ON SUMS OF INTEGERS TAKEN FROM A FIXED SEQUENCE

by

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INTRODUCTION

Given a set S of real numbers, it is natural to inquire as to the structure of the set of all real numbers which can be formed by taking sums of elements from S. In particular, one may require that the sums be finite sums although a substantial amount of work has been done for the case of infinite sums, e.g., in the study of generalized limits and summability.

We shall be primarily concerned with the case in which S is a set of positive integers although, again, a number of interesting results are available when S consists of unit fractions, for example, as well as for other special sets (cf. [12], [16], [17], [18], [42]). Henceforth, unless specified otherwise, we shall assume $S = \{s_1, s_2, \ldots\}$ is a (possibly finite) set of positive integers. In general, we are forming

$$\Sigma(S) = \left\{ \sum_{k=1}^{\infty} \alpha_k s_k \ : \ \alpha_k \text{ is a nonnegative integer, } \sum_{k=1}^{\infty} \alpha_k < \infty \right\}.$$

Of course, this is just the subsemigroup generated by S considered as a subset of the commutative semigroup of positive integers under addition.

There are several directions one might proceed.

1. <u>No additional restrictions on the α_k</u>. In this case it is well-known [35] that $\Sigma(S)$ consists of all sufficiently large multiples of $\gcd(s_1, s_2, \ldots)$. When this gcd is 1, then $\Sigma(S)$ contains all sufficiently large integers and we let $\theta(S)$ denote the largest integer not in $\Sigma(S)$. The determination of $\theta(S)$ seems to be a difficult question in general ([2], [24], [27], [35]). If $S = \{a,b\}$ with $\gcd(a,b) = 1$ then $\theta(S) = ab - a - b$ ([35]). However, $\theta(S)$ is not known for even the set $\{a,b,c\}$ with $\gcd(a,b,c) = 1$ although other special cases are known, e.g., for $S = \{n, n+1, n+2, n+6\}$, $\theta(S) = n\left[\frac{n}{6}\right] + 2\left[\frac{n}{6}\right] + 2\left[\frac{n+1}{6}\right] + 5\left[\frac{n+2}{6}\right] + \left[\frac{n+3}{6}\right] + \left[\frac{n+4}{6}\right] + \left[\frac{n+5}{6}\right] - 1$ (cf. [35]). Other special cases for which $\theta(S)$ is known include the following ([10]): For a given $k \geq 0$, let $S = \{0 < s_1 < \cdots < s_n = 2n+k\}$. Then for n sufficiently large,

$$\theta(S) \leq \begin{cases} 2n + 4k - 1 & \text{for} \quad n - k \equiv 1(3) \\ 2n + 4k + 1 & \text{for} \quad n - k \not\equiv 1(3) \end{cases}$$

and this bound is best possible. However, we shall not pursue this topic further in this paper.

2. <u>For some given m,</u> $\sum_{k=1}^{\infty} \alpha_k \leq m$. In this case, if $\theta(S) = 0$, S is known as a basis of order m. A number of

strong results have been established in this area, beginning with Lagrange's theorem that the squares from a basis of order 4 and Hilbert's proof of the Waring conjecture that the n^{th} powers form a basis of order m for some $m = g(n)$ ([23]). More recently, the names of Erdös, Kneser, Linnik, Mann, and Vinogradov, among others, have been prominent in connection with this subject (cf. [22]), but space does not permit us to pursue this further. For the reader who would like to have his name added to this distinguished list, I might suggest that he show that the set of primes together with 1 forms a basis of order 3.

3. <u>All</u> α_k <u>are 0 or 1</u>. It is this case we wish to make the main topic of the paper. We make a slight modification in our notation. For the sequence $S = (s_1, s_2, \ldots)$ let

$$P(S) \equiv \left\{ \sum_{k=1}^{\infty} \varepsilon_k s_k : \varepsilon_k = 0 \text{ or } 1, \sum_{k=1}^{\infty} \varepsilon_k < \infty \right\}.$$ Let us call

S <u>complete</u> if all sufficiently large integers belong to P(S). (In this case $\theta(S)$ has the obvious meaning.) Finally, call S <u>strongly complete</u> if S remains complete after any finite number of terms have been deleted from S.

Note that a sequence which forms a basis of order m can only grow algebraicly, whereas a complete sequence may grow exponentially, e.g., $s_n = 2^n$. Thus, the set of complete sequences is (in fact) much richer than the set of bases.

COMPLETE SEQUENCES

The first results in this area were established by R. Sprague in 1948. For the sequence $S = (s_1, s_2, \ldots)$ Sprague showed [41] that for $s_n = n^2$, S is complete with threshold $\theta(S) = 128$. Further, using results of the Tarry-Escott problem, he proved [42] that for each k, the sequence defined by $s_n = n^k$ is complete.

These methods were also employed by E. Krubeck in 1953 [29] to show that for any polynomial f(x) with integer coefficients and positive leading coefficient, if the sequence S is defined by $s_n = f(n)$ then the difference between consecutive elements of P(S) is bounded. (More precise information for this sequence will be discussed later.)

The results of [42] were also generalized by H.-E. Richert [38]. Basically, he showed that if a sufficiently long segment of integers belongs to P(S) and S does not grow too irregularly then S is complete. Richert used this to establish $\theta(S) = 33$ for $s_n = \frac{n(n+1)}{2}$ [38] and $\theta(S) = 6$ when S is the sequence of primes [37].

In 1952, C. G. Lekkerkerker [30] pointed out that the well known sequence of Fibonacci numbers, given by $s_1 = s_2 = 1$, $s_{n+2} = s_{n+1} + s_n$, is not only complete but, in fact, each positive integer has a unique representation in the form $\Sigma \varepsilon_k s_k$ where $\varepsilon_k \varepsilon_{k+1} = 0$. Further extensions in this direction may be found in [3], [7], [25], [26], [28].

Up to this point the results discussed have been limited either by the very special properties required of S or by the fact that it is necessary to hypothesize that P(S) contains a long segment of integers before completeness can be established. These restrictions make it difficult to show, for example, that a sequence is strongly complete.

The first strong result concerning complete sequences was given by K. F. Roth and G. Szekeres in 1954 [39]. They proved that if S is eventually increasing then the following two conditions imply that S is complete:

(i) $\lim_{k\to\infty} \log s_k/\log k$ exists;

(ii) $\inf_{\alpha} \left\{(\log k)^{-1} \sum_{i=1}^{k} \|s_i\alpha\|^s\right\} \to \infty$ as $k \to \infty$,

where $\|\theta\|$ denotes the distance of θ to the nearest integer and $s_k/2 < \alpha \leq 1/2$.

In fact, with the use of a saddle-point method, an asymptotic formula for the number of representations of n as an element of P(S) is derived in [39]. Note that the above conditions imply S is strongly complete. In particular, it follows (using results of L. K. Hua to verify (ii)) that if f is a polynomial mapping integers to integers with a positive leading coefficient and such that for all primes p there exists an m such that $p \nmid mf(m)$ then the sequence $(f(p_1), f(p_2), \ldots)$ is strongly complete where p_n denotes the n^{th} prime.

The next advance was made by B. J. Birch [1] in 1959 who showed that for relatively prime integers p and q (greater than 1), the increasing sequence formed from all the terms $p^a q^b$, with a and b ≥ 0, is complete. His proof is elementary and settles a conjecture made earlier by Erdös.

The following year a far-reaching generalization of Birch's result was published by J. W. S. Cassels [6]. For an increasing sequence S, let S(n) denote the number of terms of S not exceeding n. Cassels' result can be stated as follows: Suppose $S = (s_1, s_2, \ldots)$ is an increasing sequence of positive integers satisfying

$$(*) \quad \lim_{n\to\infty} \frac{S(2n) - S(n)}{\log \log n} = \infty;$$

$$(\ast\ast) \quad \sum_{k=1}^{\infty} \|s_k \alpha\| = \infty \text{ for all } \alpha \in (0,1).$$

Then S is strongly complete.

The proof given by Cassels is ingenious and uses the Hardy-Littlewood method. The conditions (*) and (**) bear a certain similarity to the conditions (i) and (ii) of Roth and Szerkeres [39] although Cassels was unaware of their results at the time his paper was written. Not only does this theorem imply the result of Birch, it also shows that for any polynomial f mapping integers to integers the sequence given by $s_n = f(n)$ is strongly complete provided only that f satisfies the obvious necessary conditions, i.e., f has a positive leading coefficient and no prime divides all the function values f(n). Furthermore, Cassels' theorem applies to sequences which grow rapidly, e.g., such that $s_n \sim \exp((n/\log n)^{1-\varepsilon})$, $\varepsilon > 0$.

In 1962, the author gave a number of theorems dealing with complete sequences, several of which will now be mentioned. All polynomials f mapping reals to reals for which $S(f) = (f(1), f(2), \ldots)$ is complete were characterized in [15]. Specifically, if f(x) is expressed as $\sum_{k=0}^{n} \alpha_k \binom{x}{k}$ then S(f) is complete if and only if:

(1) $\alpha_k = p_k/q_k$ for integers p_k, q_k with $(p_k, q_k) = 1$ and $q_k \neq 0$, $0 \leq k \leq n$;

(2) $\alpha_n > 0$;

(3) $\gcd(p_0, p_1, \ldots, p_n) = 1$.

This proof is elementary.

It had been conjectured by Erdös [9] that if $t > 0$, $1 < \alpha < 2$, the sequence $S(t,\alpha)$ defined by $s_n = [t\alpha^n]$ is complete. It was shown in [14] that this is not quite correct, e.g., $S(1,\alpha)$ is complete if and only if $1 \leq \alpha < \sqrt[3]{5}$. The subset R of the square $0 < t \leq 1$, $1 \leq \alpha \leq 2$, for which $S(t,\alpha)$ is complete was determined in [14]. R is rather complicated; for example, for any k there are vertical lines L such that $R \cap L$ has more than k components.

In [19] it was shown that the sequence S given by $s_n = F_n - (-1)^n$ (where F_n is the n^{th} Fibonacci number) is strongly complete but fails to remain complete if <u>any</u> infinite subsequence is removed. Very few sequences are currently known to enjoy this unlikely property. This proof in [19] is elementary but somewhat involved.

Also in 1962, Erdös [8] proved the following theorem dealing with slowly growing sequences. Suppose $S = (s_1, s_2, \dots)$ is a strictly increasing sequence of positive integers satisfying

(i) For some $\alpha \le (\sqrt{5} + 1)/2$, and some c, $s_n \le cn^{\alpha}$.

(ii) P(S) contains an element from every arithmetic progression.

Then S is complete.

Erdös further conjectured that this would be true if s_n were only required to satisfy $s_n \le cn^{2-\varepsilon}$ for some $\varepsilon > 0$.

This conjecture was settled by Jon Folkman in 1966 [13] in a very ingenious paper in which the following result is established. Suppose S is a nondecreasing sequence satisfying (1) or a strictly increasing sequence satisfying (2) for some c and some α, $0 \le \alpha < 1$:

(1) $s_n \le cn^{\alpha}$;

(2) $s_n \le cn^{1+\alpha}$.

Then P(S) contains an infinite arithmetic progression. Furthermore if P(S) intersects every arithmetic progression then S is complete.

Folkman's proof is elementary and quite clever.

In 1968, S. A. Burr [5], using the results of [13], showed that if $s_n = f(n) + t(n)$ where f is a polynomial of degree ≥ 1 and having a positive leading coefficient and for some β, $0 \leq \beta < 1/2$, $t(n) = O(n^{\beta})$, then P(S) contains an infinite arithmetic progression. Further, if for all primes p, there exist infinitely many s_k not divisible by p then S is strongly complete. Thus, if polynomial sequences are perturbed slightly, completeness properties are not significantly affected.

However, the following was recently established by Erdös [9]: For any sequence S of positive integers, if g is any positive function for which $\sum_n 1/g(n) < \infty$ then there exists a sequence S′ such that $|s_n - s_n'| \leq g(n)$ for n sufficiently large and P(S′) does not contain an infinite arithmetic progression. It follows in particular that if a sequence of polynomial values f(n) is perturbed by as much as $O(n^{1+\varepsilon})$ then completeness properties may be significantly affected.

In this same direction, J. W. S. Cassels [6] established the following result. For every $\varepsilon > 0$ and $\eta > 0$ there exists a sequence S containing infinitely many terms in every arithmetic progression such that $s_{n+1} - s_n = O(s_n^{\frac{1}{2}+\eta})$ and P(S) contains less than εx integers $\leq x$ for all sufficiently large x. It also follows from his arguments that if $f(x) = \sum_{k=0}^{n} \alpha_k x^k$ is a monic polynomial defining a P-V number (cf. [36]) then any sequence S satisfying $\sum_{k=0}^{n} \alpha_k s_{t+k} = 0$, $t \geq 0$, is <u>not</u> strongly complete. This includes, for example, the case $s_n = F_n$ or even finite repetitions of this sequence. It is true though that if each term is repeated sufficiently often (varying with the term) then the sequence is strongly complete. For example, in a recent note of Erdös and the author [11], it is shown that for the sequence S formed by taking m_k copies of F_k, if $m_k(2/(1+\sqrt{5}))^k$ decreases then S is strongly complete if and only if $\sum_{k=1}^{\infty} m_k(2/(1+\sqrt{5}))^k = \infty$. On the other hand, it is always true that S is not strongly complete if $\sum_{k=1}^{\infty} m_k(2/(1+\sqrt{5}))^k < \infty$.

An alternative approach to some of these sequences has recently been given by S. A. Burr [4] in which a computer is used to provide the induction step needed in the proof of the lack of strong completeness. Thus, it is seen that even rather well-behaved sequences may fail to be complete.

If $S = (s_1, s_2, \ldots)$ is strongly complete then for each n, $\theta_S(n)$, the greatest integer which does not belong to $P((s_{n+1}, s_{n+2}, \ldots))$, is a well-defined integer. The study of the dependence of $\theta_S(n)$ on n seems to be very difficult, even for relatively simple sequences S. For example, if S is the sequence of squares, then it has been shown by Ju. v. Linnik and the author [21] that for $k = 1,2,\ldots$, the sequence of functions

$$g_k(x) = \frac{\theta_S(4^k x)}{(4^k x)^2}, \quad 1 \leq x \leq 4, \ 4^k x \text{ an integer, converges to}$$

a function $g(x)$, $1 \leq x \leq 4$, which is the union of a (large) finite number of portions of parabolas. In particular, $g(x)$ lies between 4 and 5 and, in fact, attains the value 5 exactly 18 times for $1 \leq x \leq 4$. The proofs rely on deep results of Linnik [32] and A. V. Malyšev [33], [34] on the distribution of lattice points on quadratic surfaces in 4 and more dimensions.

Computer results of Lin [31] indicate that $\theta_S(n)/s_n$ often seems to approach a limit, e.g., when S is the increasing sequence of primes, $\theta_S(n)/s_n$ appears to be tending to 3. Of course, $\lim_n \theta_S(n)/s_n = 3$ would imply the well known Goldbach conjecture. A few isolated values of $\theta_S(1)$ are known (cf. [31], [20]), some of which are listed below.

s_n	$\theta_S(1)$
$\frac{n(n+1)}{2}$	33
n^2	128
$n^2 + 1$	51
$n^2 + 2$	91
$n^2 + 3$	120
$n^2 + 4$	92
$n^2 + 5$	117
$(n+1)^2 - 1$	156
n^3	12758
$n^3 + 1$	8293
n^4	5134240
$a(n+1) + b$ with $(a,b) = 1$	$\frac{(a-2)a(a+1)}{2} + ab + 1$

Table 1

Some Values of $\theta_S(1)$

SOME OPEN QUESTIONS

We conclude with a number of open questions related to complete sequences.

1. (J. Folkman) Let $S = (s_1, s_2, \dots)$ be a <u>nondecreasing</u> sequence of integers which satisfies $s_n < cn$ for all n. Does $P(S)$ contain an infinite arithmetic progression? This is true if $s_n < cn^{1-\varepsilon}$ and can fail for $s_n < cn^{1+\varepsilon}$; cf. [13].

2. Let $S = (s_1, s_2, \dots)$ with $s_n = [t\alpha^n]$. For what pairs (t,α), $t > 0$, $1 < \alpha < 2$, is S complete? For $0 < t \leq 1$, this is known [14]. Even in the range $1 < t < 2$, it is not known what happens. Conceivably, S is complete for all $1 < \alpha < \frac{1+\sqrt{5}}{2}$ and $t > 0$.

3. For a sequence S let $C(n)$ and $N(n)$ denote the conditions:

$C(n)$: If any n entries are removed from S to form S' then S' is complete.

$N(n)$: If any n entries are removed from S to form S' then S' is not complete.

For what values of $m < n$ are there sequences which satisfy both $C(m)$ and $N(n)$? For example, $S = (1,2,4,\dots,2^n,\dots)$ satisfies $C(0)$ and $N(1)$, while $S = (1,1,2,3,5,8,13,\dots)$ satisfies $C(1)$ and $N(2)$. In particular, is there a sequence which satisfies $C(2)$ and $N(3)$?

4. Let $S = (s_1, s_2, \dots)$ be an increasing sequence which is strongly complete but such that any subsequence

S' formed from S by deleting an infinite number of entries is not complete (e.g., see [19]). What can be said about the structure of S? For example, is it true $s_{n+1}/s_n \to \frac{1+\sqrt{5}}{2}$?

5. (P. Erdös) Given $\varepsilon > 0$, is there a strongly complete sequence $S = (s_1, s_2, \ldots)$ for which $s_{n+1}/s_n > 2 - \varepsilon$ for n sufficiently large? Can $s_{n+1}/s_n \to 2$ as $n \to \infty$?

(S. A. Burr) 6. Suppose $S = (s_1, s_2, \ldots)$ is a sequence of positive integers of the form $s_n = f(n) + \gamma_n$ where $f(x)$ is a polynomial and $\gamma_n = O(n)$. Is S subcomplete? This is known to be true if $\gamma_n = O(n^{\frac{1}{2}-\varepsilon})$ (cf. S. A. Burr, On the completeness of perturbed polynomial values (to appear in Pac. Jour. of Math.)). On the other hand examples exist with $\gamma_n = O(n^{1+\varepsilon})$ for which S is <u>not</u> subcomplete.

7. It is known [18] that the sequence with n^{th} term given by $s_n = n + 1/n$ is strongly complete. What about $s_n = n^2 + 1/n$? What about $s_n = f(n) + 1/n$ where $f(n)$ forms a strongly complete sequence?

8. Suppose $0 < \alpha_1 < \ldots < \alpha_k \le x$ is a sequence of real numbers with k maximal such that any two sums $\sum_{j=1}^{k} \epsilon_j \alpha_j$, $\epsilon_j = 0$ or 1, differ by at least 1. It is true that $k \le \frac{\log x}{\log 2} + O(1)$? (This strengthens a well-known conjecture of Erdös.)

9. If $S = (s_1, s_2, \ldots)$ is a strongly complete increasing sequence, let $\theta_s(n)$ denote the largest integer x for which $x \notin P(s_{n+1}, s_{n+2}, \ldots)$. It is of considerable interest to study the behavior of $\theta_s(n)$ for various sequences S. For the sequence in which $s_n = n^2$, recent results of R. L. Graham and Ju. V. Linnik show that $\theta_s(n)/s_n$ asymptotically oscillates between 4 and 5 in a predictable but complicated way. For other sequences, computer results (S. Lin, Computer experiments on sequences which form integral bases, Computations problems in abstract algebra, Pergammon Press (1969)) indicate that $\lim \theta_s(n)/s_n = \alpha_s$ exists, e.g., $\alpha_s \overset{?}{=} 3$ for s_n = the n^{th} prime.

10. Let p be a prime and suppose $a_1, \ldots, a_k$ are distinct nonzero elements of Z_p. <u>Conjecture</u>: There always exists an arrangement $a_{i_1}, \ldots, a_{i_k}$ of the a_i such that all partial sums $\sum_{j=1}^{t} a_{i_j}$, $1 \le t \le k$, are distinct modulo p.

11. Let p be a prime and suppose $a_1, \ldots, a_p \in Z_p$ such that for some r, $\sum_{b \in B \subseteq A} b \equiv 0 \pmod p$ implies $|B| = r$. <u>Conjecture</u>: The a_i assume at most 2 different values.

12. Let α and β be positive reals with α/β irrational. Let S denote the sequence $([\alpha], [\beta], [2\alpha], [2\beta], \ldots, [2^n\alpha], [2^n\beta], \ldots)$. Is S complete? What if 2 is replaced by some γ, $1 < \gamma < 2$?

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