Monochromatic Equilateral Right Triangles on the Integer Grid

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Summary. For any coloring of the $N \times N$ grid using fewer than $\log \log N$ colors, one can always find a **monochromatic** equilateral right triangle, a triangle with vertex coordinates $(x, y), (x + d, y)$, and $(x, y + d)$.

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1 Introduction

The celebrated theorem of van der Waerden [Wae27] states that for any natural numbers k and r, there is a number $W(k, r)$ such that for any coloring of the first $W(k, r)$ natural numbers by r colors, there is always a monochromatic arithmetic progression of length k. Answering a question of Erdős and Turán [ET36], Roth [Roth53] proved a density version of van der Waerden's theorem for $k = 3$. He proved that $r_3(N)$, the cardinality of the largest subset of $\{1, \ldots, N\}$ containing no three distinct elements $x, x + d, x + 2d$ in arithmetic progression, is $O(N/\log \log N)$. This was not only the first proof for the conjecture of Erdős and Turán, but also the first efficient bound on $W(3, r)$. One of the goals of the present paper is to give a combinatorial proof of such a bound, proving that $W(3, r) \leq 2^{2^{cr}}$. The best known bound for $W(3, r)$ is the one which follows from Bourgain's [Bou99] result $r_3(N) = O(N(\log \log N/\log N)^{1/2})$, which is better than ours, but uses heavy tools from analysis. Van der Waerden's Theorem was extended by Gallai, proving that in any finite coloring of $\mathbb{Z}^2$, some color contains arbitrarily large square subarrays. The simplest density version of this extension is to prove that there is always a triangle in a dense $N \times N$ grid with vertex coordinates

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$(x, y), (x + d, y)$, and $(x, y + d)$, if N is large enough compared to the density. This was first asked by Erdős and Graham in [EG80]. The first proof of the statement was given by Ajtai and Szemerédi [AS74] and later a much more general theorem, the so called Multidimensional Szemerédi Theorem [Sze75] was presented by Fürstenberg and Katznelson [FK79]. The proofs gave no (or very weak) bounds for the maximum density of subsets of the grid avoiding such triangles. The best bound is due to Shkredov [Shk], who proved that if the density of a subset of the $N \times N$ grid is at least $1/(\log\log\log N)^c$ then it contains a triangle. Our main result is the following.

Theorem 1. *There is a universal* $c > 0$, *such that for any coloring of the* $N \times N$ *grid by no more than* $c \log\log N$ *colors, there is always a monochromatic triangle with vertex coordinates* $(x, y), (x + d, y)$, *and* $(x, y + d)$.

Corollary 2 (van der Waerden's Theorem, $k = 3$ case). *For any coloring of* $[N]$ *by no more than* $c \log\log N$ *colors, there is always a monochromatic arithmetic progression of length 3. Using the usual notation,* $W(3, k) \leq 2^{2^{ck}}$.

Proof. Every coloring of the set $\mathbb{Z}$ of integers defines a coloring of $\mathbb{Z}^2$ by giving the color of $x - y$ to the point with coordinates (x, y). In this way, a monochromatic triple with vertex coordinates $(x, y), (x + d, y)$, and $(x, y + d)$, defines a monochromatic arithmetic progression $x - y - d, x - y, x - y + d$.

It is worth mentioning that the traditional combinatorial proof using color focusing gives

$$W(3, k) \leq \left. k^{k^{k^{\cdot^{\cdot^{\cdot^{k^{4k}}}}}}} \right\} (k - 1),$$

a tower-type bound.

2 Proof of Theorem 1

Let us suppose that the points of the $N \times N$ grid are colored by L colors, and there is no monochromatic equilateral right triangle. We will show that L must be large. Let us examine the coloring of the elements of the points on the diagonal of the grid, i.e., the points with coordinates (x, y) such that $x + y = N + 1$. Select the most popular color, denoted by c_1. The set of points of the diagonal with color c_1 is denoted by S_1. For any pair $p = (a, b), q = (c, d)$, elements of S_1, the points (a, d) and (c, b) cannot have the color c_1. The Cartesian product defined by the points of S_1 has the property that only the diagonal has points with color c_1. The lower-triangular part of the Cartesian product is denoted by T_1, i.e.,

$$T_1 = \{(x, y) : \exists\, s, t \ni (x, t), (s, y) \in S_1, s > x\}$$

Note that $s_1 := |S_1| \geq \frac{N}{L}$. We now define the color c_{i+1}, the set S_{i+1}, and T_{i+1} recursively, based on c_i, S_i, and T_i (where $i \geq 1$).

Suppose the pointset T_i avoids the colors $c_1, c_2, \ldots, c_i$. There is a line with slope -1, which contains many points of T_i. Let m be such that

$$|\{(x,y) : x + y = m\} \cap T_i| \geq \frac{|T_i|}{N}.$$

Select the points with the most popular color, c_{i+1}, in T_i along the line $x + y = m$. The set of these points will be S_{i+1}, and

$$T_{i+1} = \{(x,y) : \exists\, s,t \ni (x,t),(s,y) \in S_{i+1}, s > x\}.$$

Thus, the pointset T_{i+1} avoids the colors $c_1, c_2, \ldots, c_{i+1}$. Note that we have the inequality

$$s_{i+1} = |S_{i+1}| \geq \frac{\binom{s_i}{2}}{(L-i)N}.$$

If we reach Step L with $s_L \geq 2$ then we have a contradiction, since we run out of colors for T_L.
From the formula above, one can already get a feeling for the magnitude of the bound. However, for the formal proof of Theorem 1, we prove the following.

Lemma 3. *If $s_1 \geq N/r$, $s_{i+1} \geq \frac{1}{(r-i)N}\binom{s_i}{2}$ and $N = N(r) = (2r)^{2^r}$ then $s_r \geq 2$.*

Proof. We prove by induction on i that for $1 \leq i \leq r$, we have:

(a) $s_i \geq \frac{N}{2^{2^{i-1}-1} r^{2^i-1}}$,
(b) $s_i \geq r/i$.

This is clearly true for $i = 1$. Suppose it is true for some $i < r$. Then

$$s_{i+1} \geq \frac{1}{(r-i)N}\binom{s_i}{2} = \frac{s_i^2}{2rN} \cdot \frac{r}{r-i} \cdot \frac{s_i - 1}{s_i}$$

But

$$\frac{r}{r-i} \cdot \frac{s_i - 1}{s_i} \geq 1$$

since $s_i \geq r/i$ by induction. Hence, we have

$$s_{i+1} \geq \frac{s_i^2}{2rN} \geq \frac{1}{2rN} \cdot \frac{N^2}{2^{2^i-2} r^{2^{i+1}-2}} = \frac{N}{2^{2^i-1} r^{2^{i+1}-1}}$$

which is (a) for $i+1$. It is easy to see that (b) also holds for $i+1$ as well. The inequality for s_r is now

$$s_r \geq \frac{(2r)^{2^r}}{2^{2^{r-1}-1} r^{2^r-1}} \geq 2^{2^{r-1}+1} r \geq 2.$$

This completes the proof of the lemma and Theorem 1.

We note here that with a similar but somewhat more complicated argument, we can prove that there are many monochromatic corners when the number of colors is small. In particular, we can show:

Theorem 4. *For any integer $r > 0$, if the lattice points in the $N \times N$ grid are arbitrarily r-colored, and $N > 2^{2^{3r}}$ then there are always at least $\delta(r)N^3$ monochromatic "corners", i.e., triples of points $(x, y), (x + d, y), (x, y + d)$ for some $d > 0$, where $\delta(r) = (3r)^{-2^{r+2}}$.*

We note that this is similar in spirit to the results of [FGR88] where it is shown that in fact a **positive fraction** of the objects being colored must occur monochromatically. The proof follows that of Theorem 1 and is omitted.

We should also point out that this approach can be used to prove directly a quantitative version van der Waerden's theorem for 3-term arithmetic progressions, namely that if $\mathbf{Z}_p$ is colored by at most $c \log \log p$ colors, then some monochromatic 3-term arithmetic progression must be formed. Similarly, analogous results can be obtained for the occurrence of monochromatic affine lines in $GF(3)^n$ using this approach.

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