PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 73, Number 1, January 1979

A GENERAL RAMSEY PRODUCT THEOREM

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ABSTRACT. Call a family $\mathcal{F}$ of subsets of a set U *Ramsey* if no partition of U into finitely many parts can split every $F \in \mathcal{F}$. We show that under very general conditions an arbitrary collection of Ramsey families in fact has a much stronger uniform Ramsey property.

A family $\mathcal{F}$ of finite subsets of a set U is said to be *Ramsey* if for all integers $r < \infty$ and all mappings χ: $U \to \{1, 2, \ldots, r\} \equiv [1, r]$, there exists an $F \in \mathcal{F}$ which is *homogeneous*, i.e., such that for some $i \in [1, r]$ $F \subseteq \chi^{-1}(i)$. Given an arbitrary mapping P: $U \times U \to U$, a family $\mathcal{F}$ is said to be a *P-ideal* of U if

$$F \in \mathcal{F} \Rightarrow P(F, u) \in \mathcal{F}, \qquad P(u, F) \in \mathcal{F},$$

for all $u \in U$, where P is extended to $2^U \times 2^U$ in the usual way, i.e., for X, $Y \subseteq U$,

$$P(X, Y) \equiv \{P(x, y): x \in X, y \in Y\}.$$

The following somewhat unexpected result shows that the Ramsey property holds simultaneously for arbitrary collections of Ramsey families under quite general conditions.

THEOREM. *Let $\{\mathcal{F}_\alpha\}_{\alpha \in A}$ be an arbitrary family of Ramsey P-ideals of U where P: $U \times U \to U$ is arbitrary. Then for any $r < \infty$ and any mapping χ: $U \to [1, r]$, there exists $i \in [1, r]$ and $F_\alpha \in \mathcal{F}_\alpha$, $\alpha \in A$, such that $F_\alpha \subseteq \chi^{-1}(i)$ for all $\alpha \in A$.*

PROOF. We first show by induction that for any integer m and any *finite* subcollection $\mathcal{F}_{\alpha_1}, \ldots, \mathcal{F}_{\alpha_t}$ of $\{\mathcal{F}_\alpha\}_{\alpha \in A}$, there is a *finite* set $F = [\mathcal{F}_{\alpha_1}, \ldots, \mathcal{F}_{\alpha_t}] \subseteq U$ such that for any mapping χ: $F \to [1, m]$, there is an $i \in [1, m]$ and $F_j \in \mathcal{F}_{\alpha_j}$ such that $F_j \subseteq \chi^{-1}(i)$ for $1 \leqslant j \leqslant t$. For $t = 1$, this follows at once from a well-known compactness principle (see [1]). Let $\bar{t} > 1$ be fixed and suppose the assertion holds for all $t < \bar{t}$. Also, the assertion is immediate for $m = 1$. Thus, let $\bar{m} > 1$ be fixed and suppose the assertion also holds for $t = \bar{t}$ and all $m < \bar{m}$. Let $\mathcal{F}_{\alpha_1}, \ldots, \mathcal{F}_{\alpha_{\bar{t}}}$ be an arbitrary fixed subcollection of $\{\mathcal{F}_\alpha\}_{\alpha \in A}$. By induction, the sets

$$X = [\mathcal{F}_{\alpha_1}, \ldots, \mathcal{F}_{\alpha_{\bar{t}-1}}]_{\bar{m}}, \qquad Y = [\mathcal{F}_{\alpha_{\bar{t}}}]_{m^*} \quad \text{where } m^* = \bar{m}^{|X|},$$

Received by the editors November 1, 1977 and, in revised form, April 21, 1978.
AMS (MOS) subject classifications (1970). Primary 05A05, 05A17.
Key words and phrases. Ramsey's Theorem, partitions.
0002-9939/79/0000-0030/$01.75

and

$$F^* = P(X, Y)$$

exist and are finite.

Let χ: $U \to [1, \bar{m}]$ be an arbitrary fixed mapping of U into $[1, \bar{m}]$. Define a new mapping χ^* on Y so that

$$\chi^*(y) = \chi^*(y'), \qquad y, y' \in Y,$$

iff

$$\chi(P(x, y)) = \chi(P(x, y')) \quad \text{for all } x \in X.$$

Since

$$|P(X, y)| \leq |X| \quad \text{for all } y \in Y$$

then we can take χ^* to be a mapping of Y into $[1, m^*]$. By the definition of Y, there exists $F_{\bar{t}} \in \mathcal{F}_{\alpha_{\bar{t}}}$ such that for some $i \in [1, m^*]$, $F_{\bar{t}} \subseteq \chi^{*-1}(i)$. Let $f \in F_{\bar{t}}$.

We now define another mapping χ': $X \to [1, \bar{m}]$ by letting

$$\chi'(x) = \chi(P(x, f)), \qquad x \in X.$$

Note that the value of χ' is actually independent of the choice of f.

By the definition of X, there exists $k \in [1, \bar{m}]$ and $F_j \in \mathcal{F}_{\alpha_j}$ such that $F_j \subseteq \chi'^{-1}(k)$, $1 \leq j \leq \bar{t} - 1$. Therefore,

$$P(F_j, f) \subseteq \chi^{-1}(k), \qquad 1 \leq j \leq \bar{t} - 1,$$

and so

$$P(F_j, F_{\bar{t}}) \subseteq \chi^{-1}(k), \qquad 1 \leq j \leq \bar{t} - 1,$$

since

$$\chi(P(x, f)) = \chi(P(x, f')), \qquad x \in X, \quad f, f' \in F_{\bar{t}}.$$

But

$$P(F_j, f) \in \mathcal{F}_{\alpha_j}, \qquad 1 \leq j \leq \bar{t} - 1,$$

since F_{α_j} is a P-ideal, and $P(x, F_{\bar{t}}) \in \mathcal{F}_{\alpha_{\bar{t}}}$ for the same reason. Since all t of these sets are in $\chi^{-1}(k)$ then we have shown that $P(X, Y)$ can be taken as $[\mathcal{F}_{\alpha_1}, \ldots, \mathcal{F}_{\alpha_{\bar{t}}}]_{\bar{m}}$. This completes the induction step and the first assertion is proved.

Now, suppose the theorem fails. Thus, for some r there is a mapping χ: $U \to [1, r]$ and families $\mathcal{F}_{\beta_i} \in \{\mathcal{F}_\alpha\}_{\alpha \in A}$ such that

$$F_i \not\subseteq \chi^{-1}\{i\} \quad \text{for all } F_i \in \mathcal{F}_{\beta_i}, \qquad 1 \leq i \leq r. \tag{1}$$

By the preceding assertion, the (finite) set

$$[\mathcal{F}_{\beta_1}, \ldots, \mathcal{F}_{\beta_r}]_r \subseteq U$$

exists. Thus, for some $k \in [1, r]$ and $F'_j \in \mathcal{F}_{\beta_j}$,

$$F'_j \subseteq \chi^{-1}(k), \qquad 1 \leq j \leq r.$$

In particular, $F'_k \subseteq \chi^{-1}(k)$ and $F'_k \in \mathcal{F}_{\beta_k}$. This contradicts (1) and the theorem is proved. □

Typical applications of this theorem can produce significant strengthenings of many of the standard Ramsey-type results. For example, an old result of Gallai (see [3]), generalizing the theorem of van der Waerden on arithmetic progressions (see [2], [4]), asserts that for any finite subset C of $\mathbf{E}^n$, in any partition of $\mathbf{E}^n$ into finitely many classes, some class always contains a subset C' which is similar to C. Using the product theorem of this note, taking U to be $\mathbf{E}^n$ and for $\bar{x}, \bar{y} \in \mathbf{E}^n$, defining $P(\bar{x}, \bar{y}) = \bar{x} + \bar{y}$, we see, in fact, that in any partition of $\mathbf{E}^n$ into finitely many classes, one class must contain similar copies of *every* finite subset of $\mathbf{E}^n$.

By taking $U = \mathbf{Z}^+$, the set of positive integers, and $P(x, y) = xy$, we obtain the following classical theorem of Rado [3]. Call a system $\mathfrak{S}$ of homogeneous, linear equations *regular*, if for any partition of $\mathbf{Z}^+$ into finitely many classes, $\mathfrak{S}$ has a solution entirely in one class. (Such systems were completely characterized by Rado.) Then, in fact, by the product theorem, for any partition of $\mathbf{Z}^+$ into finitely many classes, some class contains solutions to *every* regular system of equations.

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