RECENT RESULTS IN GRAPH DECOMPOSITIONS

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INTRODUCTION

The subject of graph decompositions is a vast and sprawling topic, one which we certainly cannot begin to cover in a paper of this length. Indeed, recently a number of survey articles and several books have appeared, each devoted to a particular subtopic within this domain (e.g., see [Fi-Wi], [Gr-Rot-Sp], [So 1], [Do-Ro]).

What we will attempt to do in this report is twofold. First, we will try to give a brief overall view of the landscape, mentioning various points of interest (to us) along the way. When possible, we will provide the reader with references in which much more detailed discussions can be found. Second, we will focus more closely on a few specific topics and results, usually for which significant progress has been made within the past few years. We will also list throughout various problems, questions and conjectures which we feel are interesting and/or contribute to a clearer understanding of some of the current obstacles remaining in the subject.

Notation

By a graph[(1)] G we will mean a (finite) set V = V(G), called the ***vertices*** of G together with a set E = E(G) of (unordered) pairs of vertices of G, called the ***edges*** of G.

Let $\mathcal{H}$ denote a family of graphs. By an ***$\mathcal{H}$-decomposition*** of G we mean a partition of E(G) into disjoint sets $E(H_i)$ such that each of the graphs H_i induced by the edge set $E(H_i)$ is isomorphic to a graph in $\mathcal{H}$. Ordinarily we will just say that G has been decomposed into $H_i \in \mathcal{H}$. If an $\mathcal{H}$-decomposition of G exists, we denote this by writing $G \in \langle\mathcal{H}\rangle$.

By far the most work in graph decompositions has been carried out on the general problem of determining for fixed families $\mathcal{H}$ (usually singletons), necessary and sufficient conditions that $G \in \langle\mathcal{H}\rangle$. We begin by discussing several examples of this type.

Complete Graphs

For a fixed k, let $\mathcal{H}$ consist of the single graph K_k, the complete graph on k vertices. If $K_v^{(\lambda)}$ denotes the complete ***multigraph*** of multiplicity $\lambda \geq 1$ on a set of v vertices, i.e., each pair of vertices occurs as an edge exactly λ times, then the determination of necessary and sufficient conditions for

$$K_v^{(\lambda)} \in \langle\{K_k\}\rangle \tag{1}$$

to hold is among the oldest problems in combinatories. Such a decomposition is easily seen to be equivalent to the existence of a (v,k,λ)-configuration, a combinatorial structure consisting of a family of k-element subsets B of a v-element set V in which every 2-element subset of V occurs in exactly λ of the B's (see [Hal] or [Ry]). It is not difficult to show that necessary conditions for (1) to hold are:

$$\lambda(v-1) \equiv 0 \pmod{k-1}; \tag{i}$$

$$\lambda v(v-1) \equiv 0 \pmod{k(k-1)}. \tag{ii}$$

For $k = 3$ and $\lambda = 1$ such configurations are known as Steiner triple systems. It was shown by Kirkman in 1847 that in this case (i) and (ii) are also sufficient for (1) (see e.g., [Ra-Wi 1] or the extensive bibliography in [Do-Ro]).

On the other hand, for the values $v = n^2 + n + 1$, $k = n+1$, $\lambda = 1$, (1) holds iff there exists a projective plane of order n (PP(n)) (see [Ry]). In this case conditions (i) and (ii) are not sufficient since, for example, if such a plane exists then n must satisfy the celebrated condition of Bruck and Ryser [Bru-Ry], namely, $n \equiv 0$ or $1 \pmod 4$ and $n = x^2 + y^2$ for integers x and y. The first n for which the existence of PP(n) is undecided is $n = 10$. It has recently been shown that if PP(10) exists it must have very little symmetry (see [An-Hal]).

The strongest general result known for (1) is the theorem of Wilson [Wi 3].

Theorem. For fixed k and λ, (1) holds if (i) and (ii) hold, provided v is sufficiently large.

Thus, conditions (i) and (ii) are asymptotically sufficient.

More generally, let $I \subseteq \mathbb{Z}^+$, the positive integers, and consider the decomposition

$$K_v^{(\lambda)} \in \langle\{K_i : i \in I\}\rangle. \tag{2}$$

In this case, Wilson [Wi 4] has proved an asymptotic result analogous to the preceding theorem. Let

$\alpha(I)$ denote g.c.d. $\{i-1 : i \in I\}$,

$\beta(I)$ denote g.c.d. $\{i(i-1) : i \in I\}$.

Theorem (Wilson). For fixed λ and $I \subseteq \mathbb{Z}^+$, if $v \geq v_0(I,\lambda)$ and v satisfies

$$\lambda(v-1) \equiv 0 \pmod{\alpha(I)} \tag{i'}$$

$$\lambda v(v-1) \equiv 0 \pmod{\beta(I)} \tag{ii'}$$

then (2) holds.

As before it is not hard to check that (i') and (ii') are *necessary* conditions for (2).

Typical results in this class of a more precise nature are:

Theorem (Hanani [Han 2]).

$K_v \in \langle\{K_3,K_4,K_6\}\rangle$ iff $v \equiv 0$ or $1 \pmod 3$, $v \geq 3$,

$K_v \in \langle\{K_4,K_5,K_8,K_9,K_{12}\}\rangle$ iff $v \equiv 0$ or $1 \pmod 4$, $v \geq 4$.

Theorem (Wilson [Wi 2]).

$K_v \in \langle\{K_3,K_5\}\rangle$ iff $v \equiv 1 \pmod 2$, $v \geq 3$.

$K_v \in \langle\{K_4,K_7,K_{10},K_{19}\}\rangle$ iff $v \equiv 1 \pmod 3$, $v \geq 4$.

Theorem (Brouwer [Br 3]).

$K_v \in \langle\{K_4,K_7\}\rangle$ iff $v \equiv 1 \pmod 3$, $v \geq 4$, $v \neq 10, 19$

$K_v \in \langle\{K_3,K_4,K_5,K_6,K_8\}\rangle$ iff $v \geq 3$.

For a summary of many results of this type the reader is referred to the recent doctoral dissertation of Sotteau [So 1].

If H consists of the set of *all* complete graphs then of course $G \in \langle H\rangle$ for all G. In this case, however, it is of interest to know how many factors are required in the decomposition. In this case, Erdős, Goodman and Posa [Er-Go-Po] have shown that any graph on n vertices can be decomposed into at most $\lfloor n^2/4\rfloor$ edge-disjoint complete graphs. In fact, they showed the same bound applies if one takes $H = \{K_2,K_3\}$ (and that this bound can be achieved).

If instead of minimizing the number of factors in a decomposition

$$E(G) = \sum_i E(K_{n_i}) \qquad (*)$$

we instead ask for the minimum value of

$$\sum_i v(K_{n_i}) = \sum_i n_i$$

then Chung [Ch 4] has shown that for any graph on n vertices, there is a decomposition (*) with

$$\sum_i n_i \leq \lfloor n^2/2 \rfloor,$$

settling an earlier conjecture of Katona and Tarján. Furthermore, the only graph for which the bound is achieved is $K_{\lfloor \frac{n}{2}\rfloor, \lceil \frac{n}{2}\rceil}$.

CYCLES AND PATHS

An extensive literature exists on decompositions of (complete) graphs into a fixed cycle C_k. Necessary conditions for

$$G \in \langle\{C_k\}\rangle \qquad (3)$$

are:

$$n - 1 \text{ is even, } k \leq n; \qquad \text{(iii)}$$

$$n(n-1) \equiv 0 \pmod{2k} \qquad \text{(iv)}$$

It is an old conjecture that these conditions are *sufficient* for (3) to hold but this is not yet known. Cases for which (iii) and (iv) *are* sufficient include the following:

(a) $n = k$, [Ber]
(b) $k \equiv 0 \pmod 4$, [Ko 1]
(c) $k \equiv 2 \pmod 4$, [Ro 1]
(d) $n \equiv 0 \pmod k$, [J 3]
(e) $n - 1 \equiv 0 \pmod{2k}$, [J 4]
(f) $k = 2p^{\alpha}$, p prime, [Al-Va].

The reader should consult [So 1] and [Ga] for a more complete discussion of known results.

In the case of triangles C_3 ($=K_3$), Nash-Williams has raised the following conjecture:

Conjecture [1]. If all vertices of G have even degrees at least $\frac{3}{4}$ v(G) then $G \in \langle\{C_3\}\rangle$.

A variation of cycle (and complete graph) decompositions which has received some attention is that of a *resolvable* decomposition. In this case it is required that it be possible to partition the edge-disjoint cycles (partitioning E(G)) into classes, with the cycles in each class forming a partition of V(G). For example, the celebrated solution by Ray-Chaudhuri and Wilson [Ra-Wi 1] of the Kirkman schoolgirl problem shows that K_{6r+3} always has such a decomposition into C_3's. This general problem often is referred to as the "Oberwolfach" problem [Gu]. Partial results can be found in [He-Ko-Ro], [Hu-Ko-Ro] and [He-Ro].

Similar but less complete results are available for the case $K_n^{(\lambda)} \in \langle\{C_k\}\rangle$, $\lambda > 1$ (the reader should consult [So 1] for a summary). Of course, $k = 3$ is the previously mentioned case of Steiner triple systems.

Partial results for decompositions of complete multipartite graphs $K_{n,n,\ldots,n}$ into cycles are available in [So 1], [Co-Har], [So 3]. It is known for example, that

$$K_{r,s} \in \langle\{C_{2t}\}\rangle$$

iff $r \equiv s \equiv 0 \pmod 2$, $r \geq t$, $s \geq t$ and $rs \equiv 0 \pmod{2t}$.

Relatively little is known for the case that $\mathcal{H} = \langle\{C_i : i \in I\}\rangle$ for a subset $I \subseteq \mathbb{Z}^+$. The strongest conjecture as to what may be true is the following:

Conjecture (Alspach [Al]): Suppose n is odd and $m_i \geq 3$, $1 \leq i \leq r$, are integers satisfying

$$2 \sum_{i=1}^{r} m_i = n - 1.$$

Then

$$E(K_n) = \sum_{i=1}^{r} E(C_{m_i}).$$

If $\mathcal{H} = \{C_i : i \geq 3\}$ consists of *all* cycles then it is easy to see that $G \in \langle\mathcal{H}\rangle$ iff all deg(v), $v \in V(G)$, are even. In this case, however, there are a number of interesting conjectures concerning ***minimal*** cycle decompositions.

Conjecture (Hajos (see *[Lo]*)). Every graph G on n vertices with all degrees even can be decomposed into at most $\left\lfloor \frac{n}{2} \right\rfloor$ edge-disjoint cycles.

If true the bound of $\left\lfloor \frac{n}{2} \right\rfloor$ would be best possible because of $_n$.
The best result known in this direction is the theorem of Lovász *[Lo]*:

Theorem. If v(G) = n then G can be decomposed into at most $\left\lfloor \frac{n}{2} \right\rfloor$ edge-disjoint paths and cycles.

This is also the strongest partial result known towards the following beautiful question of Gallai:

Conjecture: If v(G) = n then G can be decomposed in at most $\left\lfloor \frac{n}{2} \right\rfloor$ paths.

Suppose $\ell(G)$ denotes the minimum number of ***linear forests*** (= union of paths) into which a graph G can be decomposed.

Conjecture (*[Pe]* and *[Ak-Ex-Har]*).

$$\ell(G) \leq \left\lceil \frac{1}{2} (\Delta(G)+1) \right\rceil$$

where $\Delta(G)$ denotes the maximum degree in G.

In *[Pe]* it is shown that

$$\left\lceil \frac{1}{2}\Delta(G) \right\rceil \leq \ell(G) \leq \lceil (2\Delta(G)+1)/3 \rceil$$

so that if valid, the bound in the conjecture would be close to best possible.

We mention in passing that the special case of $\ell(G)$ in which each of the paths is required to be a single edge has been intensively studied. In this case $\ell(G)$ is usually denoted by $\chi'(G)$ and called the ***chromatic index*** of G. It is just the minimum number of colors needed for coloring the edges of G so that neighboring edges have distinct colors. The reader is referred to the excellent monograph of Fiorini and Wilson *[Fi-Wi]* which summarizes much of what is currently known about $\chi'(G)$. The well known theorem of Vizing asserts that $\chi'(G) = \Delta(G)$ or $\Delta(G) + 1$ for all graphs G (where $\Delta(G)$ denotes the maximum degree of G). It is known *[Er-Wi]* that almost all graphs G have $\chi'(G) = \Delta(G)$ although Holyer *[Hol]* has shown that the problem of determining whether $\chi'(G) = \Delta(G)$ or $\Delta(G) + 1$ is NP-complete.

Along somewhat different lines, the following old conjecture of S. Lin to the best of the authors' knowledge still remains unsettled.

Conjecture (Lin *[Lin]*). If v(G) = n, e(G) > n and $G \in \langle\{C_n\}\rangle$ then G has at least two distinct representations as $\sum E(C_n)$.

No doubt the number of such representations is bounded below by some increasing function of e(G).

It has been noted by Sloane *[Sl]* (using a result of Tutte *[Tu]*) that the edge disjoint union of any two hamiltonian cycles of a graph always contains a third hamiltonian cycle.

A related conjecture of Nash-Williams is still open:

Conjecture [5]. If $\deg(v) = 2k$ for all $v \in V(G)$ and $v(G) \leq 4k$ then G can be decomposed into at most k hamiltonian cycles.

It has been recently shown by Jackson *[J 1]* that under these hypotheses G always contains at least $\left\lceil \frac{k}{3} \right\rceil$ edge-disjoint hamiltonian cycles.

Kotzig *[Ko 2]* has proposed the following related

Conjecture: If G_i can be decomposed into p_i hamiltonian cycles, $1 \leq i \leq r$, then the cartesian product $\prod_{i=1}^{r} G_i$ can be decomposed into $\sum_{i=1}^{r} p_i$ hamiltonian cycles.

This was shown to be true for $p_1 = p_2 = 1$ by Kotzig *[Ko 2]*, $p_1 = p_2 = p_3 = 1$ by Foregger *[For]* and $r = 2$, $p_2 \leq p_1 \leq 2p_2$, by Aubert and Schneider *[Au-Schn]*.

For further discussions of these and related questions (especially the analogues for directed graphs) the reader is referred to *[J 2]*.

A SINGLE ARBITRARY GRAPH

Suppose $\mathcal{H} = \{H\}$ where we assume (as usual) that H contains no isolated vertex. Let us denote by deg(x) the ***degree*** of a vertex in a graph. Define D(H) by

$$D(H) \equiv \sum_{v \in V(H)} z_v \deg(v) : z_v = 0,1,2,\ldots ,$$

i.e., D(H) is the set of nonnegative integer linear combinations of the deg(v), $v \in V(H)$.

As before the obvious necessary conditions for

$$G \in \langle\{H\}\rangle \qquad (4)$$

can be easily stated:

$$e(G) \equiv 0 \pmod{e(H)}; \qquad (v)$$

For each vertex $x \in V(G)$, $\deg(x) \in D(H)$. (vi)

As in previous cases, most of the work on this problem has been concerned with the choices $G = K_v^{(\lambda)}$, and in particular, $G = K_v$. The strongest asymptotic results are given by the following beautiful theorem of Wilson:

Theorem (Wilson *[Wi 2]*, *[Wi 4]*).

For all $\lambda > 0$ and H there exists (a least) $v(H,\lambda)$ so that if:

(a) $v \geq v(H,\lambda)$,

(b) $\lambda v(v-1) \equiv 0 \pmod{2e(H)}$,

(c) $\lambda(v-1) \equiv 0 \pmod{d}$ where $d = \text{g.c.d. } \{\deg(v) : v \in V(H)\}$,

then

$$K_v^{(\lambda)} \in \langle\{H\}\rangle$$

In other words, for $K_v^{(\lambda)}$ the necessary conditions (v), (vi) are sufficient for v sufficiently large.

A detailed analysis of such decompositions of K_v for each graph H with $v(H) \leq 5$ has been carried out by Bermond, Huang, Rosa and Sotteau

[Berm-Hu-Ro-So], extending earlier work of Bermond and Schonheim [Berm-Sc] who treated all H with $v(H) \leq 4$. In general they found that the exact values of $v(H,1)$ they obtained were always rather small, in particular, much smaller than the bounds implicit in the constructions of Wilson.

A variation which has received some attention recently is to decide whether a graph G occurs nontrivially in $\langle\{H\}\rangle$ for *any* graph H (where nontrivial means $G \neq H$). From an algorithmic point of view, it has been shown by Graham and Robinson [Gr-Rob] that the problem is NP-complete, even if G is a tree and a decomposition into two isomorphic subgraphs is required (see [Gar-Jo] for a discussion of NP-completeness).

For $G = K_v$, it was shown by Harary, Robinson and Wormald [Hara-Rob-Wo 1] that the necessary condition for the decomposition of K_v into t isomorphic edge-disjoint subgraphs is sufficient, namely,

$$v(v-1)/2 \equiv 0 \pmod t.$$

Similar results for other classes of graphs (including directed graphs) can be found in [Hara-Rob-Wo 2], [Hara-Rob-Wo 3], [Wa].

TREES

The following conjecture is usually attributed to Ringel [Ri] (see [Ro 2]).

Conjecture: For any tree T with $e(T) = n$,

$$K_{2n+1} \in \langle\{T\}\rangle.$$

Kotzig strengthened Ringel's conjecture and conjectured that every K_{2n+1} has a *cyclic* decomposition into trees isomorphic to a fixed tree T with $e(T) = n$. This is equivalent to asserting that every tree T is *graceful*, i.e., there exists a 1 - 1 labelling $\lambda: V(T) \to \{0,1,\ldots,e(T)\}$ such that all the values $|\lambda(i)-\lambda(j)|$, $e = \{i,j\} \in E(T)$ are distinct. Although still unresolved, this conjecture has stimulated numerous papers dealing with various special cases. A discussion of much of this work can be found in the survey papers of Bloom [Bl], and Huang, Kotzig and Rosa [Hu-Ko-Ro 2].

An analogous concept, that of a *harmonious* graph, in which each edge $\{i,j\}$ is assigned the value $\lambda(i) + \lambda(j)$ modulo $e(G)$ and all edge values are required to be distinct, has been studied recently. The connection of this concept with coding theory and additive number theory is covered in [Gr-Sl 1] and [Gr-Sl 2]. A particularly stubborn problem is the following.

Problem: Is it true that for an absolute $\varepsilon > 0$, every harmonious graph G with n vertices must have

$$e(G) < \left(\frac{1}{2} - \varepsilon\right)n^2?$$

In other words, if $\{a_1,\ldots,a_n\} \subseteq \mathbb{Z}_e$ has the property that every element $z \in \mathbb{Z}_e$ can be written as $z \equiv a_i + a_j \pmod{e}$ then is it true that $e < \left(\frac{1}{2} - \varepsilon\right)n^2$?

It is known that harmonious graphs with n vertices and $\frac{5}{18}(1+o(1))n^2$ edges exist. Also, it is not hard to show that almost all graphs are neither graceful nor harmonious.

In *[Ch 3]*, Chung considers the problem of decomposing a connected graph G into a minimum number $\tau(G)$ of trees. She shows that at most $\left\lceil\frac{v(G)}{2}\right\rceil$ are ever required and that this bound is achieved, for example, for complete graphs. A related result of Nash-Williams *[Nash 1]* proves that the minimum number of *forests* (i.e., acyclic graphs) a graph G can be decomposed into is exactly

$$\max\left\{\frac{e(H)}{v(H)-1}:\ H \text{ is an induced subgraph of } G\right\}.$$

(Related work occurs in *[Re]*, *[Bei]*, *[Ak-Ha]*).

The consideration of $\tau(G)$ was suggested by results of M. and T. Forreger who considered the related quantity $\tau'(G)$, defined to be the minimum number of subsets into which V(G) can be partitioned so that each subset induces a tree. They show *[For-For]* that

$$\tau'(G) \le \left\lceil\frac{1}{2}\, v(G)\right\rceil.$$

The relationship between $\tau(G)$ and $\tau'(G)$ is not yet completely understood. Examples are known for which

$$\frac{\tau(G)}{\tau'(G)} > \frac{1}{8}\, v(G)$$

and

$$\frac{\tau'(G')}{\tau(G')} > \frac{1}{4}\, v(G').$$

These are probably not the extreme values these ratios can achieve.

Question: What are the extreme values of $\frac{\tau(G)}{\tau'(G)v(G)}$ and $\frac{\tau'(G)}{\tau(G)v(G)}$?

An extended discussion for decompositions of K_n into trees is given by Huang and Rosa in *[Hu-Ro 2]*. In particular they determine which $K_n \in \langle\{T\}\rangle$ for all trees with $e(T) \le 8$.

In this connection, Gyarfas and Lehel have raised the following striking conjecture:

Conjecture [Gy-Le]. If $\{T_i : 1 \le i \le n-1\}$ is an arbitrary set of trees with $e(T_i) = i$ then $E(K_n)$ can be decomposed into $\sum_{i=1}^{n-1} E(T_i)$.

This is known to hold, for example, if all the T_i are either stars or paths *[Za-Lui]*, *[St]*. However, at present it is not even known that the *degree sequences* of the T_i can always be arranged so that the (vector) sum is the degree sequence of K_n, i.e., $(n-1,\ldots,n-1)$.

COMPLETE BIPARTITE GRAPHS

Decompositions of graphs G into complete bipartite subgraphs behave in a somewhat different manner than for other graphs. One reason for this, for example, is the fact that the *spectrum* of G (i.e., the set of eigenvalues of the adjacency matrix of G), strongly limits the minimum number of complete bipartite factors in such a decomposition of G. More precisely, if $n^+(G)$ and $n^-(G)$ denote the numbers of positive and negative eigenvalues, respectively, of A(G) (which always has all eigenvalues real), then

$$E(G) = \sum_{i=1}^{t} E(K_{r_i,s_i})$$

implies

$$t \le \max\{n^+(G), n^-(G)\}$$

(see [Gr-Pol] or [Ho]). No analogous bounds can exist for decompositions into complete graphs.

Very little work has been done for the general decomposition problem $G \in \langle\{K_{r,s}\}\rangle$. The cases in which (r,s) = (2,2) and (2,3) are treated in [Hu] and (r,s) = (2,4) and (3,3) are treated in [Hu-Ro 1] under the additional restriction that the decomposition be ***balanced***, i.e., each vertex of G appears in the same number of factors. This type of restriction on general compositions has been investigated in a number of papers (see [So 1] for a discussion of this work).

An interesting variation of decomposition into complete bipartite graphs has recently been considered by Chung, Erdös and Spencer [Ch-Er-Sp]. Define the function $\alpha(n)$ to be the least integer such that any graph G on n vertices can be decomposed into complete bipartite subgraphs

$$E(G) = \sum_i E(K_{r_i,s_i})$$

with

$$\sum_i v(K_{r_i,s_i}) = \sum_i r_i s_i \le \alpha(n)$$

Theorem

$$\frac{\alpha(n)\log n}{n^2} = O(1).$$

In order to provide the reader with an idea for the type of techniques useful in estimates of this kind (and since the results are not available in the literature) we sketch a proof.

We first show

$$\alpha(n) \ge (1-\varepsilon)\frac{n^2}{2e\log n} \qquad (5)$$

for any $\varepsilon > 0$ and all sufficiently large n. Consider a random[(2)] graph G with n vertices and $\lfloor n^2/2e \rfloor$ edges. The probability that G contains $K_{a,b}$ is bounded above by

$$\binom{n}{a}\binom{n}{b}e^{-ab} < e^{(a+b)\log n - ab}.$$

Let S denote the set of all unordered pairs {a,b} satisfying

$$1 \le a,\ b \le n \text{ and } \frac{a+b}{ab} < \frac{1-\varepsilon}{\log n}$$

The probability that G contains a $K_{a,b}$ with $\frac{a+b}{ab} < \frac{1-\varepsilon}{\log n}$ is bounded above by

$$\sum_{\{a,b\}\varepsilon S}\binom{n}{a}\binom{n}{b}e^{-ab} < \sum_{\{a,b\}\varepsilon S} e^{-\varepsilon ab}$$

$$< \sum_{\{a,b\}\varepsilon S} e^{-\varepsilon \log^2 n}$$

$$< n^2 e^{-\varepsilon \log^2 n} < 1$$

for large n. Thus, there exists a graph G with n vertices an $\lfloor n^2/2e \rfloor$ edges which does not contain any such $K_{a,b}$ as a subgraph.

Let

$$E(G) = \sum_i E(K_{r_i,s_i})$$

be a decomposition of G having $\sum(r_i+s_i)$ minimal. For any edge {u,v} in G, define

$$f(u,v) = \frac{r_i+s_i}{r_is_i}$$

where $\{u,v\} \varepsilon K_{r_i,s_i}$. Then

$$\sum_i(r_i+s_i) = \sum_{\{u,v\}} f(u,v).$$

By hypothesis, any K_{r_i,s_i} occurring in the decomposition has

$$\frac{r_i+s_i}{r_is_i} \ge \frac{1-\varepsilon}{\log n}.$$

Thus

$$f(u,v) \ge \frac{1-\varepsilon}{\log n}$$

for any $\{u,v\} \varepsilon E(G)$ and consequently

$$\alpha(n) \ge \frac{(1-\varepsilon)n^2}{2e \log n}$$

which proves (5).

We next show

$$\alpha(n) < (1+\varepsilon)\frac{n^2}{2 \log n} \qquad (6)$$

for any $\varepsilon > 0$ and all n sufficiently large. We first need the following:

Proposition. For any $\varepsilon > 0$ and $\rho > 0$, any graph G on n vertices and $\rho\binom{n}{2}$ edges contains a subgraph isomorphic to $K_{r,s}$ for some r, s with $r > \varepsilon\rho n$ and $s > (1-\varepsilon)\rho^r n$.

Proof of Proposition: Suppose $v(G) = n$, $e(G) \ge \rho\binom{n}{2}$ and G does not contain $K_{r,s}$ as a subgraph where $r = \lceil \varepsilon\rho n \rceil$ and $s = \lceil (1-\varepsilon)\rho^r n \rceil$. Thus, if we consider the adjacency matrix $A(G) = (a_{ij})$ of G then

$$\sum_{j=1}^{n} \sum_{1\le i_1<\dots<i_r\le n} a_{i_1,j}\dots a_{i_r,j} \le (s-1)\binom{n}{r}. \qquad (6)$$

The left-hand side of (6) is minimized by choosing all n of the sums $\sum_{i=1}^{n} a_{ij}$ as equal possible.

Thus, since

$$\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij} = 2e(G) = \rho n(n-1)$$

then

$$(s-1)\binom{n}{r} \ge \binom{\rho(n-1)}{r} n. \qquad (7)$$

However, this is incompatible with the assigned values of r and s.

Continuing the proof of (6), the proposition guarantees that a graph G_0 on n vertices and $\rho\binom{n}{2}$ edges contains a subgraph H_0 isomorphic to K_{r_0,s_0} where

$$r_0 = \lfloor(1-\varepsilon_0) \log n/\log (1/\rho)\rfloor,$$

$$s_0 = \lfloor(r_0^2/\log (1/\rho)\rfloor.$$

where $\varepsilon_0 > \frac{2 \log \log n}{\log n}$. We will decompose G_0 into complete bipartite subgraphs by a "greedy" algorithm. Given G_0, we find a subgraph H_0 isomorphic to K_{r_0,s_0} and we let G_1 be the subgraph of G_0 with edge set $E(G_1) = E(G_0) - E(H_0)$. Next, we find a subgraph H_1 isomorphic to K_{r_1,s_1} and we let G_2 be the subgraph of G_1 with edge set $E(G_1) - E(H_1)$. We continue in this fashion until at most $\varepsilon_1 n^2/\log n$ edges remain. Therefore,

$$E(G_0) = \sum_{i\ge 0} E(K_{r_i,s_i}) \cup S$$

where S is a set of at most $\varepsilon_1 n^2/\log n$ edges.

We will prove by induction on the number of edges that for given ε_1, $0 < \varepsilon_1 < \varepsilon_0$, and n sufficiently large

$$\sum_{i\ge 0} (r_i+s_i) \le (1+\varepsilon_1) \frac{n^2}{2 \log n} \int_0^{\rho} \log (1/x)dx + 2\varepsilon_1 n^2/\log n \qquad (8)$$

Since

$$\sum_{i\ge 0} (r_i+s_i) = r_0 + s_0 + \sum_{i\ge 1} (r_i+s_i)$$

then by induction

$$\sum_{i\geq 0} (r_i+s_i) + |S| \leq \frac{(1-\varepsilon_1)(\log n)^2}{(\log(1/\rho))^3} + (1+\varepsilon_1) \frac{n^2}{2\log n} \int_0^{\rho'} \log(1/x)dx$$

$$+ 2\varepsilon_1 n^2/\log n$$

where $\rho' = (e(G_0)-r_0s_0)/\binom{n}{2}$ and n is sufficiently large. However, straightforward calculation shows that

$$\frac{(1-\varepsilon_1)(\log n)^2}{(\log(1/\rho))^3} + (1+\varepsilon_1) \frac{n^2}{2\log n} \int_0^{\rho'} \log(1/x)dx$$

$$\leq (1+\varepsilon_1) \frac{n^2}{2\log n} \int_0^{\rho} \log(1/x)dx + 2\varepsilon_1 \frac{n^2}{\log n}$$

Thus,

$$\alpha(n) \leq (1+\varepsilon_1) \frac{n^2}{2\log n} \int_0^{1} \log(1/x)dx + 2\varepsilon_1 \frac{n^2}{\log n}$$

$$\leq (1+\varepsilon) \frac{n^2}{2\log n}$$

for any preassigned $\varepsilon > 0$, provided n is sufficiently large. This proves (6).

The theorem follows by combining (5) and (6). □

We mention in passing the following:

Problem. Find an explicit construction for a graph G on n vertices and cn^2 edges (or even $cn^2/\log n$ edges) which contains no $K_{m,m}$ as a subgraph with $m = c' \log n$.

H-FREE GRAPHS

At the other end of the spectrum, a large number of papers have appeared within the last 10 years which deal with the following question. For a fixed graph H, we say that a graph G is ***H-free*** if G contains no subgraph isomorphic to H. Define $\alpha(G;\overline{H})$ to be the minimum number of factors possible in an H-free decomposition of G.

(BIG) PROBLEM

Determine (or estimate) $\alpha(G;\overline{H})$ for various families of G and H.

When G is a complete graph then $\alpha(G;\overline{H})$ is what might be thought of as an inverse "Ramsey number". In particular, if $r(H;k) = r$ denotes the least integer so that any k-coloring of the edges of K_r always forms a monochromatic subgraph isomorphic to H, then

$\alpha(K_{r(H;k)};\overline{H}) = k + 1$

There are a rather large number of recent survey papers covering this interesting topic, e.g., *[Bu 1]*, *[Bu 2]*, *[Be-Ch-Le]*, *[Par]*, *[Gr]*, *[Bo]*, *[Gr-Rot-Sp]*, *[Ne-Rod]*. Rather than duplicate their contents, we will restrict ourselves to mentioning several of what we consider to be the most attractive open problems in the area.

Question (Erdös). Does $\lim_{n\to\infty} r(K_n;2)^{1/n}$ exist? If so, what is its value? (It is known that it must be between $\sqrt{2}$ and 4 (see *[Gr-Rot-Sp]* or *[Er-Sp]*).

Question *[Er-Gr]*. Is it true that if T_m is a tree with m vertices then for fixed k, $r(T_m;k) = (1+o(1))mk$? It is known *[Er-Gr]* that it lies between $\frac{1}{2}(1+o(1))mk$ and $2(1+o(1))mk$.

Question: Is $\lim_{k\to\infty} r(K_3;k)^{1/k} < \infty$?

It is known *[Ch 2]* that the limit exists and is greater than 3.1 (see *[Ch 1]*).

Define a family $\mathcal{G}$ of graphs to be ***L-set*** if for some absolute constant $c = c(\mathcal{G})$,

$r(G;2) \leq cv(G)$ for all $G \in \mathcal{G}$.

Define the (local) edge density $\rho(G)$ of a graph G by

$$\rho(G) = \max_{H\subseteq G} \frac{e(H)}{v(H)}.$$

(Strong) Conjecture (Erdös). If $\rho(G)$ is bounded for $G \in \mathcal{G}$ then $\mathcal{G}$ is an L-set.

Conjecture (Erdös). If G_m has chromatic number m then $r(G_m;2) \geq r(K_m;2)$.

Question: Is it true that if H is any C_4-free graph then for any k there exists another C_4-free graph G_k so that $\alpha(G_k;H) > k$?

This is known *[Ne-Rod]* to be true for K_m-free graphs and C_{2m+1}-free graphs.

Problem *[Er-Fa-Ro-Sc]*. If $\alpha(G;\overline{P}_n) > 2$ then how small can e(G) be (where P_n denotes the path of length n)?

It is rather embarrassing that at present we can rule out neither $e(G) > cn^2$ nor $e(G) < cn$!

There are many other beautiful problems still open in Ramsey graph theory which unfortunately we must restrain ourselves (because of space limitations) from discussing. Many can be found in *[Ne-Rod]*, *[Gr]*, *[Bu 1]*, *[Bu 2]*.

We close this section with a final problem which has been annoying a number of people for (what seems to us to be) an unreasonable length of time. Let $\mathcal{C}$ denote the set of odd cycles. It is not hard to see that

$$\alpha(K_{2^n};\overline{\mathcal{C}}) = n, \quad \alpha(K_{2^n+1};\overline{\mathcal{C}}) > n.$$

In other words, it is possible to decompose K_{2^n} into n *bipartite* graphs but this is not possible for K_{2^n+1}. Let L(n) denote the least integer such that in every decomposition of K_{2^n+1} into n subgraphs, some subgraph has an odd cycle of length at most L(n).

Question: Does $L(n) \to \infty$ as $n \to \infty$?

SIMULTANEOUS DECOMPOSITION

Given two graphs G and G' with e(G) = e(G'), by a *U-decomposition of G and G'* we mean a pair of partitions $E(G) = \sum_{i=1}^{r} E_i$, $E(G') = \sum_{i=1}^{r} E_i'$, such that as graphs, E_i and E_i' are isomorphic for all i. The function *U(G,G')* is defined to be the minimum value of r for which a U-decomposition of G and G' into r parts exists. U-decompositions always exist when e(G) = e(G') since we can choose all the E_i and E_i' to be single edges.

A number of standard graph invariants can be placed into this framework. For example, if G' consists of e(G) disjoint edges then U(G,G') is just the chromatic index of G (mentioned earlier). When G' is a *star* of degree e(G) then U(G,G') is known as the edge-dominating number of G. Similarly, $\min_{G'} U(G,G')$ has been called the thickness, arboricity or biparticity of G (see *[Har]*, *[Har-Hs-Mi]*) when G' ranges over all planar graphs, acyclic graphs or bipartite graphs, respectively.

Several recent papers have dealt with the quantity

$$U(n) \equiv \max_{G,G'} U(G,G')$$

where v(G) = v(G') = n (and, of course, e(G) = e(G')). The basic result is this.

Theorem [Ch-Er-Gr-Ul-Ya].

$$U(n) = \frac{2}{3} n + o(n).$$

The bound is achieved by (approximately) taking G to be a star of degree n and G' to be $\frac{n}{3}$ disjoint triangles.

A rather surprising phenomenon occurs for the analogous function $U_m(n)$, defined by simultaneously decomposing k graphs $G_1,\ldots,G_k$ into mutually isomorphic subgraphs (so that $U(n) = U_2(N)$). In this case:

Theorem [Ch-Er-Gr 1]. For all $k \geq 3$,

$$U_k(n) = \frac{3}{4} n + o(n)$$

where the o(n) term depends only on k.

It was completely unexpected that the coefficient $\frac{3}{4}$ would be independent of k, for $k \geq 3$.

Three graphs which drive $U(G_1,G_2,G_3)$ up to $\frac{3}{4}$ n are:

G_1 = a star of degree n;

$G_2 = \frac{n}{3}$ K_3's;

$G_3 = \left(\frac{n-\sqrt{n}}{2}\right)$ disjoint edges together with $K_{\sqrt{n}}$.

If the graphs under consideration are restricted to be bipartite then corresponding function $U_k^*(n)$ satisfies:

$U_2^*(n) = \frac{n}{2} + o(n)$,

$U_k^*(n) = \frac{3}{4} n + o(n)$.

L.Babai has raised the following tantalizing question.

Question: Is it true that if for some $\varepsilon > 0$, $v(G) = v(G') = n$ and $e(G) = e(G') > \varepsilon n^2$ then $U(G,G') = o(n)$?

(Weak) supporting evidence for an affirmative answer is that all known examples for achieving $U(G,G') = \frac{3}{4} n + o(n)$ have, in fact, a *linear* number of edges.

The question "Is $U(G,G') = 1$?," known as the graph isomorphism problem, has been actively studied recently from an algorithmic point of view. It is known (see [Lu], [Ba]) that for any fixed bound on $\Delta(G)$, there is a polynomial time algorithm (in $v(G)$) for testing isomorphism to G. The general problem for arbitrary graphs has not been shown to be NP-complete (and, many researchers feel that it is not). However, the related question "Is $U(G,G') = 2$?" has been proved by F. Yao [Ya] to be NP-complete.

For trees T, T', the question "Is $U(T,T') = k$?" has a polynomial time solution for $k = 1$ and is undecided for $k > 1$.

OTHER DIRECTIONS

In this final section we indicate some of the variations on our central theme which can be found in the literature.

Suppose $\mathcal{H}_2$ denotes the class of all graphs of diameter exactly 2. In [Bos-Er-Ro], Bosák, Erdös and Rosa show (among other things) that for any $k > 2$, there is a complete graph $K_{r(k)} \in \langle \mathcal{H} \rangle$ which has a decomposition with exactly k factors (extending earlier work of [Bos-Ro-Zn]). A number of papers (e.g., [Zn 1], [Zn 2], [Pa], [To]) have dealt with questions of a similar type for graphs of diameter d, especially in the case that all the factors are required to be isomorphic (see [Ko-Ro] for a survey of these results).

Many of the problems and/or results described in earlier sections have *directed* analogues. For example, suppose K_n^* denotes the complete symmetric directed graph on n vertices. A necessary condition that $K_n^* \in \langle\{C_k^*\}\rangle$ (where C_k^* denotes a directed k-cycle) is that $n \geq k$ and $n(n-1) \equiv 0 \pmod{k}$.

Conjecture (Bermond): These conditions are sufficient for $K_n^* \in \langle\{C_k^*\}\rangle$ except for $(n,k) = (4,4)$, $(6,3)$ and $(6,6)$.

Sotteau [So 2] has shown, for example, that if $k \geq 5$ is odd, $n \geq k$ and $n \equiv 0$ or $1 \pmod{k}$ then $K_n^* \in \langle\{C_k^*\}\rangle$, settling the above conjecture when k is an odd prime power, or k is odd and n is a prime power. Many references to this and related work can be found in [So 1].

In another direction, one might ask the analogous questions for hypergraphs (and indeed, people have). Typical results range from the difficult area of t-designs (see, e.g., [Ra-Wi 3], [Gr-Li-L], [Wi 1], [Br 1]), the beautiful theorem of Baranyai on complete hypergraph decompositions, [Bar], [Ca], (and more generally, hypergraph designs (see [Br-Sch] for a survey) directed hypergraph decompositions [Ge], and U-decompositions of hypergraphs [Ch-Er-Gr 2], to name a few. An especially stubborn problem of this type (and one for which Erdös is offering US $500) is the following problem.

Problem. (Erdös, Faber, Lovász [Er]). Suppose $\mathcal{F}$ is a family of n n-sets such that for any F, F' $\in \mathcal{F}$, F $\neq$ F', we have $|F \cap F'| \leq 1$. Is it true that it is always possible to partition the underlying set $\bigcup_{F \in \mathcal{F}} F$ of vertices into n classes $C_1, \ldots C_n$ so that $|C_i \cap F| \leq 1$ for all i and all F $\in \mathcal{F}$?

Some partial results can be found in [Hi].

In most of these variations, one might also ask when *resolvable* decompositions are possible, i.e., so that the vertex sets of the factors can be grouped to form partitions of the factored graph. This topic also has a wide literature, some of which can be found in [Han-Ray-Wi], [Han 1], [Ra-Wi 2], [J 2], [Ka], and especially [So 1].

FOOTNOTES

1. Usually we will adopt the graph-theoretic terminology in [Har].
2. For a detailed treatment of the use of the probabilistic method, see [Er-Sp].

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