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A REMARK ON STEINER MINIMAL TREES

BY

R. L. GRAHAM AND F. K. HWANG

Abstract. For a finite set of points X in the plane, the Steiner minimal tree for X is the shortest network containing as nodes the points of X and possibly some other points as well. In this note we show that the ratio of the lengths of the Steiner minimal tree for X and the minimal spanning tree for X can never be less than $1/\sqrt{3}$. This result is also valid with X contained in any n-dimensional Euclidean space.

Introduction. The general problem of finding Steiner minimal trees on a set of points is a very old and interesting problem [22, 18, 13] and one which has been of considerable interest in network design and operations research. For general background on the problem, the reader is referred to the excellent survey of Gilbert and Pollak [13].

In this note we shall establish a new lower bound for the ratio of the length of a Steiner minimal tree on a set of points in n-dimensional Euclidean space to the length of the minimal spanning tree on this set (see the following section for definitions). In particular, we show that this ratio can never be less than $1/\sqrt{3} = .5771\cdots$. It was previously known [19] to be bounded by 1/2. It is conjectured that for the plane the correct bound is $\sqrt{3}/2 = .8660\cdots$.

DEFINITIONS. We begin by defining our terms.

If X is a finite subset of a metric space, a *spanning tree* $T(X)$ on X is simply a collection of pairs $\{x_i, x_j\}$ (called *edges*) satisfying:

For any $x, x' \in X$, $x \neq x'$, there is a unique sequence of distinct vertices $P(x, x') = (x_0, x_1, \cdots, x_k)$ with $x_0 = x$, $x_k = x'$ and $\{x_i, x_{i+1}\} \in T(X)$ for $0 \leq i < k$.

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The *length of* $T(X)$, denoted by $l(T(X))$, is defined by

$$l(T(X))=\sum_{\{x,x'\}\in T(X)} d(x,x')\,,$$

where $d(x,x')$ denotes the *distance* between x and x'. $T^*(X)$ is said to be a *minimal* spanning tree on X if $T^*(X)$ is a spanning tree on X and

$$l(T^*(X))\le l(T(X))$$

for all spanning trees $T(X)$ on X.

Finally, $S(X)$ is said to be a *Steiner tree on* X if for some $Y\supseteq X$, $S(X)=T(Y)$ is a spanning tree on Y. If $l(S^*(X))\le l(S(X))$ for all Steiner trees $S(X)$ on X, then $S^*(X)$ is said to be a *Steiner minimal tree on* X.

If v is a vertex of $S^*(X)$ which does not belong to X, then v is called a *Steiner point* of $S^*(X)$. We remark that in any Euclidean space, a Steiner point of $S^*(X)$ must be incident to exactly three edges of $S^*(X)$, each of which meets the other two at angles of 120° (cf. [23]).

The main result. Let E^n denote n-dimensional Euclidean space.

THEOREM. *If* $X\subseteq E^n$ *is finite then*

$$l(S^*(X))/l(T^*(X))\ge 1/\sqrt{3}\,. \tag{1}$$

Proof. If $|X|$, the cardinality of X, is small then the theorem is known to hold for X. This is shown for $|X|=3$ in [13] and $|X|=4$ in [20] (Kallman [17] studies the general case in the Euclidean plane allowing at most one Steiner point) where in both cases the stronger lower bound of $\sqrt{3}/2$ is proved. Henceforth, we may assume $|X|=m>4$. Assume that (1) holds for all $X'\subseteq E^n$ with $|X'|<m$. Suppose $S^*(X)$ is not a full Steiner tree (one containing $m-2$ Steiner points). Then we can decompose it into a union of full Steiner trees [13] and by induction the theorem is true on each of the subsets of vertices. Since the union of the minimal spanning trees on the subsets of vertices is a spanning tree on X (though not necessarily minimal), the theorem is a fortiori true. Therefore, we may assume that $S^*(X)$ is a full Steiner tree. Thus

there must exist a Steiner point which is adjacent to two points x_1 and $x_2 \in X$ (see Figure 1) each of degree 1.

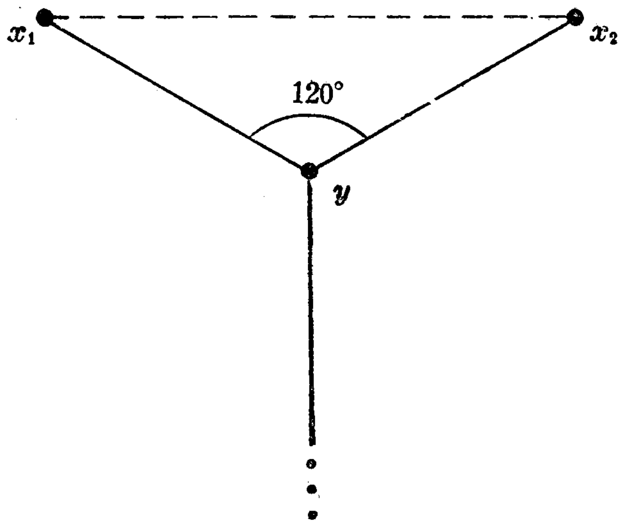

FIGURE 1

It is also known (see [13]) that the angle between the edges $\{x_1, y\}$ and $\{x_2, y\}$ must be 120°. Hence, assuming (without loss of generality) that $d(x_1, y) \geq d(x_2, y)$, an easy calculation shows that

$$d(x_1, x_2) \leq \sqrt{3}\, d(x_1, y)\,. \tag{2}$$

Let $X' = X - \{x_1\}$, $X'' = X' \cup \{y\}$. By induction,

$$\begin{aligned}(1/\sqrt{3})\, l(T^*(X')) &\leq l(S^*(X')) \leq l(S^*(X'')) \\ &\leq l(S^*(X)) - d(x_1, y)\end{aligned} \tag{3}$$

since $S^*(X) - \{x_1, y\}$ is a spanning tree on X''. But

$$l(T^*(X)) \leq d(x_1, x_2) + l(T^*(X'))$$

since $T^*(X') \cup \{x_1, x_2\}$ is a spanning tree on X. Hence, by (2) and (3),

$$\begin{aligned}(1/\sqrt{3})\, l(T^*(X)) &\leq (1/\sqrt{3})\, d(x_1, x_2) + (1/\sqrt{3})\, l(T^*(X')) \\ &\leq d(x_1, y) + l(S^*(X)) - d(x_1, y) \\ &= l(S^*(X))\end{aligned}$$

and the induction step is completed. This proves the theorem. ∎

REMARKS. Examples of sets X are known [4] in a high-dimensional Euclidean space for which

$$l(S^*(X))/l(T^*(X)) \to (3/2)^{1/2}/(2^{3/2} - 1) = .66984\,.$$

A well-known conjecture[1] [12] asserts that for $X \subseteq E^2$ the following inequality holds:

Conjecture: $l(S^*(X))/l(T^*(X)) \geq \sqrt{3}/2$.

If true, the conjecture would be best possible, as the set consisting of the three vertices of an equilateral triangle shows.

At present no efficient method is known for computing a Steiner minimal tree on a large general set X, even when $X \subseteq E^2$ (cf. [3, 5, 6, 8, 9, 18]). It has recently been shown [10] that even the determination of Steiner minimal trees in the Euclidean plane is at least as hard as any "NP-complete" problem (see [2] for an explanation of this term) and so probably one will have to settle for efficient heuristics for finding relatively short Steiner trees. Recent results of Shamos [22] show that if $X \subseteq E^2$, $|X| = m$, a minimal spanning tree on X may be constructed in $O(m \log m)$ steps. By the result of this paper, this will bring us to within a factor of $1/\sqrt{3}$ of $l(S^*(X))$ (which is not too bad in this type of problem).

A problem closely related to the one considered in this note is the Steiner minimal tree problem with the *rectilinear* (or "Manhattan") metric, denoted by d_R. For $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n) \in E^n$, this is defined by

$$d_R(x, y) = \sum_{k=1}^{n} |x_k - y_k|\,.$$

It has been proved by Hwang [16] that for $X \subseteq E^2$, we always have

$$l_R(S^*(X))/l_R(T^*(X)) \geq 2/3$$

(where the subscript R denotes the corresponding quantities using the rectilinear metric) and that this bound is best possible. It may be that the corresponding bound in E^n is $n/(2n-1)$ but this has not yet been established. Efficient special-case rectilinear Steiner minimal trees are discussed in [1]. It has been shown by Garey and

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(1) One of the authors is currently offering $100 for a proof or disproof of this conjecture.

Johnson [12] that the rectilinear Steiner minimal tree problem in E^2 is NP-complete.

Analogous results have recently been proved [21] for the "travelling salesperson" problem. In this case an $O(n \log n)$ algorithm can be used to construct a Hamiltonian circuit on a set $X \subseteq E^2$ with $|X| = m$ which has length guaranteed always to be at most twice the minimal one. It is conjectured by Tarjan [24] that no efficient (= polynomial-time bounded) algorithm can always come to within a factor of $2 - \varepsilon$ of the optimal solution, for a fixed $\varepsilon > 0$.

In a similar vein, Garey and Johnson [11] recently showed that if, for some fixed $\varepsilon > 0$, a polynomial algorithm can always determine the chromatic number of any graph to within a factor of $2 - \varepsilon$, then in fact the exact chromatic number can also be determined in polynomial time, a highly unlikely possibility in view of the NP-completeness of the problem of finding the chromatic number of a graph. They conjecture that a similar result holds when $2 - \varepsilon$ is replaced by any fixed factor α.

Added in proof. Tarjan's conjecture has recently been disproved by N. Cristofides, who has constructed a polynomial bounded algorithm for the travelling salesperson problem which always comes to within a factor of 3/2 of the optimal solution. We also point out that it has very recently been shown by F. K. Hwang and F. R. K. Chung that for the case of E^2,

$$\frac{l(S^*(T))}{l(X^*(X))} \geq \frac{2\sqrt{3} + 2 - \sqrt{7 + 2\sqrt{3}}}{3} = .74309\cdots.$$

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BELL LABORATORIES, MURRAY HILL, NEW JERSEY 07974, U. S. A.