SPECIAL REPORT

ANSWERING ROTA'S QUESTION

by R.L. Graham

Figure 1

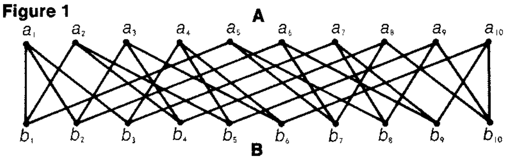
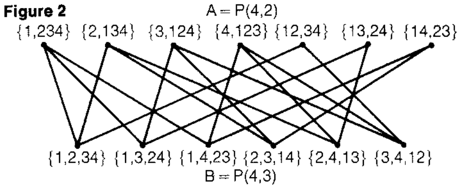


Figure 2



A basic concept in mathematics is the idea of a matching, which can be described as follows. Suppose there exist two collections (or sets) of objects, A and B, so that to each object *a* that belongs to A, there is associated some subset contained in B. A matching of A into B is simply a selection, for each object *a* in A, of some unique object *b* in that subset of B. A typical interpretation is to imagine A to be a set of men and B to be a set of women. For each man *a* in A, there exists a subset that corresponds to the set of women he knows. A matching of A into B is just a way of pairing up each man with a specific woman from his associated subset.

The most famous result dealing with matchings is the so-called Marriage Theorem, discovered by the English mathematician Phillip Hall in 1935. It asserts that, in order for there to be a matching of a set A of men into a set B of women, it is both necessary and sufficient that, for any given number *k*, any subset of *k* men know altogether at least *k* women. An example of this is shown in Figure 1.

It can be checked in the figure that every *k* men know at least *k* women. For example, the three men a_1, a_3, and a_8 know the seven women b_1, b_2, b_3, b_4, b_6, b_7, and b_{10}. Therefore, by the Marriage Theorem this implies that there must exist a matching of A into B. Although the theorem does not indicate how to find it, one such matching is a_1—b_1, a_2—b_4, a_3—b_2, a_4—b_3, a_5—b_7, a_6—b_9, a_7—b_8, a_8—b_{10}, a_9—b_5, and a_{10}—b_6.

Mathematicians usually refer to the type of structure shown in Figure 1 as a bipartite graph. The partition graph G(*n*,*k*), one of the most fundamental classes of bipartite graphs, is formed as follows. For any given numbers *n* and *k*, the set of objects (or points) A of G(*n*,*k*) consists of all possible partitions of the numbers from 1 to *n* into exactly *k* pieces, or blocks. This set A is denoted by P(*n*,*k*). For example, for $n = 4$ and $k = 2$, A = P(4,2) and consists of the seven partitions [1,234], [2,134], [3,124], [4,123], [12,34], [13,24], and [14,23]. Similarly, the set of points B of G(*n*,*k*) is defined as P(*n*,*k* + 1); *i.e.*, all possible partitions of the numbers from 1 to *n* into exactly $k + 1$ blocks. In the above example, B = P(4,3) and is the set of the six partitions [1,2,34], [1,3,24], [1,4,23], [2,3,14], [2,4,13], and [3,4,12].

The lines of this partition graph are placed between a partition p_A in A and a partition p_B in B provided each block of p_B is contained in some block of p_A. In this case it is said that p_B refines p_A. See Figure 2. For example, the partition [1,2,34] in B refines [12,34] in A, but it does not refine the partition [13,24] because the block 34 of [1,2,34] is not contained in any block of [13,24].

A long-standing question, first raised some 15 years ago by the mathematician G.-C. Rota, is whether for every such partition graph G(*n*,*k*) there is always a matching of the smaller of its two sets A and B into the larger set. Although for many years it was generally believed that the answer was in the affirmative, researchers were frustrated in their attempts to prove mathematically that there would always be such a matching for every choice of *n* and *k*. It should be pointed out that even for moderate values of *n* and *k*, the size of P(*n*,*k*), which is denoted by S(*n*,*k*), is awesome.

In 1977 Rota's question was finally settled in a very unexpected way by E. Rodney Canfield of the University of Georgia, who showed that the answer to Rota's question is in the negative! In particular he proved that, once *n* becomes large enough, none of the partition graphs will have the desired matching, whenever *k* is chosen so that S(*n*,*k*) assumes its largest values.

It seemed unlikely, however, that the smallest value of *n* for which this happens would ever be known. The best estimates Canfield offered for the first time his technique works is when *n* approximately equals 6.52608×10^{24}. For such values of *n*, the corresponding values of S(*n*,*k*) can be truly astronomical, exceeding, for example, 10 raised to the 10^{20} power. It may well be that Canfield's answer to Rota's question belongs to the growing collection of mathematical results that serve as pointed reminders of the limits of human thought.

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