{F_{2s+2}}\right) = \frac{F_{2s-1} F_{2s+3}}{F_{2s} F_{2s+2}} > 1.$$

Subcase 2. $t \geqslant s + 2$.

Then

$$a - b \geqslant F_{2t+1} - F_{2s+1} + (\epsilon_t^{(a)} - 1) F_{2t} +$$

$$+ (\epsilon_{s+1}^{(a)} - 1) F_{2s+2} - (\epsilon_s^{(a)} - 1) F_{2s} =$$

$$= F_{2t+2} - (2 - \epsilon_t^{(a)}) F_{2t} - (2 - \epsilon_{s+1}^{(a)}) F_{2s+2} - (\epsilon_s^{(b)} - 1) F_{2s}$$

and

$$y(b) - y(a) \geqslant$$

$$\geqslant \frac{\epsilon_s^{(b)}}{F_{2s}} - \left(\frac{1}{F_{2s+2}} + \ldots + \frac{1}{F_{2t}}\right) - \frac{\epsilon_t^{(a)} - 1}{F_{2t}} - \frac{\epsilon_{s+1}^{(a)} - 1}{F_{2s+2}} \geqslant$$

$$\geqslant \frac{\epsilon_s^{(b)} - 1}{F_{2s}} + \frac{1}{F_{2s+2}} + \frac{1}{F_{2t}} - \frac{1}{F_{2t+2}} -$$

$$- \frac{\epsilon_t^{(a)} - 1}{F_{2t}} - \frac{\epsilon_{s+1}^{(a)} - 1}{F_{2s+2}} =$$

$$= \frac{\epsilon_s^{(b)} - 1}{F_{2s}} + \frac{2 - \epsilon_{s+1}^{(a)}}{F_{2s+2}} + \frac{2 - \epsilon_t^{(a)}}{F_{2t}} - \frac{1}{F_{2t+2}}$$

by Lemma 2. Therefore,

$$a - b \geqslant F_{2t+2} - F_{2t} - F_{2s+2} - F_{2s} \geqslant F_{2t+1} - F_{2s+2} - F_{2s} \geqslant$$

$$\geqslant F_{2s+5} - F_{2s+2} - F_{2s} = F_{2s+4} + F_{2s-1} > F_{2s+4}.$$

If $\epsilon_s^{(b)} > 1$ or $\epsilon_{s+1}^{(a)} < 2$ then

$$(a - b)(y(b) - y(a)) >$$

$$> F_{2s+4}\left(\frac{1}{F_{2s+2}} - \frac{1}{F_{2t+2}}\right) \geqslant F_{2s+4}\left(\frac{1}{F_{2s+2}} - \frac{1}{F_{2s+6}}\right) =$$

$$= \frac{F_{2s+4}(F_{2s+6} - F_{2s+2})}{F_{2s+2}F_{2s+6}} \geqslant \frac{F_{2s+4}^2}{F_{2s+2}F_{2s+6}} > 1.$$

On the other hand, if $\epsilon_s^{(b)} = 1$ and $\epsilon_{s+1}^{(a)} = 2$ then

$$(a - b)(y(b) - y(a)) \geqslant$$

$$\geqslant (F_{2t+2} - F_{2t})\left(\frac{1}{F_{2t}} - \frac{1}{F_{2t+2}}\right) = \frac{F_{2t+1}^2}{F_{2t}F_{2t+2}} > 1.$$

This completes the proof of (35) for case (ii).

(iii) $t > v = u = s$.

Then

$$\bar{\epsilon}^{(a)} = (\epsilon_t^{(a)}, 0, 0, \ldots, 0, 0),$$

$$\bar{\epsilon}^{(b)} = (0, \epsilon_{t-1}^{(b)}, 1, 1, 1, \epsilon_s^{(b)})$$

where $\epsilon_t^{(a)}, \epsilon_{t-1}^{(b)}, \epsilon_s^{(b)} > 0$ and $\bar{\epsilon}^{(b)}$ does not have two 2's in it.

Subcase 1. $t = s + 1$.

Then

$$\bar{\epsilon}^{(a)} = (\epsilon_{s+1}^{(a)}, 0),$$

$$\bar{\epsilon}^{(b)} = (0, \epsilon_s^{(b)}).$$

However, this case has already been considered in Subcase 1 of (i).

Subcase 2. $t \geqslant s + 2$.

Then

$$\begin{aligned} a - b \geqslant {} & \epsilon_t^{(a)} F_{2t} - \\ & - (F_{2t-1} - F_{2s-1} + (\epsilon_{t-1}^{(b)} - 1)F_{2t-2} + (\epsilon_s^{(b)} - 1)F_{2s}) = \\ & = (\epsilon_t^{(a)} - 1)F_{2t} + F_{2t-2} + F_{2s-1} - \\ & - (\epsilon_{t-1}^{(b)} - 1)F_{2t-2} - (\epsilon_s^{(b)} - 1)F_{2s} = \\ & = (\epsilon_t^{(a)} - 1)F_{2t} + (2 - \epsilon_{t-1}^{(b)})F_{2t-2} - (\epsilon_s^{(b)} - 1)F_{2s} + F_{2s-1} \end{aligned}$$

and

$$y(b) - y(a) \geqslant \frac{\epsilon_s^{(b)}}{F_{2s}} + \frac{1}{F_{2s+2}} + \ldots + \frac{\epsilon_{t-1}^{(b)}}{F_{2t-2}} - \frac{\epsilon_t^{(a)}}{F_{2t}} \geqslant$$

$$\geqslant \frac{1}{F_{2s-1}} + \frac{\epsilon_s^{(b)} - 1}{F_{2s}} - \frac{1}{F_{2t-2}} + \frac{1}{F_{2t}} + \frac{\epsilon_{t-1}^{(b)} - 1}{F_{2t-2}} - \frac{\epsilon_t^{(a)}}{F_{2t}} =$$

$$= \frac{1}{F_{2s-1}} + \frac{\epsilon_s^{(b)} - 1}{F_{2s}} - \frac{2 - \epsilon_{t-1}^{(b)}}{F_{2t-2}} - \frac{\epsilon_t^{(a)} - 1}{F_{2t}}.$$

If $\epsilon_t^{(a)} > 1$ or $\epsilon_{t-1}^{(b)} < 2$ then

$$a - b \geqslant F_{2t-2} - F_{2s} + F_{2s-1} \geqslant F_{2s+4} - F_{2s-2} \geqslant F_{2s+3}$$

and

$$y(b) - y(a) \geqslant \frac{1}{F_{2s-1}} - \frac{1}{F_{2t-2}} - \frac{1}{F_{2t}} \geqslant$$

$$\geqslant \frac{1}{F_{2s-1}} - \frac{1}{F_{2s+2}} - \frac{1}{F_{2s+4}} \geqslant \frac{1}{F_{2s+3}}$$

by (ii) in Lemma 2. Therefore, $(a-b)(y(b)-y(a)) \geqslant 1$ in this case.

On the other hand, if $\epsilon_t^{(a)} = 1$ and $\epsilon_{t-1}^{(b)} = 2$ then $\epsilon_s^{(b)} = 1$ and

$$(a-b)(y(b)-y(a)) \geqslant F_{2s-1} \frac{1}{F_{2s-1}} = 1.$$

This completes the proof of (35) for the case (iii).

In summary, we have shown (35) holds in all possible cases and the proof of the theorem is complete. ∎

The proof of Theorem 2 is now immediate. All that is required is to observe that x_n^* as defined is just αy_n. In fact, (35) actually implies the stronger equality (4), namely,

$$\inf_{n \geqslant 1} \inf_{m \geqslant 0} n\,|x_{m+n}^* - x_m^*| = \alpha$$

CONCLUDING REMARKS

In a certain sense, sequences $\bar{x}$ for which

(40) $$\inf_{n \geqslant 1} \inf_{m \geqslant 0} n\,|x_{m+n} - x_m| = \alpha$$

are ”spread out” as much as possible. However, we do not currently know of any such $\bar{x}$ which is essentially different from $\bar{x}^*$. Do they exist? It is curious that in fact, x_n^* is nowhere dense.

A natural guess for a sequence $\bar{x}$ for which $C(\bar{x})$ is large is one given by $x_n = \{n\theta\}$ for a suitable irrational θ. In fact, such sequences are not too bad, the best one (of course) being given by $x'_n = \{n\tau\}$. In this case

$$C(\bar{x}') = \frac{3 - \sqrt{5}}{2} = 0.381966 \ldots < \alpha = 0.39441967 \ldots .$$

However, the first n terms of $\bar{x}'$ and $\bar{x}^*$ are always order isomorphic which may help explain the good behavior of $\bar{x}'$. We have not shown that the permutations they induce are the only ones which achieve equality in Theorem 2 (and, in fact, for small n, such permutations are *not* unique up to reflection. For example, $\pi = (1, 3, 2)$ and $\pi' = (3, 1, 2)$ have $u(\pi) = u(\pi') = \frac{3}{2}$).

The connection of our results with diophantine approximation is the following. It is well known (see [5]) that for any $\epsilon > 0$, the inequality

$$\left| \tau - \frac{p}{q} \right| < \frac{1 - \epsilon}{\sqrt{5}\, q^2}$$

can hold for only finitely many q with $(p, q) = 1$. This can be restated as

$$\text{(41)} \qquad q \| q\tau \| \geqslant \frac{1 - \epsilon}{\sqrt{5}}$$

for all sufficiently large q. The reason that this does not contradict Theorem 1 is that (41) is not valid for $q = 1$ since $\| \tau \| = \frac{3 - \sqrt{5}}{2}$ and the value of $C(\bar{x})$ can definitely be affected by what happens infinitely often for $n = 1$. What (41) would imply in this framework is

$$C'(\bar{x}') = \liminf_{n \to \infty} \liminf_{m \to \infty} n \,| x'_{m+n} - x'_m | \geqslant \frac{1}{\sqrt{5}}$$

(with equality, in fact). However, this is clearly not best possible for arbitrary sequences $\bar{x}$ in $[0, 1]$ since $C'(\bar{x})$ can be arbitrarily large. For example, by choosing

$$\bar{X}^{(t)} = (\underbrace{x_0^*, x_0^*, \ldots, x_0^*}_{t}, \underbrace{x_1^*, x_1^*, \ldots, x_1^*}_{t}, \ldots)$$

we find

$$C'(\bar{X}^{(t)}) \geqslant t\alpha.$$

In this case of two (or more) dimensions, problems of this type are well known to offer substantial difficulties. Even in the case of two-dimensional diophantine approximation (with the sup norm), the value of

$$\sup_{\theta,\varphi} \liminf_{q} q^{\frac{1}{2}} \max \{\| q\theta \|, \| q\varphi \|\}$$

is not known. Is is known (see [1]) to be at least $\sqrt{\frac{2}{7}}$. This implies that the analogue $C_2(\bar{x})$ of $C(\bar{x})$ for a sequence $\bar{x} \in [0,1] \times [0,1]$ with the sup norm d, namely,

$$C_2(\bar{x}) = \inf_{n \geqslant 1} \liminf_{m \geqslant 0} n^{\frac{1}{2}} d(x_{m+n}, x_m),$$

can remain above $\sqrt{\frac{2}{7}} - \epsilon$ for any $\epsilon > 0$, for a suitable sequence $\bar{x}$. This is probably not the best possible value, however. It would be very interesting to know just what the truth is in this case, as well as in higher dimensions.

REFERENCES

[1] J.W.S. Cassels, Simultaneous diophantine approximation, *J. London Math. Soc.*, 30 (1955), 119–121.

[2] N.G. de Bruijn – P. Erdős, Sequences of points on a circle, *Indag. Math.*, 11 (1949), 46–49.

[3] P. Erdős – R.L. Graham, *Old and new problems and results in combinatorial number theory,* Monographie No. 28 de l'Enseigment Math., Univ. Genève, Geneva, 1980.

[4] L. Kuipers – H. Niederreiter, *Uniform distribution of sequences,* John Wiley and Sons, New York, 1974.

[5] I. Niven, *Diophantine approximations,* John Wiley and Sons, New York, 1963.

[6] A. Ostrowski, Eine Verschärfung des Schubfächerprinzips in einem linearen Intervall, *Arch. Math.*, 8 (1957), 1–10.

[7] A. Ostrowski, Bemerkungen zu meiner Mitteilung: Eine Verschärfung des Schubfächerprinzips in einem linearen Intervall, *Arch. Math.*, 8 (1957), 330.

[8] A. Schönhage, Zum Schubfächerprinzip im linearen Intervall, *Arch. Math.*, 8 (1957), 327–329.

[9] G.H. Toulmin, Subdivision of an interval by a sequence of points, *Arch. Math.*, 8 (1957), 158–161.

F.R.K. Chung – R.L. Graham

Bell Laboratories, Murray Hill, New Jersey, USA.