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On Subgraph Number Independence in Trees

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For finite graphs F and G, let $N_F(G)$ denote the number of occurrences of F in G, i.e., the number of subgraphs of G which are isomorphic to F. If $\mathscr{F}$ and $\mathscr{G}$ are families of graphs, it is natural to ask then whether or not the quantities $N_F(G)$, $F \in \mathscr{F}$, are linearly independent when G is restricted to $\mathscr{G}$. For example, if $\mathscr{F} = \{K_1, K_2\}$ (where K_n denotes the complete graph on n vertices) and $\mathscr{F}$ is the family of all (finite) *trees*, then of course $N_{K_1}(T) - N_{K_2}(T) = 1$ for all $T \in \mathscr{F}$. Slightly less trivially, if $\mathscr{F} = \{S_n : n = 1, 2, 3,...\}$ (where S_n denotes the *star* on n edges) and $\mathscr{G}$ again is the family of all trees, then $\sum_{n=1}^{\infty}(-1)^{n+1} N_{S_n}(T) = 1$ for all $T \in \mathscr{G}$. It is proved that such a linear dependence can *never* occur if $\mathscr{F}$ is finite, no $F \in \mathscr{F}$ has an isolated point, and $\mathscr{G}$ contains all trees. This result has important applications in recent work of L. Lovász and one of the authors (Graham and Lovász, to appear).

INTRODUCTION

It is a trivial observation (in fact, almost a definition) that in any finite tree T, the number of vertices of T always exceeds the number of edges of T by exactly 1. In [1], it was asked to what extent this can happen for graphs in general. That is, given a finite family $\mathscr{F}$ of graphs G, when can there be a fixed linear dependence between the number of occurrences of the $G \in \mathscr{F}$ as subgraphs of a tree T which is valid for all finite[1] trees T? In this paper, we answer this question. In particular, this can never happen if none of the $G \in \mathscr{F}$ have isolated points.

[1] All graphs considered in this paper are finite. For terminology, see [3].

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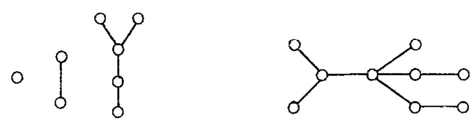
SOME NOTATION

For a graph G, we let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. If H is a *labeled* graph (i.e., with distinguishable vertices) and G is an unlabeled graph, we define $N_G(H)$ to be the number of occurrences of G in H, i.e., the number of ways a subset of $|E(G)|$ edges can be selected from $E(H)$ together with i vertices from $V(H)$ if G has i isolated vertices, so that the resulting subgraph of H is isomorphic to G. Of course, the product of $N_G(H)$ and the order of the automorphism group of G is just $E_G(H)$, the number of ways of embedding G into H (considering G as a labeled graph). For example, if G and H are as shown in Fig. 1 then $N_G(H) = 28$ and $E_G(H) = 112$.



G H

FIGURE 1

Note that if the isolated point is removed from G to form G' then $N_{G'}(H) = 14 = \frac{1}{2}N_G(H)$. Of course, in general, if G is formed from a graph G' having no isolated points by adding i isolated points, then

$$N_G(H) = \binom{|V(H)| - |V(G')|}{i} N_{G'}(H). \tag{1}$$

THE MAIN RESULT

The primary result of this paper can be stated as follows.

THEOREM. *Let $\mathscr{F}$ be a finite family of forests (i.e., acyclic graphs), each having no isolated points, and suppose there exist real numbers A_F, $F \in \mathscr{F}$, and A_0 such that the equation*

$$\sum_{F \in \mathscr{F}} A_F N_F(T) = A_0 \tag{2}$$

is valid for all trees T. Then $A_F = 0$ for all $F \in \mathscr{F}$.

Remark. Since any subgraph of a tree is a forest, there is no loss of generality in assuming $\mathscr{F}$ is a family of forests.

Proof. We may assume without loss of generality that among all families for which an equation of the form (2) is possible, $\mathscr{F}$ has the least number of elements. The basic idea of the proof is to construct a very large tree W^* for which *one* of the quantities $N_F(W^*)$ is much larger than all the others, thereby forcing its coefficient A_F to be 0. However, this contradicts the minimality of $|\mathscr{F}|$.

If T is a tree with a distinguished vertex v, we let $T^{(k)}$ denote the tree formed from T by adjoining k disjoint paths of length k to v (see Fig. 2). Similarly, if F is a forest with components $T_1, \ldots, T_n$ having distinguished vertices $v_1, \ldots, v_n$, respectively, then $F^{(k)}$ denotes the forest with components $T_1^{(k)}, \ldots, T_n^{(k)}$.

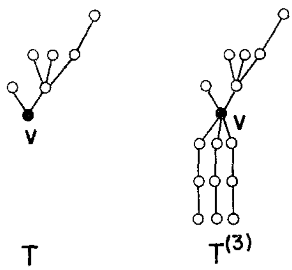


FIGURE 2

We now define a forest $W = W(\mathscr{F})$ with components W_i and distinguished vertices $w_i \in V(W_i)$, $1 \leqslant i \leqslant t$, as follows:

(i) Some $F \in \mathscr{F}$ occurs as a subgraph of $W^{(k)}$ for some k.

(ii) $|E(W)|$ is minimal among all W satisfying (i).

Note that by (ii) every component of $W_i - \{w_i\}$ has a vertex of degree $\geqslant 3$. Define $\mathscr{F}'$ to be the set $\{F \in \mathscr{F} : F \subseteq W^{(k)} \text{ for some } k\}$.

Next, we choose s to be a large fixed integer, depending only on $\mathscr{F}$, to be determined later. For (large) integers n, define n_k by

$$n_k = [n^{1+s^{-k}}], \qquad 1 \leqslant k \leqslant s(s+t).$$

We are finally ready to define the tree $W^* = W^*(n)$ for each sufficiently large n.

1. W^* will have a subset of $2s + t - 1$ vertices, called *special* vertices, denoted by $X = \{x_1, \ldots, x_s\}$, $Y = \{y_1, \ldots, y_{s-1}\}$ and $\{w_1, \ldots, w_t\}$.

2. For $1 \leqslant k \leqslant s$, x_k has n_k paths of length 1 attached to it.

3. For $1 \leqslant k \leqslant s - 1$, y_k has n_{ks+j} paths of length j attached to it for $1 \leqslant j \leqslant s$.

4. For $1 \leqslant k \leqslant t$, w_k has $n_{s(s+k-1)+j}$ paths of length j attached to it for $1 \leqslant j \leqslant s$.

5. Also attached to w_k is a copy of W_k with w_k being the distinguished vertex of W_k.

6. The special vertices are joined sequentially by paths of length s, i.e., between adjacent vertices in the sequence $(x_1, \ldots, x_s, y_1, \ldots, y_{s-1}, w_1, \ldots, w_t)$ are placed paths of length s.

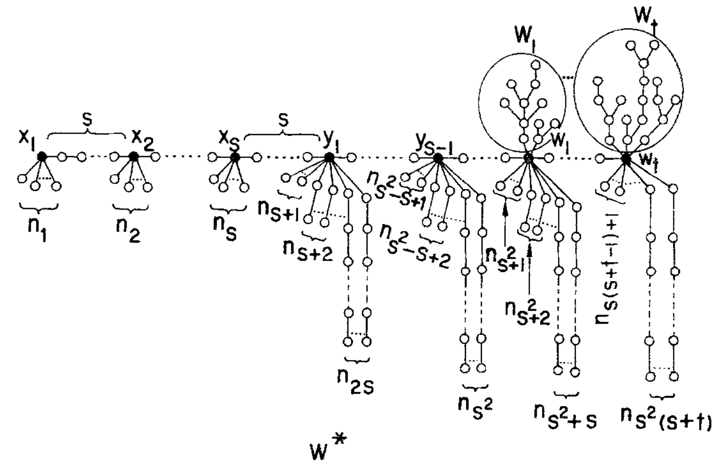


FIGURE 3

This completes the construction of W^*. In Fig. 3 we illustrate the structure of W^*.

By hypothesis, we have

$$\sum_{F \in \mathscr{F}} A_F N_F(W^*(n)) = A_0$$

for all n. However, since by the definition of $\mathscr{F}'$, no $F \in \mathscr{F} - \mathscr{F}'$ occurs as a subgraph of $W^{(k)}$ for any k, it is not difficult to see that $N_F(W^*(n)) = 0$ for these F, provided we have chosen s and n sufficiently large. Hence, we have

$$\sum_{F \in \mathscr{F}'} A_F N_F(W^*(n)) = A_0 \tag{3}$$

for all sufficiently large n. It is important to note that by the minimality assumptions we have made, any embedding of any $F \in \mathscr{F}'$ into W^* must use *all* the edges of *all* the W_i, $1 \leqslant i \leqslant t$, in W^*, again, provided s and n are sufficiently large. We claim it will be sufficient to prove the following result.

FACT. For any distinct $F, F' \in \mathscr{F}'$, either

$$N_F(W^*)/N_{F'}(W^*) > n^{s-s^3}$$

or

$$N_{F'}(W^*)/N_F(W^*) > n^{s^{-s^3}}$$

for n sufficiently large.

Suppose the fact holds. Since we must have $|\mathscr{F}'| > 1$, then there is some element $F^* \in \mathscr{F}'$ such that

$$N_{F^*}(W^*)/N_F(W^*) > n^{s^{-s^3}}$$

for all $F \in \mathscr{F}' - \{F^*\}$. By (3) we have

$$A_{F^*} + \sum_{F \in \mathscr{F}' - \{F^*\}} A_F \left(\frac{N_F(W^*)}{N_{F^*}(W^*)}\right) = \frac{A_0}{N_{F^*}(W^*)}. \tag{4}$$

But as $n \to \infty$, all terms in (4) tend to zero except A_{F^*} which is nonzero by hypothesis. This contradiction would then prove the theorem.

Proof of fact. Let F and F' be two distinct elements of $\mathscr{F}'$. Partition the components of F into three classes: F_1, the set of *stars*, i.e., trees with at most one vertex of degree $\geqslant 2$; F_2, the nonstars which are *starlike*, i.e., nonstar trees with at most one vertex of degree $\geqslant 3$; and F_3, the *nonstarlike* trees, i.e., those having at least two vertices of degree $\geqslant 3$. Define F_1', F_2', and F_3' in an analogous way for F'. As we have noted earlier, F_3 must consist of t trees $T_1, \ldots, T_t$ where T_k is formed from W_k by adjoining a (nonempty!) set of paths to w_k (with a similar remark applying to F_3').

We need one more concept. A *weak attachment* α *of* F *to* W^* is formed as follows. A vertex u_i is selected from each component C_i of F. These u_i are mapped by an injection α into the set of special vertices of W^* with the restrictions that:

$$\alpha(u_i) = \begin{cases} x_j \text{ for some } j \text{ if } C_i \in F_1, \\ y_j \text{ for some } j \text{ if } C_i \in F_2, \\ w_j \text{ for some } j \text{ if } C_i \in F_3. \end{cases}$$

A weak attachment α of F to W^* is said to be *proper* if α can be extended to an embedding of F into W^*. We let $|\alpha|$ denote the number of ways α can be extended to an embedding of F into W^*. Note that in a proper weak attachment α of F to W^*, u_i must be a vertex of C_i of maximal degree if $C_i \in F_1 \cup F_2$ (except if C_i is a path). Define the sequence $\tau(\alpha) = (\tau_1, \tau_2, \ldots, \tau_{s(s+t)})$ as follows:

$$\tau_k = \begin{cases} \text{number of paths of lenth 1 leaving } u_i \text{ for} \\ \quad \alpha(u_i) = x_k,\ 1 \leqslant k \leqslant s, \\ \text{number of paths of length } j \text{ leaving } u_i \text{ for } \alpha(u_i) = y_l, \\ \quad \text{where } k = ls + j \text{ for } 1 \leqslant j \leqslant s,\ 1 \leqslant l \leqslant s-1, \\ \text{number of paths of length } j \text{ leaving } u_i \text{ for } \alpha(u_i) = w_m, \\ \quad \text{where } k = s^2 + (m-1)s + j \text{ for } 1 \leqslant j \leqslant s,\ 1 \leqslant m \leqslant t. \end{cases}$$

It is then clear that

$$| \alpha | < K_0 \prod_{k=1}^{s(s+t)} n_k^{\tau_k}, \tag{5}$$

where K_0, $K_1, \ldots,$ denote constants depending on s and not on n. The sequences $\tau(\alpha)$ can be linearly ordered as follows.

For $\tau(\alpha) = (\tau_1, \tau_2, \ldots, \tau_{s(s+t)})$ and $\tau(\alpha') = (\tau_1', \tau_2', \ldots, \tau'_{s(s+t)})$, we define $\tau(\alpha') > \tau(\alpha)$ if either:

(i) $\sum_{k=1}^{s(s+t)} \tau_k' > \sum_{k=1}^{s(s+t)} \tau_k$; or

(ii) $\sum_{k=1}^{s(s+t)} \tau_k' = \sum_{k=1}^{s(s+t)} \tau_k$ and $\tau(\alpha')$ is lexicographically greater than $\tau(\alpha)$, i.e., for some m, $\tau_k' = \tau_k$ for $1 \leqslant k < m$ and $\tau_m' > \tau_m$.

We let $\tau^{(F)} = (\tau_1^{(F)}, \ldots, \tau_{s(s+t)}^{(F)})$ denote a *maximal* sequence $\tau(\alpha)$ in this ordering as α ranges over all proper weak attachments of F to W^*. The proof of the fact will depend on the following assertion.

Claim. If $\tau^{(F')} > \tau^{(F)}$ then $N_{F'}(W^*)/N_F(W^*) > n^{s^{-s^3}}$ for n sufficiently large.

Proof of claim. Suppose $\tau^{(F')} > \tau^{(F)}$. It is easily seen that

$$N_{F'}(W^*) \geqslant \prod_{k=1}^{s(s+t)} \binom{n_k}{\tau_k^{(F')}} > K_1 \prod_{k=1}^{s(s+t)} n_k^{\tau_k^{(F')}}.$$

On the other hand, it is not hard to show that

$$N_F(W^*) < K_2 \prod_{k=1}^{s(s+t)} n_k^{\tau_k^{(F)}}. \tag{7}$$

To see this, we consider F as a labeled forest and we show that

$$N_F(W^*) < K_3 \prod_{k=1}^{s(s+t)} n_k^{\tau_k^{(F)}}$$

for a suitable constant $K_3 = K_3(s)$.

First, the nonstarlike trees in F_3 can only be embedded into the W_i parts of W^* and, since the total number of proper weak attachments of F_3 to W^* is bounded by a function of s, then the embedding of the nonstarlike trees of $\mathscr{F}'$ contributes a factor of at most $K_4 \prod_{k=s^2+1}^{s(s+t)} n_k^{\tau_k'}$, where $\tau'(\beta) = (\tau'_{s^2+1}, \ldots, \tau'_{s(s+t)})$ is a (maximal) sequence derived from some proper weak attachment β of F_3 to W^*.

Next, consider an embedding of a starlike tree $T \in F_2$ which is not a star. Suppose T is formed by adjoining m_k paths of length k, $1 \leqslant k \leqslant s$, to the "center" vertex u. Although it may be possible to embed T into W^* by mapping u onto some $x_i \in X$ (e.g., when at most two of the m_k, $k \geqslant 2$, are nonzero), when this is done we must use edges in one of the paths of length s connecting x_i to adjacent special vertices of W^*, and so, there are at most $K_5 n_1^{(-1+\Sigma_{k=1}^{s} m_k)}$ such embeddings. However, this factor is negligible compared to the corresponding factor of $K_6 n^{\Sigma_{k=1}^{s} m_k}$ which we obtain if we embed T by mapping u onto some $y_i \in Y$ since

$$\frac{n_{s^2}^{m}}{n_1^{m-1}} > K_7 \frac{n^{m(1+s^{-s^2})}}{n^{m(1+s^{-1})-1}} > K_8 n^{1/2}$$

provided s has been chosen sufficiently large for $\mathscr{F}$ and n is sufficiently large.

Finally, we consider a star $S \in F_1$, say, consisting of m paths of length 1 adjoined to a vertex u. If $m \geqslant 3$, then in any embedding of F into W^*, u must be mapped onto some vertex in $X \cup Y$ since these are the only available vertices of degree $\geqslant 3$. However, since $n_k/n_{k+1} \to \infty$ as $n \to \infty$, the dominant contribution will certainly come from the embeddings which map u onto some $x_i \in X$ (in fact, the smaller the index i, the better). If $m \leqslant 2$, then there are many ways of embedding S into W^*, for example, so that u does not map onto a special vertex of W^*. Again, however, the dominant term clearly comes from those embeddings which take u onto some special vertex $x_i \in X$.

Thus, all except a negligible fraction of the embeddings of F into W^* are extensions of proper weak attachments α of F to W^*. Note that if α and α' are proper weak attachments of F to W^* and $\tau(\alpha') > \tau(\alpha)$, then by definition, either

$$\sum_{k=1}^{s(s+t)} \tau_k' > \sum_{k=1}^{s(s+t)} \tau_k$$

or

$$\sum_{k=1}^{s(s+t)} \tau_k' = \sum_{k=1}^{s(s+t)} \tau_k$$

and for some $m \leqslant s(s+t)$, $\tau_k' = \tau_k$ for $1 \leqslant m < k$, and $\tau_m' > \tau_m$.

In the first case,

$$\prod_{k=1}^{s(s+t)} n_k^{\tau_k'} > K_9 \prod_{k=1}^{s(s+t)} n^{\tau_k'(1+s^{-k})}$$

$$= K_9 \cdot n^{\Sigma_{k=1}^{s(s+t)} \tau_k'} \cdot n^{\Sigma_{k=1}^{s(s+t)} \tau_k' s^{-k}}$$

$$\geqslant K_9 n^{1+\Sigma_{k=1}^{s(s+t)} \tau_k}$$

$$> K_{10} n^{1/2s} \prod_{k=1}^{s(s+t)} n_k^{\tau_k}$$

for s and n sufficiently large. In the second case,

$$\prod_{k=1}^{s(s+t)} n_k^{\tau_k'} > K_9 n^{\sum_{k=1}^{s(s+t)} \tau_k'} \cdot n^{\sum_{k=1}^{s(s+t)} \tau_k' s^{-k}}$$

$$= K_9 n^{\sum_{k=1}^{s(s+t)} \tau_k} \cdot n^{\sum_{k=1}^{m-1} \tau_k s^{-k}} \cdot n^{\sum_{k=m}^{s(s+t)} \tau_k' s^{-k}}$$

But

$$\sum_{k=m}^{s(s+t)} \tau_k' s^{-k} \geqslant (\tau_m + 1)\, s^{-m} = \tau_m s^{-m} + s^{-m}$$

and

$$\sum_{k=m}^{s(s+t)} \tau_k s^{-k} \leqslant \tau_m s^{-m} + \sum_{k=m+1}^{s(s+t)} s^{1/2} \cdot s^{-k}$$

$$\leqslant \tau_m s^{-m} + 2s^{-m-1/2}.$$

Hence, in either case,

$$\prod_{k=1}^{s(s+t)} n_k^{\tau_k'} \Big/ \prod_{k=1}^{s(s+t)} n_k^{\tau_k} > K_{11} n^{s^{-m}-2s^{-m-1/2}} > K_{11} n^{s^{-2^{s2}}} > K_{11} n^{1/s^{s^3}}. \tag{8}$$

But since there are at most $K_{12} = K_{12}(s)$ proper weak attachments of F to W^* then by (5), (8), and the definition of $\tau^{(F)}$ we have

$$E_F(W^*) < K_{13} \prod_{k=1}^{s(s+t)} n_k^{\tau_k^{(F)}}. \tag{9}$$

Hence, from (7) and (9), we have

$$N_{F'}(W^*)/N_F(W^*) \geqslant N_{F'}(W^*)/E_F(W^*)$$

$$> K_{14} \prod_{k=1}^{s(s+t)} n_k^{\tau_k^{(F')}} \Big/ \prod_{k=1}^{s(s+t)} n_k^{\tau_k^{(F)}} > n^{1/s^{s^3}}$$

for n sufficiently large and the claim is proved.

From the preceding discussion it is not difficult to see that if $\tau^{(F)} = \tau^{(F')}$, then F and F' are isomorphic, which contradicts the hypothesis that they are distinct elements of $\mathscr{F}'$. Therefore, we must have $\tau^{(F)} \neq \tau^{(F')}$ and so the fact always holds, provided s is sufficiently large. This completes the proof of the theorem. ∎

Concluding Remarks

As we have seen in Eq. (1), when F has isolated points, $N_F(T)$ can be written as

$$N_F(T) = P(n)\, N_{F'}(T), \tag{10}$$

where $P(n)$ is a polynomial (depending on F) in $n = |V(T)|$ and F' has no isolated points. However, such an expression, valid for all trees T, can always be written in the form

$$P(n)\, N_{F'}(T) = \sum_{F \in \mathscr{F}_{F'}(d)} A_F N_F(T) \tag{11}$$

where $\mathscr{F}_{F'}(d)$ consists of all those forests which can be formed by adjoining exactly $d = \deg P(n)$ additional edges to F'. This follows by the observation that

$$\binom{n - 1 - |E(F')|}{d} N_{F'}(T) = \sum_{F \in \mathscr{F}_{F'}(d)} N_{F'}(F)\, N_F(T) \tag{12}$$

since the left-hand side of (12) can be interpreted as counting the number of ways of selecting a copy of F' in T together with d additional edges of T. For example, if F' is the forest shown in Fig. 4a, then

$$(n - 4)\, N_{F'}(T) = 2N_{F_1}(T) + 4N_{F_2}(T) + 2N_{F_3}(T) + 3N_{F_4}(T), \tag{13}$$

where the F_i are given in Fig. 4b.

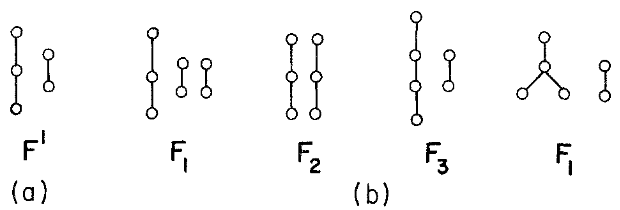


Figure 4

We remark that if $\mathscr{F}$ is allowed to be infinite, then nontrivial linear dependences among the $N_F(T)$, $F \in \mathscr{F}$, can exist. For example, if S_k denotes the star with k edges, then as we have noted earlier for $\mathscr{F} = \{S_k : k = 1, 2, \ldots\}$

$$\sum_{k=1}^{\infty} (-1)^{k+1} N_{S_k}(T) = 1 \tag{14}$$

for all trees T.

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