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Note

Covering the Positive Integers by Disjoint Sets of the Form $\{[n\alpha + \beta] : n = 1, 2, \ldots\}$

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For real numbers $\alpha > 0$ and β, let $S(\alpha, \beta)$ denote the set of integers $\{[\alpha n + \beta] : n = 1, 2, 3, \ldots\}$ where, as usual, $[x]$ denotes the greatest integer $\leqslant x$. A finite family $\{S(\alpha_i, \beta_i) : 1 \leqslant i \leqslant r\}$ of these sets is said to be an *eventual covering family* (ECF) if every sufficiently large integer occurs in exactly one $S(\alpha_i, \beta_i)$.

It is well known (e.g., see [11], [1], [6], [7]) that if all β_i are zero then the only ECF's are:

(i) $r = 1, \alpha_1 = 1$;
(ii) $r = 2$, α_1 irrational with $1/\alpha_1 + 1/\alpha_2 = 1$.

However, if the β_i are allowed to be nonzero then a greater variety of ECF's is possible. For example, $\{S(2, 0), S(2, 1)\}$, $\{S(3/2, 1), S(3, 0)\}$, $\{S(9/7, 0), S(9/2, -1/2)\}$, $\{S(7/4, 0), S(7/2, -1), S(7, -3)\}$ and $\{S(2\alpha_1, 0), S(2\alpha_1, \alpha_1), S(\alpha_2, 0)\}$ where α_1 is irrational and $1/\alpha_1 + 1/\alpha_2 = 1$, are all ECF's.

In general, the problem of characterizing all ECF's seems to be difficult. Even in the case in which all the α_i and β_i are assumed to be integers, only limited success has been achieved [10], [12]. In this case, following Erdös [2], [3], we call the ECF a family of *exact covering congruences*. It is easily seen that if $\{S(a_i, b_i) : 1 \leqslant i \leqslant r\}$ and $\{S(a_i', b_i') : 1 \leqslant i \leqslant r'\}$ are families of exact covering congruences, and $\{S(\alpha_1, \beta_1), S(\alpha_2, \beta_2)\}$ is an ECF, then

$$\left\{\bigcup_{i=1}^{r} S(\alpha_1 a_i, \alpha_1 b_i + \beta_1)\right\} \cup \left\{\bigcup_{i=1}^{r'} S(\alpha_2 a_i', \alpha_2 b_i' + \beta_2)\right\} \tag{1}$$

is an ECF. The main result of this note is the following.

THEOREM. *Any* ECF *in which some* α_i *is irrational must be of the form* (1).

Thus, because of results of Skolem (Fact 4 below), the Theorem implies that the complexity of any ECF is essentially no greater than the complexity of the two exact covering congruences from which it must be constructed.

The proof of the theorem will require several preliminary results.

FACT 1. If $1/\alpha_1$, $1/\alpha_2$ and 1 are linearly independent over the rationals then $S(\alpha_1, \beta_1)$ and $S(\alpha_2, \beta_2)$ have infinitely many common elements.

This well-known result can be proved by a straightforward application of an approximation theorem of Kronecker and appears, for example, in [2], [11] or [7].

FACT 2. If some α_i in an ECF is irrational then all α_i in the ECF are irrational.

This follows from the fact [7] that for α irrational $\{n\alpha (\text{mod } 1): n = 1, 2, 3,...\}$ is dense in $[0, 1)$.

FACT 3. Suppose $S(\alpha_1, \beta_1)$ and $S(\alpha_2, \beta_2)$ are disjoint. Then either

(i) α_1/α_2 is rational,

or

(ii) there exist positive integers a_1, a_2 such that

$$a_1/\alpha_1 + a_2/\alpha_2 = 1 \qquad \text{and} \qquad a_1\beta_1/\alpha_1 + a_2\beta_2/\alpha_2 \equiv 0 (\text{mod } 1).$$

This is a corrected form of a result of Skolem [1, Satz 6]. The original statement in [1] neglects to allow for the possibility (i); however, modulo this oversight, the proofs for (ii) hold and the result is valid (cf. [7] or [4]).

FACT 4 (Skolem [8]). If α_1 is irrational then $\{S(\alpha_1, \beta_1), S(\alpha_2, \beta_2)\}$ is an ECF if and only if

$$1/\alpha_1 + 1/\alpha_2 = 1 \qquad \text{and} \qquad \beta_1/\alpha_1 + \beta_2/\alpha_2 \equiv 0 (\text{mod } 1).$$

As one might suspect, it is not difficult to deduce this result from the preceding fact.

Proof of the Theorem. Suppose $\mathscr{F} = \{S(\alpha_i, \beta_i) : i = 1,..., m\}$ is an ECF with some α_i irrational. By Fact 2 we may assume all α_i are irrational. Partition the α_i into equivalence classes $C_1, ..., C_t$ by the condition that α_i and α_j belong to the same C_k if and only if α_i/α_j is rational. Choose a fixed representative $\alpha_i^* \in C_i$, $1 \leqslant i \leqslant t$.

First, suppose $t \geqslant 3$ Applying Fact 3 to $S(\alpha_1^*, \beta_1^*)$, $S(\alpha_2^*, \beta_2^*)$ and $S(\alpha_3^*, \beta_3^*)$, all of which must be pairwise disjoint, we have (since (i) never holds) for some choice of *positive* integers $A_1, ..., A_6$

$$A_1/\alpha_1^* + A_2/\alpha_2^* = 1, \quad A_3/\alpha_1^* + A_4/\alpha_3^* = 1, \quad A_5/\alpha_2^* + A_6/\alpha_3^* = 1. \tag{2}$$

Since $1/\alpha_1^*$ is irrational, the determinant

$$\begin{vmatrix} A_1 & A_2 & 0 \\ A_3 & 0 & A_4 \\ 0 & A_5 & A_6 \end{vmatrix}$$

must vanish. But its value is just $-(A_2A_3A_6 + A_1A_4A_5)$ which *cannot* vanish for positive A_i.

Hence, we may assume $t \leqslant 2$. Since $\mathscr{F}$ is an ECF then density considerations immediately imply

$$\sum_{i=1}^{m} 1/\alpha_i = 1. \tag{3}$$

But if $t = 1$ then (3) would have the form

$$\frac{1}{\alpha_1^*} \sum_{i=1}^{m} \alpha_1^*/\alpha_i = \frac{1}{\alpha_1^*} \sum_{i=1}^{m} r_i = 1, \tag{3'}$$

where the r_i are rational, which is clearly impossible. Therefore, we must have $t = 2$. Define R_i, $i = 1, 2$, by

$$R_i = \frac{1}{\alpha_1^*} \sum_{\alpha \in C_i} \alpha, \qquad i = 1, 2.$$

By the definition of C_i, we see that R_i is rational. Thus, (3) becomes

$$R_1/\alpha_1^* + R_2/\alpha_2^* = 1. \tag{3''}$$

Let $\alpha_{i_1} \in C_1$, $\alpha_{i_2} \in C_2$ and consider the sets $S(\alpha_{i_1}, \beta_{i_1})$ and $S(\alpha_{i_2}, \beta_{i_2})$. Since these are disjoint then by Fact 3 there exist positive integers A_{i_1}, A_{i_2} such that

$$A_{i_1}/\alpha_{i_1} + A_{i_2}/\alpha_{i_2} = 1. \tag{4}$$

Thus

$$\left(\frac{A_{i_1}\alpha_1^*}{\alpha_{i_1}}\right) \frac{1}{\alpha_1^*} + \left(\frac{A_{i_2}\alpha_2^*}{\alpha_{i_2}}\right) \cdot \frac{1}{\alpha_2^*} = 1. \tag{5}$$

Since $A_{i_1}\alpha_1{}^*/\alpha_{i_1}$ and $A_{i_2}\alpha_2{}^*/\alpha_{i_2}$ are rational and $\alpha_1{}^*$ is irrational then (3″) and (5) imply

$$R_1 = A_{i_1}\alpha_1{}^*/\alpha_{i_1}, \qquad R_2 = A_{i_2}\alpha_2{}^*/\alpha_{i_2}. \tag{6}$$

Letting $\alpha_i' = \alpha_i{}^*/R_i$, $i = 1, 2$, we have $\mathscr{F} = \mathscr{F}_1 \cup \mathscr{F}_2$ where

$$\mathscr{F}_i = \bigcup_{\alpha_j \in C_i} S(A_j\alpha_i', \beta_j), \qquad i = 1, 2.$$

By Fact 3, if $\alpha_{i_1} \in C_1$, $\alpha_{i_2} \in C_2$ then not only does (4) hold but also

$$A_{i_1}\beta_{i_1}/\alpha_{i_1} + A_{i_2}\beta_{i_2}/\alpha_{i_2} \equiv 0 \pmod 1. \tag{7}$$

By (6) and the definition of α_1', we can write (7) as

$$\beta_{i_1}/\alpha_1' + \beta_{i_2}/\alpha_2' \equiv 0 \pmod 1. \tag{7'}$$

Holding i_2 fixed and recalling that $\alpha_1{}^* \in C_1$, we have by subtraction

$$\beta_j/\alpha_1' - \beta_1{}^*/\alpha_1' \equiv 0 \pmod 1 \tag{8}$$

for $\alpha_j \in C_1$. This implies

$$\beta_j - \beta_1{}^* = M_j\alpha_1' \tag{9}$$

for $\alpha_j \in C_1$ and some choice of integers M_j. Similar arguments for C_2 show that

$$\beta_k - \beta_2{}^* = M_k'\alpha_2' \tag{9'}$$

for $\alpha_k \in C_2$ and some choice of integers M_k'.

Thus, $\mathscr{F}$ can be written as

$$\mathscr{F} = \mathscr{F}_1 \cup \mathscr{F}_2 = \left\{ \bigcup_{\alpha_j \in C_1} S(A_j\alpha_1', M_j\alpha_1' + \beta_1{}^*) \right\}$$
$$\cup \left\{ \bigcup_{\alpha_k \in C_2} S(A_k\alpha_2', M_k'\alpha_2' + \beta_2{}^*) \right\}$$

where $\bigcup_{\alpha_j \in C_1} S(A_j, M_j)$ and $\bigcup_{\alpha_k \in C_2} S(A_k, M_k')$ are families of exact covering congruences since

$$1/\alpha_1' + 1/\alpha_2' = 1, \qquad \beta_1{}^*/\alpha_1' + \beta_2{}^*/\alpha_2' \equiv 0 \pmod 1$$

imply by Fact 3 that $\{S(\alpha_1', \beta_1{}^*), S(\alpha_2', \beta_2{}^*)\}$ is an ECF. This proves the Theorem.

We remark that a result of Mirsky and Newman (cf. [2]) asserts that if $\{\bigcup_{i=1}^{r} S(a_i, b_i)\}$ is a family of exact covering congruences with $r \geqslant 2$ then $a_i = a_j$ for some $i \neq j$. This can be combined with the Theorem to yield the following result.

COROLLARY. *If* $\{S(\alpha_i, \beta_i) : i = 1,..., r\}$ *is an* ECF *with some* α_i *irrational and* $r \geqslant 3$ *then* $\alpha_i = \alpha_j$ *for some* $i \neq j$.

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