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ON THE DISTANCE MATRIX OF A TREE

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For a tree T on n vertices, let $D(T) = (d_{ij}(T))$ denote the distance matrix of T, i.e., $d_{ij}(T)$ is just the length of the unique path between the ith vertex and the jth vertex of T. Denote by $\Delta_T(x)$ the characteristic polynomial of $D(T)$, so that $\Delta_T(x) = \det(D(T) - xI)$. In this paper, we investigate a number of properties of $\Delta_T(x)$. In particular, we find simple expressions for the first few and last few coefficients of $\Delta_T(x)$.

1. Introduction

Let $\mathcal{T}_n$ denote the set of n vertex trees (i.e., connected acyclic graphs*) and let $\mathcal{T} = \bigcup_{n>0} \mathcal{T}_n$. For a fixed $T \in \mathcal{T}_n$, let the vertices of T be labelled (arbitrarily) $v_1, v_2, ..., v_n$ and define the *distance* d_{ij} between two vertices v_i and v_j to be the number of edges contained in the (unique) path between v_i and v_j in T. The *distance matrix* $D(T)$ for T is the n by n matrix which has d_{ij} as its (i, j)th entry.

It was shown in [2] that for $T \in \mathcal{T}_n$, the determinant of $D(T)$ is equal to $(-1)^{n-1}(n-1)2^{n-2}$, *independent of the structure of* T. Motivated by the observation that the determinant of $D(T)$ is merely the constant term in the *characteristic polynomial* $\Delta_T(x)$ of $D(T)$, in this paper we investigate the dependence of other coefficients of $\Delta_T(x)$ upon the structure of T.

Recall that for the *adjacency matrix*** $A(T)$ for T, if we write

$$\det(A(T) - xI) = \sum_{k=0}^{n} \alpha_k(T)x^k ,$$

* See [3] for graph theoretic terminology.
** Which has $a_{ij} = 1$ if edge $\{i, j\}$ belongs to T, $a_{ij} = 0$ otherwise.

then it can be shown [4] that

$$(1)\qquad \alpha_k(T) = \begin{cases} (-1)^{(n+k)/2} M(\tfrac{1}{2}(n-k)) & \text{if } k \equiv n \pmod 2, \\ 0 & \text{otherwise,} \end{cases}$$

where $M(t)$ denotes the number of ways that t disjoint edges can be selected from T. Writing

$$(2)\qquad \Delta_T(x) = \det(D(T) - xI) = \sum_{k=0}^{n} \delta_k(T)\, x^k,$$

one might hope that similar expressions could be given which relate each $\delta_k(T)$ directly to the number of occurrences of various subgraphs in T. The results of this investigation show that such a relation appears to exist, although it is rather more complex than that for the $\alpha_k(T)$.

2. Preliminary facts

Let $\mathcal{P}_k$ denote the set of k element subsets of $\{1, 2, ..., n\}$. For $S \in \mathcal{P}_k$, let D_S denote the k by k principal submatrix of D consisting of all elements with row and column indices belonging to S. Then from elementary determinant theory we obtain

$$(3)\qquad \delta_{n-k}(T) = (-1)^{n-k} \sum_{S \in \mathcal{P}_k} \det(D_S), \qquad 0 \leqslant k \leqslant n.$$

It follows from (3) that

$$(4a)\qquad \delta_n(T) = (-1)^n,$$

$$(4b)\qquad \delta_{n-1}(T) = 0,$$

$$(4c)\qquad \delta_{n-2}(T) = (-1)^{n-1} \sum_{i<j} d_{ij}^2,$$

$$(4d)\qquad \delta_{n-3}(T) = (-1)^{n-1} \sum_{i<j<k} 2 d_{ij} d_{jk} d_{ki}.$$

This sequence can be continued, but it is not particularly illuminating to do so.

As we remarked earlier, the value of $\delta_0(T)$ depends only on the number n of vertices of T.

Theorem 2.1 (see [2]).

(5) $$\delta_0(T) = \det(D(T)) = (-1)^{n-1}(n-1)2^{n-2}.$$

It was also shown in [2] that this theorem has the following result as an immediate corollary.

Corollary 2.2. *$D(T)$ has one positive eigenvalue and $n-1$ negative eigenvalues.*

The preceding results can be used to determine the sign of $\delta_k(T)$ for all k.

Theorem 2.3.

(6) $$\operatorname{sgn}(\delta_{n-k}(T)) = \begin{cases} (-1)^n & \text{for } k = 0, \\ 0 & \text{for } k = 1, \\ (-1)^{n-1} & \text{for } 2 \leqslant k \leqslant n. \end{cases}$$

Proof. For $k = 0$ and 1 the theorem follows from (4). Let k satisfy $2 \leqslant k \leqslant n$ and let the eigenvalues of D be denoted by $\lambda_1, -\lambda_2, -\lambda_3, \ldots, -\lambda_n$, where $\lambda_1, \lambda_2, \ldots, \lambda_n > 0$ (by Corollary 2.2). Then

(7) $$\begin{aligned} \Delta_T(x) &= (-1)^n (x - \lambda_1)(x + \lambda_2)(x + \lambda_3) \ldots (x + \lambda_n) \\ &= (-1)^n (x - \lambda_1) \sum_{k=0}^{n-1} g_k x^{n-1-k} \\ &= (-1)^n \left[x^n + \sum_{k=1}^{n-1} (g_k - \lambda_1 g_{k-1}) x^{n-k} - \lambda_1 g_{n-1} \right], \end{aligned}$$

where g_k is an elementary symmetric function, the sum of all k-fold products among $\lambda_2, \lambda_3, \ldots, \lambda_n$ (and $g_0 \equiv 1$). From (4b) and (7) we have

$$\delta_{n-1}(T) = g_1 - \lambda_1 = 0$$

so that

$$\lambda_1 = g_1 .$$

Then, since

$$g_k - g_1 g_{k-1} < 0 \quad \text{for } k = 2, 3, ..., n-1,$$

and

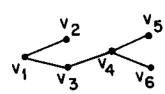
$$-g_1 g_{n-1} < 0$$

the theorem follows.

3. The canonical form

We now describe a matrix $M(T)$ which results from a sequence of elementary row and column operations applied to $D(T) - xI$. We begin by selecting a vertex v_i of degree 1 from T. Suppose v_j is the vertex adjacent to v_i. Then subtract the jth row and column from the ith row and column, respectively, and delete the vertex v_i from T. In the new tree, now, let v_k be a vertex of degree 1 adjacent to v_l. Then subtract the lth row and column from the kth row and column respectively, and delete the vertex v_k. Repeat this procedure until we are left with a single vertex, called the *root* of T, which we may assume is labeled v_1. The matrix obtained in this manner is denoted by $M(T)$. It is easy to see that $M(T)$ depends only on the choice of the root v_1 and not on the order in which the successive vertices of degree 1 were chosen.

Example.

T: 

$$D(T) - xI = \begin{bmatrix} -x & 1 & 1 & 2 & 3 & 3 \\ 1 & -x & 2 & 3 & 4 & 4 \\ 1 & 2 & -x & 1 & 2 & 2 \\ 2 & 3 & 1 & -x & 1 & 1 \\ 3 & 4 & 2 & 1 & -x & 2 \\ 3 & 4 & 2 & 1 & 2 & -x \end{bmatrix},$$

$$\Delta_T(x) = x^6 - 84x^4 - 368x^3 - 580x^2 - 368x - 80,$$

$$M(T) = \begin{bmatrix} -x & 1+x & 1+x & 1 & 1 & 1 \\ 1+x & -2(1+x) & -x & 0 & 0 & 0 \\ 1+x & -x & -2(1+x) & x & 0 & 0 \\ 1 & 0 & x & -2(1+x) & x & x \\ 1 & 0 & 0 & x & -2(1+x) & -x \\ 1 & 0 & 0 & x & -x & -2(1+x) \end{bmatrix}$$

By keeping track of the entries of $D(T) - xI$ as the preceding algorithm is performed, the reader will have little difficulty in verifying the next result.

Theorem 3.1. *The entries of the matrix $M(T) = (m_{ij})$ are given by*

$$\text{(8)} \quad m_{ij} = \begin{cases} -x & \text{if } i = j = 1, \\ 1+x & \text{if } i = 1 \text{ or } j = 1 \text{ and } v_i \text{ and } v_j \text{ are adjacent in } T, \\ 1 & \text{if } i = 1 \text{ or } j = 1 \text{ and } v_i \text{ and } v_j \text{ are not adjacent in } T, \\ -2(1+x) & \text{if } i = j \neq 1, \\ x & \text{if } i, j \neq 1 \text{ and } v_i \text{ and } v_j \text{ are adjacent in } T, \\ -x & \text{if there exists } v_k \text{ such that } d_{1i} = d_{1j} = 1 + d_{1k}, \\ 0 & \text{otherwise.} \end{cases}$$

It is the (potential) sparseness of $M(T)$ which makes this matrix useful in the study of $\Delta_T(x) = \det(M(T))$.

If π denotes an arbitrary element of $\mathcal{S}_n$, the set of permutations on $\{1, 2, ..., n\}$, let $e(\pi)$ denote the number of cycles of even length in π. Further, let $M(\pi)$ represent the product

$$\text{(9)} \qquad M(\pi) = \prod_{i=1}^{n} m_{i,\pi(i)} .$$

Then we have

$$\text{(10)} \qquad \Delta_T(x) = \sum_{\pi \in \mathcal{S}_n} (-1)^{e(\pi)} M(\pi).$$

$M(\pi)$ is a polynomial in x; we let $M_k(\pi)$ denote the coefficient of x^k in $M(\pi)$. Thus

$$\text{(11)} \qquad \delta_k(T) = \sum_{\pi \in \mathcal{S}_n} (-1)^{e(\pi)} M_k(\pi) .$$

Theorem 3.2.

$$\delta_k(T) \equiv 0 \ (\mathrm{mod}\, 2^{n-k-2}) \quad \textit{for } 0 \leqslant k \leqslant n-2. \tag{12}$$

Proof. By (11), it is sufficient to consider just those π for which $M_k(\pi) \neq 0$. For each such π, at most two non-zero factors in (9) lie in the first row or column and at most k other factors can lie off the diagonal of $M(T)$. Hence, at least $n-k-2$ factors in (9) are of the form $-2(1+x)$ by Theorem 3.1 and (12) follows.

4. The coefficients $\delta_k(T)$, $k \leqslant 3$

Define the *root cycle* of a permutation $\pi \in \mathcal{S}_n$ for T to be the cycle of π containing 1 (this corresponds to the root v_1 of T). Putting $k = 0$ in (11) we have

$$\delta_0(T) = \sum_{\pi \in \mathcal{S}_n} (-1)^{e(\pi)} M_0(\pi).$$

For each π for which $M(\pi) \neq 0$, the factors of $M(\pi)$ can include no term $\pm x$ or 0. Hence, $M(\pi)$ must have exactly one factor m_{1j}, $j \neq 1$, one factor m_{i1}, $i \neq 1$, and $n-2$ factors m_{ij}, $i = j > 1$. This implies that the root cycle of π is a 2-cycle and the remaining vertices of T occur in 1-cycles of π. Based on this observation, Table 1 presents all the data needed to compute $\delta_0(T)$.

The labels on the edges of the cycle in Table 1 indicate the corresponding entries in $M(T)$. Thus, $m_{1i} = m_{i1} = x+1$ or $m_{1i} = m_{i1} = 1$. The notation $\#(\pi)$ denotes the number of permutations which have a root cycle of the indicated form, with all remaining cycles being 1-cycles. Thus

$$\delta_0(T) = (-1)(n-1)(-2^{n-2}) = (-1)^{n-1}(n-1)2^{n-2}.$$

which agrees with (5).

Table 1

ROOT CYCLE OF π	$(-1)^{e(\pi)}$	$\#(\pi)$	$M_0(\pi)$
X+1 or 1  X+1 or 1; 1	-1	n - 1	$(-2)^{n-2}$

Looking ahead and interpreting the factor $n-1$ in the expression for $\delta_0(T)$ as the number $N_{S_1}(T)$ of subgraphs of T consisting of a single edge S_1, we have

$$(5')\qquad \delta_0(T) = (-1)^{n-1}2^{n-2}N_{S_1}(T).$$

In the following we show that the coefficients $\delta_k(T)$, $1 \leqslant k \leqslant 3$, may be expressed in terms of the number of subtrees of various kinds in T. Table 2 defines the relevant subtree counting quantities. The degree of vertex v_i in T is denoted by d_i.

Theorem 4.1.

$$(13)\qquad \delta_1(T) = (-1)^{n-1}2^{n-3}(2nN_{S_1}(T) - 2N_{S_2}(T) - 4).$$

Table 2

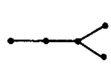
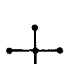
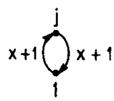
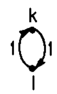
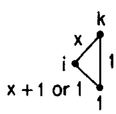
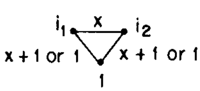
Subtree Count	Subtree Being Counted	Expression for Subtree Count
$N_{S_1}(T)$		$\frac{1}{2}\sum_i d_i = n-1$
$N_{S_2}(T)$		$\sum_i \binom{d_i}{2}$
$N_{P_3}(T)$		$\sum_{\{i,j\}\in T} (d_i-1)(d_j-1)$
$N_{S_3}(T)$		$\sum_i \binom{d_i}{3}$
$N_{P_4}(T)$		$\sum_{i<j \;\; d_{ij}=2} (d_i-1)(d_j-1)$
$N_Y(T)$		$\sum_{\{i,j\}\in T} (d_i-1)\binom{d_j-1}{2} + (d_j-1)\binom{d_i-1}{2}$
$N_{S_4}(T)$		$\sum_i \binom{d_i}{4}$

Table 3

Root Cycle of π	$(-1)^{e(\pi)}$	$\#(\pi)$	$\overline{\#(\pi)}$	$M_1(\pi)$
	$+1$	1	1	$-(-2)^{n-1}$
	-1	d_1	$\frac{2(n-1)}{n}$	$\binom{n}{1}(-2)^{n-2}$
	-1	$n-(d_1+1)$	$\frac{(n-1)(n-2)}{n}$	$\binom{n-2}{1}(-2)^{n-2}$
	$+1$	$2[(n-1)-d_1]$	$\frac{2(n-1)(n-2)}{n}$	$(-2)^{n-3}$
	$+1$	$\left[d_1(d_1-1)+\sum_{i\neq 1}(d_i-1)(d_i-2)\right]$	$\left[2N_{P_2}(T)-\frac{2(n-2)(n-1)}{n}\right]$	$-(-2)^{n-3}$

Proof. From (11) we have

$$\delta_1(T)=\sum_{\pi\in\mathcal{S}_k}(-1)^{e(\pi)}M_1(\pi).$$

For each π with $M_1(\pi)\neq 0$, the factors of $M(\pi)$ include at most one entry $m_{i,j}$ with either $i\neq j$ or $i=1$ or $j=1$. This implies that any cycle of π which is not the root cycle must be a 1-cycle and that the root cycle has length at most 3. Based on these observations, Table 3 presents all the data needed to compute $\delta_1(T)$.

In Table 3 (and Table 4 which appears later) the symbols i, j and k appearing as vertex labels in root cycle graphs have the following special meaning:

i — a generic vertex in T,
j — a vertex in T such that $d_{1j}=1$,
k — a vertex in T such that $d_{1k}\geqslant 2$.

The quantity $\overline{\#(\pi)}$ is defined by

$$\overline{\#(\pi)}=\frac{1}{n}\sum_{\substack{\text{all}\\ \text{root}\\ \text{choices}}}\#(\pi) \tag{14}$$

and is introduced to remove the virtual (but not actual) dependence of the result on the root choice. We then have

$$(15) \qquad \delta_1(T) = \sum (-1)^{e(\pi)} \overline{\#(\pi)}\, M_1(\pi),$$

where the summation extends over the 5 classes of permutations given in Table 3. Inserting the data from Table 3 and simplifying, (15) becomes

$$\delta_1(T) = (-1)^{n-1} 2^{n-3} (2nN_{S_1}(T) - 2N_{S_2}(T) - 4)$$

and the theorem is proved.

Define a *basic cycle* of a permutation π to be either the root cycle of π with respect to the tree T or any cycle of π having length greater than 1.

Theorem 4.2.

$$(16) \qquad \delta_2(T) = (-1)^{n-1} 2^{n-4} [2(n^2 - n - 4) N_{S_1}(T) - (5n - 7) N_{S_2}(T) + 6N_{S_3}(T) - 2N_{P_3}(T)] .$$

Proof. From (11) we have

$$\delta_2(T) = \sum_{\pi \in \mathcal{S}_n} (-1)^{e(\pi)} M_2(\pi).$$

For each π such that $M_2(\pi) \neq 0$, the factors of $M(\pi)$ include at most two $m_{i,j}$ with $i \neq j$ or $i = 1$ or $j = 1$. This implies that the root cycle of π is of length at most 4, and that any basic cycle of π other than the root cycle must be a 2-cycle. Moreover, π can have at most 2 basic cycles. Table 4 enumerates the various possible basic cycle structures and includes all data needed to compute $\delta_2(T)$. We then have

$$\delta_2(T) = \sum (-1)^{e(\pi)} \overline{\#(\pi)}\, M_2(\pi),$$

where the summation extends over the 14 classes of permutations given in Table 4. Inserting the data from Table 4 and simplifying, this becomes

$$\delta_2(T) = (-1)^{n-1} 2^{n-4} [2(n^2 - n - 4) N_{S_1}(T) - (5n - 7) N_{S_2}(T) + 6N_{S_3}(T) - 2N_{P_3}(T)] .$$

This proves (16).

Table 4

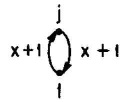
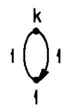
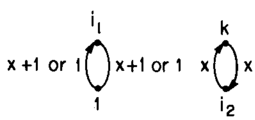
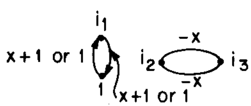
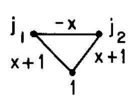
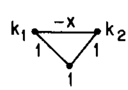
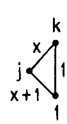
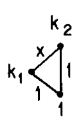
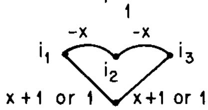
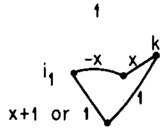
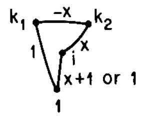
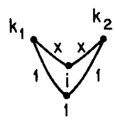
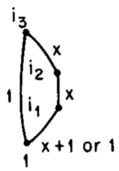
Basic cycle of π	$(-1)^{e(\pi)}$	$\overline{\#(\pi)}$	$M_2(\pi)$
	+1	1	$-\binom{n-1}{1}(-2)^{n-1}$
	−1	$\frac{2(n-1)}{n}$	$\binom{n}{2}(-2)^{n-2}$
	−1	$\frac{(n-1)(n-2)}{n}$	$\binom{n-2}{2}(-2)^{n-2}$
	+1	$\frac{(n-1)(n-2)(n-3)}{n}$	$(-2)^{n-4}$
	+1	$\frac{n-3}{2n}[2nN_{S_2}-2(n-2)(n-1)]$	$(-2)^{n-4}$
	+1	$\frac{2}{n}N_{S_2}(T)$	$-\binom{n-1}{1}(-2)^{n-3}$
	+1	$\frac{2(n-1)}{n}[N_{S_2}(T)-(n-2)]$	$-\binom{n-3}{1}(-2)^{n-3}$
	+1	$\frac{4}{n}N_{S_2}(T)$	$\binom{n-2}{1}(-2)^{n-3}$
	+1	$\frac{2}{n}[(n-2)(n-1)-2N_{S_2}(T)]$	$\binom{n-3}{1}(-2)^{n-3}$
	−1	$\frac{6}{n}[nN_{S_3}(T)-(n-1)N_{S_2}(T)+(n-1)(n-2)]$	$(-2)^{n-4}$
	−1	$\frac{2}{n}[nN_{P_3}(T)+2N_{S_2}(T)-(n-1)(n-2)]$	$-(-2)^{n-4}$

Table 4 (continued)

	-1	$\frac{4(n-1)}{n}\left[N_{S_2}(T)-(n-2)\right]$	$-(-2)^{n-4}$
	-1	$\frac{2(n-1)}{n}\left[N_{S_2}(T)-(n-2)\right]$	$(-2)^{n-4}$
	-1	$\frac{2}{n}\left[(n-2)(n-1)-2N_{S_2}(T)\right]$	$(-2)^{n-4}$

Theorem 4.3.

$$\begin{aligned}(17)\qquad \delta_3(T) = (-1)^{n-1}\,2^{n-5}\,[&\tfrac{4}{3}(n^2-4)(n-3)N_{S_1}(T)\\ &-2(3n^2-11n+9)N_{S_2}(T)+2(7n-22)N_{S_3}(T)\\ &-4(n-3)N_{P_3}(T)-2N_{P_4}(T)-24N_{S_4}(T)\\ &+4N_Y(T)+2N_{S_2}(T)^2\,].\end{aligned}$$

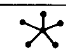
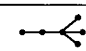
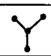
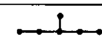
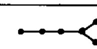
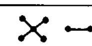
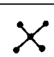
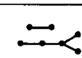
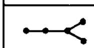
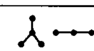
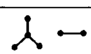
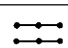
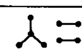
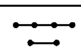
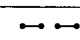
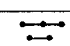
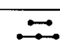
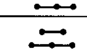
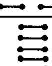
The proof of Theorem 4.3 is similar to the proofs for Theorem 4.1 and 4.2, but involves considerably more complicated and lengthy calculations and is omitted.

5. Linear combinations of subforests

It must be admitted that the expressions we have thus far derived for $\delta_k(T)$ are not particularly illuminating. In fact, the appearance of the nonlinear term $N_{S_2}(T)^2$ in $\delta_3(T)$ is rather ominous. There is, however, a different "coordinate system" in which these results may be expressed which is somewhat more encouraging. What we shall do is write the $\delta_k(T)$ as *linear* combinations of $N_F(T)$ where F ranges over $\mathcal{F}_{k+1}$, the set of *forests* (i.e., acyclic subgraphs) with at most $k+1$ edges.

Table 5 presents in tabular form the appropriate coefficients for $N_F(T)$ associated with this representation.

Table 5
Coefficients A_F for representation $\delta_k(T) = (-1)^{n-1}2^{n-k-2}\sum_{F\in\mathcal{F}_{k+1}} A_F N_F(T)$.

F	δ_0	δ_1^*	δ_2	δ_3	δ_4	F	$\delta_4(?)$
	1	4	-4	0	0		5
		2	8	-4	0		-7
		4	16	0	0		-4
			3	12	-4		-15
			0	8	-4		-20
			7	28	4		-35
			12	48	16		13
				4	16		1
				-2	8		-15
				-10	0		17
				10	40		8
				12	48		28
				4	32		16
				20	80		33
				32	128		52
							80

* δ_1 has constant term of -4.

For example,

$$\delta_2(T) = (-1)^{n-1}2^{n-4}(-4N_{S_1}(T) + 8N_{S_2}(T) + 16N_{2S_1}(T) + 3N_{S_3}(T) + 7N_{S_1+S_2}(T) + 12N_{3S_1}(T)).$$

The corresponding expression for $\delta_1(T)$,

$$\delta_1(T) = (-1)^{n-1} 2^{n-3} (4N_{S_1}(T) + 2N_{S_2}(T) + 4N_{2S_1}(T) - 4),$$

seems to be unique in having a non-homogeneous term -4.

If we assume that $\delta_4(T)$ can be written in the form

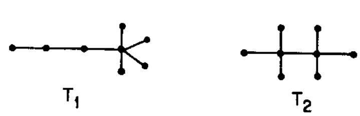
$$\delta_4(T) = (-1)^{n-1} 2^{n-6} \sum_{F \in \mathcal{F}_5} A_F N_F(T),$$

which, a priori, is by no means clear, then by calculating* $\Delta_T(x)$ for a sufficiently independent set of trees T, we can solve the resulting system of linear equations for the coefficients A_F. The coefficients determined in this manner for δ_4 are also included in Table 5.

6. Some questions

A number of questions have been left unresolved. We discuss several of these now.

(1) *Does $\Delta_T(x)$ determine T?* No examples are currently known of two nonisomorphic trees, T_1 and T_2 for which $\Delta_{T_1}(x) = \Delta_{T_2}(x)$. It is conjectured that this cannot occur. This is in contrast to the situation for $A_T(x)$, the adjacency matrix characteristic polynomial. The smallest possible example of two nonisomorphic trees with the same $A_T(x)$ was given in [1] and is shown in Fig. 1.



$A_{T_1}(x) = A_{T_2}(x) = x^8 - 7x^6 + 9x^4$

Fig. 1.

In fact, Schwenk has recently shown [5] that almost any large tree T has many cospectral mates T' (i.e., such that $A_T(x) = A_{T'}(x)$).

It is also the case that there exist nonisomorphic graphs G_1 and G_2 for which $\Delta_{G_1}(x) = \Delta_{G_2}(x)$. For example, Shrikhande [6] gives a pair of nonisomorphic graphs on 16 vertices (one of which is $L(K_{4,4})$, the line graph on the complete bipartite graph $K_{4,4}$) for which

* In the appendix, $\Delta_T(x)$ is given for all trees on $\leqslant 8$ vertices.

$$\Delta_{G_1}(x) = \Delta_{G_2}(x) = (x - 24)(x + 4)^6 x^9 .$$

However, for both of these graphs, all d_{ij}, $i \neq j$, are either 1 or 2 so that $A(T)$ and $D(T)$ are very closely related (it is also true that $A_{G_1}(x) = A_{G_2}(x)$).

The authors admit that there is presently not much evidence on which to base the conjecture of the uniqueness of $\Delta_T(x)$ (see appendix).

(2) *Can* $\delta_k(T)$ *always be expressed in the form*

$$(18) \qquad \delta_k(T) = (-1)^{n-1} 2^{n-k-2} \sum_{F \in \mathcal{F}_{k+1}} A_F N_F(T)$$

for suitable integers A_F (with the mild exception occurring for $k = 1$)? It may be the case that an alternative approach to the one we have taken for expanding $\det(D(T) - xI)$ may lead to an immediate affirmative answer to this question.

(3) *If* $\delta_k(T)$ *can be written in the form* (18), *is this expansion unique?* One strongly suspects that the answer is yes, but this has not yet been proved. It is sufficient to show that if

$$\sum_{F \in \mathcal{F}} A_F N_F(T) = C \quad \text{for all trees } T,$$

then $A_F = 0$ for all $F \in \mathcal{F}$, where $\mathcal{F}$ is some fixed finite set of non-trivial forests. If $\mathcal{F}$ is allowed to be infinite, then we can have a non-trivial linear dependence, e.g., if $\mathcal{F} = \{S_k : S_k$ is a star with k edges, $k = 1, 2, ...\}$, then

$$\sum_{S_k \in \mathcal{F}} (-1)^{k+1} N_{S_k}(T) = 1 .$$

Also, if $\mathcal{F}$ is allowed to contain the trivial forest F_0 consisting of a single vertex, then

$$N_{F_0}(T) - N_{S_1}(T) = 1 .$$

(4) *If the answer to* (3) *is in the affirmative, then what do the coefficients* A_F *signify?* It is possible that a different "coordinate system" for the $\delta_k(T)$ may yield expressions of still greater simplicity from which further properties of the $\delta_k(T)$ can be deduced.

Appendix

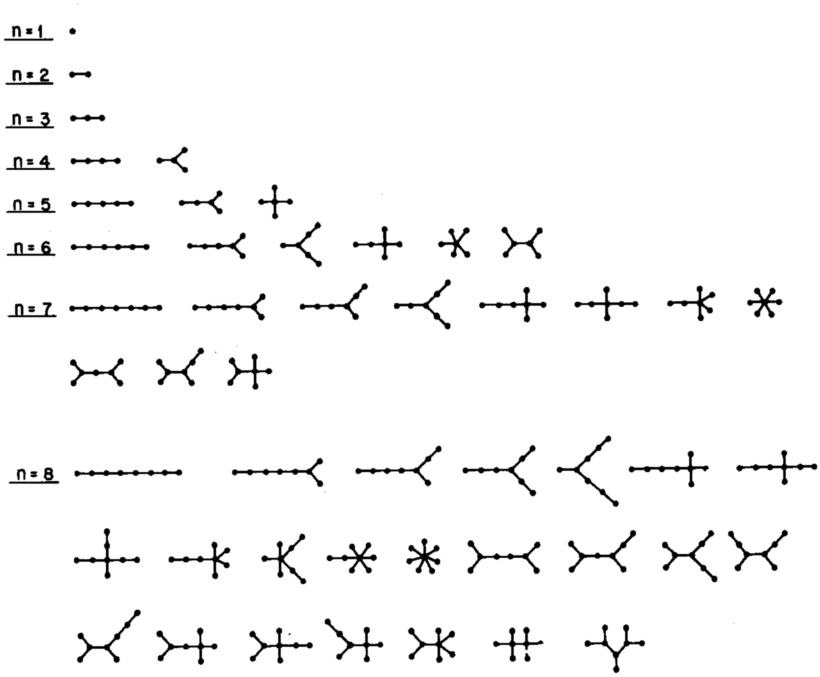
Tabulated below are distance matrix characteristic polynomial coefficients $\delta_k(t_i)$ for all trees on 8 or fewer vertices. The subscripts on the t_i refer to the order given in Table 6.

$$(-1)^{n-1}\,\delta_k(t_i)/2^{n-k-2}, \quad n \neq k$$

t_i	k	0	1	2	3	4	5	6	7	8
$n=1$	t_1	0	−1							
$n=2$	t_1	1	0	1						
$n=3$	t_1	2	6	0	−1					
$n=4$	t_1	3	16	20	0	−1				
	t_2	3	14	15	0	−1				
$n=5$	t_1	4	30	70	50	0	−1			
	t_2	4	28	58	38	0	−1			
	t_3	4	24	44	28	0	−1			
$n=6$	t_1	5	48	162	224	105	0	−1		
	t_2	5	46	145	184	84	0	−1		
	t_3	5	46	143	178	77	0	−1		
	t_4	5	44	126	148	65	0	−1		
	t_5	5	42	117	136	60	0	−1		
	t_6	5	36	90	100	45	0	−1		
$n=7$	t_1	6	70	308	630	588	196	0	−1	
	t_2	6	68	286	552	488	164	0	−1	
	t_3	6	68	284	540	464	148	0	−1	
	t_4	6	68	282	528	438	132	0	−1	
	t_5	6	64	248	438	366	122	0	−1	
	t_6	6	64	244	442	340	108	0	−1	
	t_7	6	58	200	324	208	86	0	−1	
	t_8	6	50	156	240	190	66	0	−1	
	t_9	6	66	264	476	402	134	0	−1	
	t_{10}	6	66	260	460	376	120	0	−1	
	t_{11}	6	50	156	240	190	66	0	−1	
$n=8$	t_1	7	96	520	1408	1980	1344	336	0	−1
	t_2	7	94	493	1280	1721	1134	291	0	−1
	t_3	7	94	491	1264	1672	1068	264	0	−1
	t_4	7	94	489	1246	1611	984	228	0	−1
	t_5	7	94	491	1262	1662	1056	255	0	−1
	t_6	7	90	445	1080	1365	875	227	0	−1
	t_7	7	90	441	1052	1293	790	195	0	−1

t_i \ k	0	1	2	3	4	5	6	7	8
t_8	7	90	437	1026	1227	720	172	0	−1
t_9	7	84	426	852	1011	624	164	0	−1
t_{10}	7	84	376	822	984	564	143	0	−1
t_{11}	7	76	310	636	715	432	116	0	−1
t_{12}	7	66	245	476	525	322	91	0	−1
t_{13}	7	92	466	1156	1483	952	248	0	−1
t_{14}	7	92	464	1138	1432	892	223	0	−1
t_{15}	7	92	460	1110	1356	808	191	0	−1
t_{16}	7	92	460	1112	1368	824	200	0	−1
t_{17}	7	92	462	1128	1411	872	216	0	−1
t_{18}	7	88	418	960	1159	720	188	0	−1
t_{19}	7	88	410	920	1075	640	160	0	−1
t_{20}	7	88	412	930	1096	660	167	0	−1
t_{21}	7	82	349	732	829	498	131	0	−1
t_{22}	7	84	362	764	867	520	136	0	−1
t_{23}	7	90	433	996	1185	714	179	0	−1

Table 6
Unlabeled trees on n vertices, $n \leqslant 8$.



References

[1] L. Collatz and U. Singowitz, Spektren Endlicher Graphen, Abh. Math. Sem. Univ. Hamburg 21 (1957) 63–77.

[2] R.L. Graham and H.O. Pollak, On the addressing problem for loop switching, Bell System Tech. J. 50 (8) (October 1971).

[3] F. Harary, Graph Theory (Addison–Wesley, Reading Mass., 1969).

[4] A. Mowshowitz, The characteristic polynomial of a graph, J. Combin. Theory 12(B) (1972) 177–193.

[5] A. Schwenk, On the spectrum of a graph, Doctoral Dissertation, University of Michigan, Ann Arbor, Mich. (1972).

[6] S.S. Shrikhande, The uniqueness of the L_2 association scheme, Ann. Math. Statist. 30 (1959) 781–798.