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SOME RESULTS ON MATCHING IN BIPARTITE GRAPHS*

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1. Introduction. Let G be a finite bipartite graph[1] with sets of vertices A and B and with a set of edges E between A and B. For $X \subseteq A$ let $D(X) \subseteq B$ be defined by

$$D(X) = \{b \in B : \{x, b\} \in E \text{ for some } x \in X\}.$$

In other words $D(X)$ is the set of all vertices in B connected by an edge of G to a vertex in X. If v_A and v_B are measures on A and B respectively (which by the finiteness of G will just mean the assignment of a positive weight to each vertex), we say that v_B *dominates* v_A if

$$(1) \qquad v_B(D(X)) \geqq v_A(X) \quad \text{for all} \quad X \subseteq A.$$

Let $A_1, \cdots, A_m$ and $B_1, \cdots, B_n$ be partitions of A and B respectively into non-empty disjoint subsets. Let G' denote the *quotient* bipartite graph with sets of vertices $A' = \{A_1, \cdots, A_m\}$ and $B' = \{B_1, \cdots, B_n\}$ and a set of edges E' defined such that $\{A_i, B_j\} \in E'$ if and only if $\{a, b\} \in E$ for some $a \in A_i$, $b \in B_j$. Of course, $v_{A'}$ and $v_{B'}$ denote the natural induced measures on A' and B', and D' is defined in the obvious way.

It is important to note that if all vertices have weight 1 and with $a \in A$ we associate the subset $D(\{a\}) \subseteq B$, then by the marriage theorem of P. Hall (cf. [3]) v_B dominates v_A if and only if there exists a system of distinct representatives (cf. [3]) for these subsets.[2] If $b(a)$ is the representative chosen from $D(\{a\})$, then the mapping $a \to b(a)$ is a *matching* of A into B, i.e., a $1-1$ mapping of A into B such that $\{a, b(a)\}$ is an edge of G. It is this application of our results which motivated the present study (cf. Examples 1 and 2). Our main object in this note is to develop several theorems which will enable one to show that v_B dominates v_A by examining the structure of the (hopefully much simpler) quotient graph G'.

2. The main results. Define the matrix $R = (\rho_{ij})$ by

$$\rho_{ij} = \inf_{\varnothing \neq X \subseteq A_i} \frac{v_B(D(X) \cap B_j)}{v_A(X)}, \quad 1 \leqq i \leqq m, \quad 1 \leqq j \leqq n.$$

By the definition of ρ_{ij} we have

$$(2) \qquad \rho_{ij} v_A(X \cap A_i) \leqq v_B(D(X) \cap B_j)$$

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[1] We assume G has no isolated vertices.

[2] It is true in general that v_B dominates v_A if and only if there exists a flow on the edges of G taking v_A into v_B (cf. [1]). However, the concept of a flow will not enter into the subsequent development.

for all i, j and all $X \subseteq A$. (Because G has no isolated vertices, R has positive row sums.) For any nonnegative $m \times n$ matrix R with positive row sums, define a real-valued function $\mathscr{L}$ by

$$\mathscr{L}(R) = \inf_S \max_j \left(\sum_i \sigma_{ij}\right),$$

where the inf is taken over all $m \times n$ matrices $S = (\sigma_{ij})$ with $\sigma_{ij} \geqq 0$ and $\sum_j \sigma_{ij}\rho_{ij} \geqq 1$ for all i.

FACT 1. For any matrix R on which $\mathscr{L}$ is defined, there exists a matrix $T = (\tau_{ij})$ such that $\tau_{ij} \geqq 0$,

$$\sum_j \tau_{ij}\rho_{ij} = 1 \quad \text{for all } i, \quad \sum_i \tau_{ij} \leqq \mathscr{L}(R) \quad \text{for all } j.$$

If R is strictly positive, then also $\sum_i \tau_{ij} = \mathscr{L}(R)$ for all j.

These facts follow from simple compactness and variational arguments.

THEOREM 1. *$\mathscr{L}(R) \leqq 1$ implies v_B dominates v_A.*

Proof. Let $X \subseteq A$. Then

$$\begin{aligned} v_B(D(X)) &= \sum_j v_B(D(X) \cap B_j) \\ &\geqq \sum_j \left(\sum_i \tau_{ij}\right) v_B(D(X) \cap B_j) \qquad \text{(for the matrix } T = (\tau_{ij}) \text{ in Fact 1)} \\ &= \sum_{i,j} \tau_{ij} v_B(D(X) \cap B_j) \\ &\geqq \sum_{i,j} \tau_{ij}\rho_{ij} v_A(X \cap A_i) \\ &\geqq \sum_i \sum_j (\tau_{ij}\rho_{ij}) v_A(X \cap A_i) = \sum_i v_A(X \cap A_i) = v_A(X) \end{aligned}$$

and the theorem is proved.

The following result is no surprise.

FACT 2. If v_B dominates v_A then $v_{B'}$ dominates $v_{A'}$.

This is an immediate consequence of the fact that if $X' \subseteq A$ then

$$\begin{aligned} v_{A'}(X') &= v_A(X) \\ &\leqq v_B(D(X)) \\ &\leqq v_{B'}(D'(X')), \end{aligned}$$

where $X = \bigcup_{x' \in X'} x' \subseteq A$, by the definition of the *edges* of G'.

In the other direction we have the following theorem.

THEOREM 2. *Suppose $v_B(B_j)/v_A(A_i) = \rho_{ij}$ whenever $B_j \in D'(A_i)$, where the ρ_{ij} are defined previously. Then $v_{B'}$ dominates $v_{A'}$ implies v_B dominates v_A.*

Proof. Choose $\varnothing \neq X \subseteq A$ and let $\alpha_i = v_A(X \cap A_i)/v_A(A_i)$ for $1 \leqq i \leqq m$. We can assume that the A_i have been labeled so that $0 \leqq \alpha_1 \leqq \alpha_2 \leqq \cdots \leqq \alpha_m$.

By the hypothesis that v_B dominates v_A and the fact that $\alpha_{j+1} - \alpha_j \geqq 0$ we have

$$(3)\qquad \begin{aligned} \alpha_1 v_{A'}(A_1 \cup \cdots \cup A_m) &\leqq \alpha_1 v_{B'}(D'(A_1 \cup \cdots \cup A_m)),\\ (\alpha_2 - \alpha_1) v_{A'}(A_2 \cup \cdots \cup A_m) &\leqq (\alpha_2 - \alpha_1) v_{B'}(D'(A_2 \cup \cdots \cup A_m)),\\ &\vdots\\ (\alpha_{j+1} - \alpha_j) v_{A'}(A_{j+1} \cup \cdots \cup A_m) &\leqq (\alpha_{j+1} - \alpha_j) v_{B'}(D'(A_{j+1} \cup \cdots \cup A_m)),\\ &\vdots\\ (\alpha_m - \alpha_{m-1}) v_{A'}(A_m) &\leqq (\alpha_m - \alpha_{m-1}) v_{B'}(D'(A_m)). \end{aligned}$$

If we sum the *left-hand sides* of inequalities (3), we obtain

$$(4)\qquad \begin{aligned} &\alpha_1 v_{A'}(A_1) + \alpha_2 v_{A'}(A_2) + \cdots + \alpha_m v_{A'}(A_m)\\ &= \alpha_1 v_A(A_1) + \alpha_2 v_A(A_2) + \cdots + \alpha_m v_A(A_m) && \text{(by definition of } v_{A'})\\ &= v_A(X \cap A_1) + v_A(X \cap A_2) + \cdots + v_A(X \cap A_m) && \text{(by definition of } \alpha_i)\\ &= v_A(X). \end{aligned}$$

Similarly if we sum the *right-hand side* of (3) we obtain (after minor computations)

$$(4')\qquad \sum_j \left[v_B(B_j) \max_{B_j \in D'(A_i)} (\alpha_i)\right].$$

However, since $\rho_{ij} = v_B(B_j)/v_A(A_i)$ whenever $B_j \in D'(A_i)$ by hypothesis,

$$\frac{v_A(X \cap A_i)}{v_A(A_i)} \leqq \frac{v_B(D(X) \cap B_j)}{v_B(B_j)} \quad \text{for} \quad B_j \in D'(A_i).$$

Hence,

$$\frac{v_B(D(X) \cap B_j)}{v_B(B_j)} \geqq \max_{B_j \in D'(A_i)} \frac{v_i(X \cap A_i)}{v_A(A_i)} = \max_{B_j \in D'(A_i)} (\alpha_i)$$

and

$$(4'')\qquad \sum_j \left[v_B(B_j) \max_{B_j \in D'(A_i)} (\alpha_i)\right] \leqq \sum_j v_B(D(X) \cap B_j) = v_B(D(X)).$$

Therefore, combining (4), (4′), (4″), we have

$$v_A(X) \leqq \sum_j \left[v_B(B_j) \max_{B_j \in D'(A_i)} (\alpha_i)\right] \leqq v_B(D(X)),$$

i.e., v_B dominates v_A, and the theorem is proved.

COROLLARY. *Suppose each vertex in A_i has weight w_i and is connected to the same number p_{ij} of vertices in B_j, and each vertex in B_j has weight $\bar{w}_j$ and is connected to the same number q_{ij} of vertices in A_i. Then $v_{B'}$ dominates $v_{A'}$ implies v_B dominates v_A.*

Proof. For $\varnothing \neq X \subseteq A_i$, there are $p_{ij}|X|$ edges connected to vertices in B_j. Since each vertex of B_j can accept at most q_{ij} of these edges,

$$|D(X) \cap B_j| \geqq \frac{p_{ij}}{q_{ij}}|X|$$

and

$$\frac{v_B(D(X) \cap B_j)}{v_A(X)} = \frac{\bar{w}_j|D(X) \cap B_j|}{w_i|X|} \geqq \frac{\bar{w}_j p_{ij}}{w_i q_{ij}}$$
$$= \frac{v_B(B_j)}{v_A(A_i)}.$$

Since X was arbitrary, the hypothesis of Theorem 2 is satisfied and the corollary is proved. We shall present an application of this result in the final section.

To state the final result in this section, we require an additional definition. Let $\Delta = (\delta_{ij})$ denote an arbitrary $m \times n$ matrix with positive real entries. Let $\Delta^* = (\delta^*_{ij})$ be the $n \times m$ matrix defined by

$$\delta^*_{ij} = \delta_{ji}^{-1}, \qquad 1 \leqq i \leqq n, \quad 1 \leqq j \leqq m.$$

THEOREM 3. $\mathscr{L}(\Delta)\mathscr{L}(\Delta^*) \leqq 1$.

Proof. By Fact 1, there exists a matrix $T = (\tau_{ij})$ which satisfies $\tau_{ij} \geqq 0$, $\sum_j \tau_{ij}\delta_{ij} = 1$ for all i and $\sum_i \tau_{ij} = \mathscr{L}(\Delta)$ for all j. Define the $n \times m$ matrix $T^* = (\tau^*_{ji})$ by

$$\tau^*_{ji} = \frac{\tau_{ij}\delta_{ij}}{\mathscr{L}(\Delta)}.$$

Then $\tau^*_{ji} \geqq 0$ and

$$\sum_i \tau^*_{ji}\delta^*_{ji} = \sum_i \frac{\tau_{ij}\delta_{ij}\delta_{ij}^{-1}}{\mathscr{L}(\Delta)} = \frac{1}{\mathscr{L}(\Delta)}\sum_i \tau_{ij} = \frac{1}{\mathscr{L}(\Delta)} \cdot \mathscr{L}(\Delta) = 1.$$

However,

$$\sum_j \tau^*_{ji} = \sum_j \frac{\tau_{ij}\delta_{ij}}{\mathscr{L}(\Delta)} = \frac{1}{\mathscr{L}(\Delta)}$$

so that $\mathscr{L}(\Delta^*) \leqq 1/\mathscr{L}(\Delta)$ and the theorem is proved.

Note that if $v_B(D(a) \cap B_j)$, $a \in A_i$, is independent of a and $v_A(D^{-1}(b) \cap A_i)$, $b \in B_j$, is independent of b, then $\rho_{ij} = v_B(B_j)/v_A(A_i)$ and R^* is the R matrix corresponding to an attempt to show v_A dominates v_B. Hence, if $\rho_{ij} > 0$ for all i, j, then $\mathscr{L}(R)\mathscr{L}(R^*) \leqq 1$, which implies either $\mathscr{L}(\Gamma) \leqq 1$ or $\mathscr{L}(R^*) \leqq 1$, i.e., either v_B dominates v_A or v_A dominates v_B.

We mention in passing that it can be shown in fact that $\mathscr{L}(\Delta)\mathscr{L}(\Delta^*) = 1$ if and only if $\sigma_{ij} = \xi_i\eta_j$, i.e., $\sigma_{ij}\sigma_{kl} = \sigma_{il}\sigma_{kj}$ for all i, j, k, l.

3. Applications. In this section we apply the preceding results to several specific bipartite graphs.

Example 1. Let S be a set of s elements. For $\binom{s}{k} \leqq \binom{s}{l}$ form the graph G by letting $A = \{X \subseteq S : |X| = k\}$, $B = \{Y \subseteq S : |Y| = l\}$ and $\{X, Y\}$ is an edge of G if and only if $X \subseteq Y$ or $Y \subseteq X$. By choosing the trivial partitions $A = A_1$, $B = B_1$ and assigning weights of 1 to all vertices of G, we find that the corollary applies. Since G' has only two vertices, it is obvious that $v_{B'}$ dominates $v_{A'}$ and hence, v_B dominates v_A. This implies the well-known result that there is a matching of A into B.

The same arguments can be applied to the k- and l-dimensional subspaces of an s-dimensional vector space over a finite field, yielding the result that there is a matching between these two sets (which is also well known).

Example 2. Let S be a set of s elements. For $0 < k < s$, form the graph $G_k^s = G$ by letting $B = \{$partitions π_k of S into k nonempty subsets$\}$, $A = \{$partitions π_{k+1} of S into $k+1$ nonempty subsets$\}$ and $\{\pi_k, \pi_{k+1}\}$ is an edge of G if and only if π_{k+1} is a *refinement* of π_k. Assume that all vertices of G are assigned weight 1. In general it is not known whether or not there is a matching between the two sets of vertices of G. The cardinality of B is given by $S(s, k)$, a Stirling number of the second kind (cf. [2]), and these numbers very quickly become unpleasantly large. We demonstrate the power of our methods for the case $s = 12$, $k = 5$. For these parameters $|B| = S(12, 5) = 1379400$ and $|A| = S(12, 6) = 1323652$. A natural choice for the partition $A_1 \cup \cdots \cup A_m = A$ is obtained by letting two partitions of S be in the same block A_i if and only if there is a 1–1 correspondence between the *cardinalities* of the blocks of the two partitions. Thus, the cardinalities generate the same *number-theoretic* partition of 12 into 6 parts. With $B_1 \cup \cdots \cup B_n = B$ defined similarly we find that $|A'| = 11$, $|B'| = 13$ and the matrix R is given by Table 1, where it is easily seen by the corollary that

$$\rho_{ij} = \frac{v_B(B_j)}{v_A(A_i)} = \frac{|B_j|}{|A_i|} \quad \text{for} \quad B \in D'(A_i).$$

TABLE 1

	11118	11127	11136	11145	11226	11235	1·1244	11334	12225	12234	12333	22224	22233
111117	$\frac{5}{8}$	10	0	0	0	0	0	0	0	0	0	0	0
111126	$\frac{1}{28}$	$\frac{4}{7}$	$\frac{4}{3}$	0	3	0	0	0	0	0	0	0	0
111135	$\frac{1}{56}$	0	$\frac{2}{3}$	1	0	6	0	0	0	0	0	0	0
111144	$\frac{1}{35}$	0	0	$\frac{8}{5}$	0	0	6	0	0	0	0	0	0
111225	0	$\frac{2}{21}$	0	$\frac{1}{3}$	$\frac{1}{2}$	2	0	0	1	0	0	0	0
111234	0	$\frac{1}{35}$	$\frac{1}{15}$	$\frac{1}{10}$	0	$\frac{3}{5}$	$\frac{3}{8}$	$\frac{1}{2}$	0	$\frac{3}{2}$	0	0	0
111333	0	0	$\frac{3}{10}$	0	0	0	0	$\frac{9}{4}$	0	0	3	0	0
112224	0	0	0	0	$\frac{1}{5}$	0	$\frac{1}{2}$	0	$\frac{2}{5}$	2	0	$\frac{1}{4}$	0
112233	0	0	0	0	$\frac{1}{10}$	$\frac{2}{5}$	0	$\frac{1}{3}$	0	1	$\frac{4}{9}$	0	$\frac{1}{3}$
122223	0	0	0	0	0	0	0	0	$\frac{2}{5}$	2	0	$\frac{1}{4}$	$\frac{2}{3}$
222222	0	0	0	0	0	0	0	0	0	0	0	5	0

R

The matrix $S = (\sigma_{ij})$ shown in Table 2 has $\sigma_{ij} \geqq 0$, $\sum_j \sigma_{ij}\rho_{ij} \geqq 1$ for all i, and $\sum_i \sigma_{ij} \leqq 1$ for all j.

TABLE 2

0	$\frac{1}{10}$	0	0	0	0	0	0	0	0	0	0	0
0	0	$\frac{3}{4}$	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	$\frac{1}{6}$	0	0	0	0	0	0
0	0	0	0	0	$\frac{1}{2}$	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	$\frac{2}{3}$	0	0	0
0	0	0	0	0	0	0	0	0	0	$\frac{1}{3}$	0	0
0	0	0	0	1	0	$\frac{5}{6}$	0	0	$\frac{22}{135}$	0	$\frac{4}{5}$	0
0	0	0	0	0	$\frac{1}{2}$	0	1	0	$\frac{23}{135}$	$\frac{2}{3}$	0	0
0	0	0	0	0	0	0	0	$\frac{5}{6}$	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	$\frac{1}{5}$	0

S

Hence, $\mathscr{L}(R) \leqq 1$ and by Theorem 1, v_B dominates v_A. We note that a direct verification that v_B dominates v_A, i.e., $|X| \leqq |D(X)|$ for all $X \subseteq A$, involves checking $2^{1323652}$ cases, a somewhat tedious task.

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